MATH 323 - EXERCISES 6- SOLUTIONS

1. We have

$$G(t) = \mathbb{E}\left[t^Y\right] = \sum_{y=0}^{\infty} t^y p(y)$$

so therefore differentiating the power series, we have that

$$G^{(k)}(t) = \sum_{y=k}^{\infty} y(y-1)\ldots(y-k+1)t^{y-k}p(y)$$

and hence $G^{(k)}(0) = k!p(k)$. Hence we can recover the function $p(y)$ by

$$p(y) = \frac{G^{(y)}(0)}{y!} \qquad y = 0, 1, 2, \ldots.$$

Note that from this result, we have that

$$G(t) = \sum_{y=0}^{\infty} \frac{G^{(y)}(0)}{y!} t^y$$

This is a Taylor series expansion of $G(t)$ at $t = 0$ (this is called a *Maclaurin* series): it is a polynomial in t. Now suppose $G_1(t) = G_2(t)$ for all $|t| < 1$. Consider $G_1(t) - G_2(t)$; we have that

$$G_1(t) - G_2(t) = 0 \qquad \text{for all } t,\ -1 < t < 1.$$

We have from the definition that

$$G_1(t) - G_2(t) = \sum_{y=0}^{\infty} (p_1(y) - p_2(y))t^y = \sum_{y=0}^{\infty} \frac{(G_1^{(y)}(0) - G_2^{(y)}(0))}{y!} t^y = 0.$$

Thus we must have that $p_1(y) = p_2(y)$ for all y: this follows by noting that, using the same method

$$\frac{d^k}{dt^k}\left\{G_1(t) - G_2(t)\right\}_{t=0} = k!(p_1(k) - p_2(k)) = 0$$

for $k = 0, 1, 2, \ldots$.

2. By using the formulae given in lectures:

(a) $Binomial(1, 1/2)$.

(b) Discrete Uniform on $\{1, 2, 3\}$, that is $p(y) = 1/3$ for $y = 1, 2, 3$, zero otherwise.

(c) $Binomial(3, 1/3)$

(d) $p(0) = 2/3, p(1) = 1/3$, zero otherwise.

(e) $Poisson(2)$

(f) Discrete Uniform on $\{0, 1, 2, \ldots, N\}$, that is

$$p(y) = c = \frac{1}{N+1} \qquad y = 0, 1, 2, \ldots, N,$$

and zero otherwise.

3. We have

$$G(t) = 1 - (1-t^2)^{1/2}.$$

For $-1 < t < 1$, as in question 1, the function has the Maclaurin expansion

$$G(t) = \sum_{y=0}^{\infty} \frac{G^{(y)}(0)}{y!} t^y.$$

Certainly $G(1) = 1$: by the definition this is a fundamental requirement of a pgf. Therefore

$$\sum_{y=0}^{\infty} \frac{G^{(y)}(0)}{y!} = 1$$

and it only remains to show that $G^{(k)}(0) \geq 0$ for all k as then we have that $G(t)$ is the pgf for

$$p(y) = \frac{G^{(y)}(0)}{y!} \qquad y = 0, 1, 2 \ldots.$$

Now, we have using the Maclaurin series expansion, or the binomial theorem for the exponent $1/2$, that the required series expansion is

$$\begin{aligned}(1-t^2)^{1/2} &= 1 + \frac{1}{2}(-t^2) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(-t^2)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-t^2)^3 + \cdots \\ &= 1 - \frac{1}{2}t^2 - \frac{1}{8}t^4 - \frac{3}{24}t^6 - \cdots\end{aligned}$$

that is, the jth coefficient is

$$\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{1}{2}-(j-1)\right)}{j!}(-t^2)^j.$$

Therefore

$$G(t) = \frac{1}{2}t^2 + \frac{1}{8}t^4 + \frac{3}{24}t^6 + \cdots$$

and as all the coefficients are positive, we must have that $G(t)$ is a valid pgf.

