MATH 323 - EXERCISES 7- SOLUTIONS

1. (a) Density function must integrate to 1 over $\mathcal{Y} = [0,1]$, so

$$\int_0^1 f(y)\,dy = 1 \Longrightarrow \int_0^1 cy^2 + y\,dy = 1 \Longrightarrow \left[c\frac{y^3}{3} + \frac{y^2}{2}\right]_0^1 = 1 \Longrightarrow c = \frac{3}{2}$$

(b) Distribution function F given for $0 \le y \le 1$ by

$$F(y) = \int_0^y f(t)\,dt = \frac{y^3 + y^2}{2}$$

and $F(y) = 0$ for $y < 0$, and $F(y) = 1$ for $y > 1$.

(c) $P(Y < 1/2) = F(1/2) = 3/16$.

(d) From definition of conditional probability

$$P(Y > 1/2 \mid Y > 1/4\,) = \frac{P(Y > 1/2,\ Y > 1/4\,)}{P(Y > 1/4\,)} = \frac{P(Y > 1/2\,)}{P(Y > 1/4\,)} = \frac{1 - F(1/2)}{1 - F(1/4)} = \frac{104}{123}$$

(e) If S is the sum random variable, then $S \sim Binomial(200, p)$ where

$$p = P(Y_i = 1) = P(Y < 1/6) = F(1/6) = \frac{7}{432}$$

Then by direct calculation

$$P(S \le 3) = P(S = 0) + P(S = 1) + P(S = 2) + P(S = 3) = 0.593$$

2. Density function must integrate to 1 over $\mathcal{Y} = [-1,1]$, so

$$\int_{-1}^1 f(y)\,dy = 1 \Longrightarrow c = \frac{3}{4}$$

Distribution function F given for $-1 \le y \le 1$ by

$$F(y) = \int_{-1}^y f(t)\,dt = \left[\frac{3y}{4} - \frac{y^3}{4}\right]_{-1}^1 = \frac{3}{4}\left[\left(y - \frac{y^3}{3}\right) + \frac{2}{3}\right] = \frac{3}{4}\left(y - \frac{y^3}{3}\right) + \frac{1}{2}$$

and $F(y) = 0$ for $y < -1$, and $F(y) = 1$ for $y > 1$.

3. (a) Density function must integrate to 1 over $\mathcal{Y} = [0, \pi/2]$, so

$$\int_0^{\pi/2} f(y)\,dy = 1 \Longrightarrow \int_0^{\pi/2} cy(\pi - y)\,dy = 1 \Longrightarrow c = \frac{12}{\pi^3}$$

(b) The area is $X = \frac{1}{2}\sin Y$, so $\mathcal{Y} = [0, \pi/2] \Longrightarrow \mathcal{X} = [0, 1/2]$, and distribution function is F_X given by

$$F_X(x) = P(X \leq x\ ) = P\left(\frac{1}{2}\sin Y \leq x\ \right) = P(Y \leq \sin^{-1}(2x)) = F(\sin^{-1}(2x))$$

as $\sin$ is monotone on $[0, \pi/2]$. Hence by differentiation with respect to x,

$$f_X(x) = \frac{2}{\sqrt{1-4x^2}} f(\sin^{-1}(2x)) = \qquad 0 \leq x \leq 1/2$$

and zero otherwise.

4. By differentiation, $f(y) = 2ye^{-y^2},\ y > 0$, and zero otherwise

5. Need to consider ranges of integration carefully;

$$F(y) = \begin{cases} \displaystyle\int_0^y t\ dt & = y^2/2 \qquad 0 \leq y \leq 1 \\ \displaystyle\int_0^1 t\ dt + \int_1^y (2-t)\ dt & = 2y - y^2/2 - 1 \quad 1 \leq y \leq 2 \end{cases}$$

and $F(y) = 0$ for $y < 0$, and $F(y) = 1$ for $y > 2$. Hence $P(0.8 < Y < 1.2) = F(1.2) - F(0.8) = 0.36$.

6. (a) If $\Phi(.)$ denotes the standard Normal cdf, then

$$P(\ T > 3\,|A) = 1 - P(\ T \leq 3\,|A) = 1 - \Phi((3-2)/(3/4)) = 1 - \Phi(4/3) = 1 - 0.90878 = 0.0912.$$

(b) By the Theorem of Total Probability

$$P(\ T > 3\ ) = P(\ T > 3\,|A)P(A) + P(\ T > 3\,|B)P(B) = (1 - \Phi(4/3)) \times \frac{3}{10} + (1 - \Phi(-4/3)) \times \frac{7}{10}$$

so $P(T > 3) = 0.6635$.

(c) By Bayes Theorem

$$P(A|\ T > 3\ ) = \frac{P[\ T > 3\,|A)P(A)}{P(\ T > 3\ )} = \frac{0.0912 \times 0.3}{0.6635} = 0.0412.$$

7. (a) $Y \sim \mathit{Exponential}(\beta) \Longrightarrow F(y) = 1 - e^{-y/\beta}$ for $y > 0$, so $F(y) = 1/2 \Longrightarrow y = \beta \ln 2$.

(b) $\ln Y \sim \mathit{Normal}(\mu, \sigma^2)$, so

$$F(y) = 1/2 \Longrightarrow P(\ Y \leq y\ ) = 1/2 \Longrightarrow P(\ (\ln Y - \mu)/\sigma \leq (\ln y - \mu)/\sigma) = 1/2$$

$$\Longrightarrow \Phi((\ln y - \mu)/\sigma) = 1/2 \Longrightarrow (\ln y - \mu)/\sigma = 0 \Longrightarrow \ln y = \mu \Longrightarrow y = e^{\mu}$$