## MATH 323 - Exercises 5- Solutions

1. Need to show that p(.) is non-negative, and sums to one over the range of *Y* identified. Sum of geometric series gives result, that is,

$$\sum_{y=0}^{\infty} p(y) = \sum_{y=0}^{\infty} \frac{1}{1+\lambda} \left(\frac{\lambda}{1+\lambda}\right)^y = \frac{1}{1+\lambda} \left(1 - \frac{\lambda}{1+\lambda}\right)^{-1} = 1$$

Distribution function *F* given, for y = 0, 1, ... by

$$F(y) = \sum_{i=0}^{y} p(i) = \sum_{i=0}^{y} \frac{1}{1+\lambda} \left(\frac{\lambda}{1+\lambda}\right)^{i} = (1-p) \sum_{i=0}^{y} p^{i} = (1-p) \frac{1-p^{y+1}}{1-p} = 1-p^{y+1}$$

where  $p = \lambda/(1 + \lambda)$ . This is the *Geometric* distribution in a slightly different form.

2.  $X = Y - n \implies \mathcal{X} = \{0, 1, 2, \ldots\}$  and for  $x \in \mathcal{X}$ 

$$p_X(x) = P(X = x) = P(Y - n = x) = P(Y = x + n) = \binom{n + x - 1}{n - 1} p^n (1 - p)^x$$

with  $p_X(x) = 0$  otherwise.

3. We can see that the approximation works for  $Y \sim Hypergeometric(N, R, n)$  by manipulating the combinatorial terms. For  $y \in \{\max(0, n - N + R), \dots, \min(n, R)\}$ 

$$p(y) = \frac{\binom{R}{y}\binom{N-R}{n-y}}{\binom{N}{n}} = \frac{R!}{y!(R-y)!} \frac{(N-R)!}{(n-y)!(N-R-n+y)!} \frac{n!(N-n)!}{N!}$$
$$= \binom{n}{y} \frac{R!}{(R-y)!} \frac{(N-R)!}{(N-R-n+y)!} \frac{(N-n)!}{N!}$$
$$\approx \binom{n}{y} R^y (N-R)^{n-y} \frac{1}{N^n} = \binom{n}{y} p^y (1-p)^{n-y}$$

where p = R/N, as for *N* and *R* large

$$\frac{R!}{(R-y)!} \approx R^y \qquad \frac{(N-R)!}{(N-R-n+y)!} \approx (N-R)^{n-y} \qquad \frac{(N-n)!}{N!} = \frac{1}{N^n}$$

Now for N and R large, with p = R/N fixed, the hypergeometric distribution is tends to the binomial distribution, and thus sampling without replacement from a large population is approximately equivalent to sampling with replacement.

4. Let U = "Number of Heads"; U represents the number of 'successes' in a sequence of binary trials, so from lectures we know that

$$U \sim Binomial(n, 1/2),$$

and Y = U - (n - U) = 2U - n. Thus

$$\mathcal{Y} = \{-n, -n+2, -n+4, \dots, n-4, n-2, n\}$$

and for  $y \in \mathcal{Y}$ ,

$$p(y) = P(Y = y) = P(2U - n = y) = P(U = (y + n)/2) = \binom{n}{(y + n)/2} \left(\frac{1}{2}\right)^n$$

with p(y) = 0 otherwise.

5. If  $Y \sim Geometric(p)$ , then

$$p(y) = (1-p)^{y-1}p$$
  $F(y) = 1 - (1-p)^y$ 

for  $y \in \{1, 2, 3...\}$ . Thus  $P(Y > n) = (1 - p)^n$ , and hence

$$P(Y = n + k \mid Y > n) = \frac{P(Y = n + k, Y > n)}{P(Y > n)} = \frac{P(Y = n + k)}{P(Y > n)}$$
$$= \frac{(1 - p)^{n + k - 1}p}{(1 - p)^n} = (1 - p)^{k - 1}p = P(Y = k)$$

- 6. Need p(y) non-negative and convergent;
  - (i) always non-negative and convergent; need k = 1
  - (ii) always non-negative and convergent if  $\alpha < -1$ ; no closed form for k.

7. (i) 
$$Y = \min\{Y_1, \dots, Y_n\}$$
, so  $\mathcal{Y} = \{0, 1\}$ .  
 $P(Y = 1) = P(\min\{Y_1, \dots, Y_n\} = 1) = P(Y_1 = 1, Y_2 = 1, \dots, Y_n = 1) = p^n$   
 $P(Y = 0) = 1 - p^n$ 

Hence

$$p(y) = \begin{cases} 1 - p^n & y = 0\\ p^n & y = 1 \end{cases}$$

(ii) 
$$Z = \max\{Y_1, \dots, Y_n\}$$
, so  $\mathcal{Z} = \{0, 1\}$ .  
 $P(Z = 0) = P(\max\{Y_1, \dots, Y_n\} = 0) = P(Y_1 = 0, Y_2 = 0, \dots, Y_n = 0) = (1 - p)^n$   
 $P(Z = 1) = 1 - (1 - p)^n$ 

Hence

$$p(z) = \begin{cases} (1-p)^n & z = 0\\ 1 - (1-p)^n & z = 1 \end{cases}$$

8. We have that the function  $\mathbb{1}_A$  is a mapping acting on S,  $\mathbb{1}_A : S \longrightarrow \{0, 1\}$  and

$$P(\mathbb{1}_A(s) = 1) \equiv P(s \in A) = P(A) = p$$

and hence  $\mathbb{1}_A$  is a Bernoulli random variable with parameter p. If Y is discrete, can always write  $\mathcal{Y} = \{y_1, y_2, \ldots\}$  as a list of those values for which p(y) > 0, and then express

$$Y = \sum_{i=1}^{\infty} \mathbb{1}_{A_i} y_i$$

where  $A_i = \{s : Y(s) = y_i\}.$