If you have not encountered the binomial expansion for fractional exponents, you may proceed as follows: we want the series expansion of $(1-t^2)$, so write

$$(1-t^2)^{1/2} = \sum_{j=0}^{\infty} a_j t^j.$$

Then squaring both sides, we have the identity

$$1 - t^2 = \left(\sum_{j=0}^{\infty} a_j t^j\right)\left(\sum_{k=0}^{\infty} a_k t^k\right) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_j a_k t^{j+k} = \sum_{l=0}^{\infty}\left\{\sum_{k=0}^{l} a_{l-k}a_k\right\} t^l$$

and in general the lth coefficient is

$$\sum_{k=0}^{l} a_{l-k}a_k = \begin{cases} 1 & l = 0 \\ -1 & l = 2 \\ 0 & \text{otherwise}\end{cases}$$

Thus we have by direct computation that $a_0 = 1$, $a_1 = 0$, $a_2 = -1/2$, $a_3 = 0$ etc.

4. We consider the mgf: we have by definitions of expectation

$$m_X(t) = \mathbb{E}\left[e^{tX}\right] = \mathbb{E}\left[e^{t(aY+b)}\right] = e^{bt}\mathbb{E}\left[e^{atY}\right] = e^{bt}m_Y(at)$$

defined on the range of t such that $\mathbb{E}\left[e^{atY}\right]$ is finite. For the pgf, we have

$$G_X(t) = m_X(\ln t) = e^{b\ln t}m_Y(a\ln t) = t^b m_Y(\ln(t^a)) = t^b G_X(t^a).$$

5. By definition, assuming that the sum is finite,

$$m(t) = \sum_{y\neq 0} e^{ty}p(y) = \sum_{y\neq 0} e^{ty}\frac{1}{2}p^{|y|-1}(1-p).$$

We split the sum as

$$\sum_{y=-\infty}^{-1} e^{ty}\frac{1}{2}p^{|y|-1}(1-p) + \sum_{y=1}^{\infty} e^{ty}\frac{1}{2}p^{|y|-1}(1-p).$$

The second term is merely $(1/2)$ times the mgf for a Geometric random variable with success probability $q = 1-p$,

$$\frac{1}{2}\frac{qe^t}{1-pe^t}.$$

For the first term

$$\sum_{y=-\infty}^{-1} e^{ty}\frac{1}{2}p^{|y|-1}(1-p) = \sum_{y=1}^{\infty} e^{-ty}\frac{1}{2}p^{y-1}(1-p) = \frac{1}{2}\frac{qe^{-t}}{1-pe^{-t}}$$

by summing the geometric series. Thus

$$m(t) = \frac{1}{2}\frac{qe^t}{1-pe^t} + \frac{1}{2}\frac{qe^{-t}}{1-pe^{-t}}.$$

6. As Y only takes non-negative values, we have that $P(Y<0)=0$, and hence

$$\sum_{y=0}^{\infty} P(Y<y)t^y = \sum_{y=1}^{\infty} P(Y<y)t^y = t\sum_{y=1}^{\infty} P(Y<y)t^{y-1} = t\sum_{y=1}^{\infty}\left\{\sum_{j=0}^{y-1}p(j)\right\}t^{y-1}$$

Exchanging the order of summation, we have that

$$\sum_{y=1}^{\infty}\sum_{j=0}^{y-1}p(j)t^{y-1} = \sum_{j=0}^{\infty}\sum_{y=j+1}^{\infty}p(j)t^{y-1}.$$

But the final sum can be rewritten

$$\sum_{j=0}^{\infty}\left\{\sum_{y=j+1}^{\infty}t^{y-1}\right\}p(j) = \sum_{j=0}^{\infty}\frac{t^j}{1-t}p(j)$$

using the sum of the geometric series for the inner bracket. Hence

$$\sum_{y=0}^{\infty} P(Y<y)t^y = t\sum_{j=0}^{\infty}\frac{t^j}{1-t}p(j) = \frac{t}{1-t}\sum_{j=0}^{\infty}t^j p(j) = \frac{t}{1-t}G(t).$$