MATH 323 - EXERCISES 4- SOLUTIONS

1. For events $A_1, A_2, B \subseteq S$ with $P(B) > 0$: we must have that S has a finite number n_S of outcomes, and suppose A_1, A_2 and B have n_1, n_2 and n_B outcomes. Suppose n_{1B} is the number in the intersection $A_1 \cap B$.

$$\text{(I)} \qquad P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = n_{1B}/n_B \geq 0$$

$$\text{(II)} \quad P(S|B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = n_B/n_B = 1$$

$$\text{(III)} \quad P(A_1 \cup A_2|B) = \frac{P((A_1 \cup A_2) \cap B)}{P(B)} = \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)}$$

$$= (n_{1B}/n_B) + (n_{2B}/n_B) = P(A_1|B) + P(A_2|B)$$

as $(A_1 \cup A_2) \cap B = (A_1 \cap B) \cup (A_2 \cap B)$ which are disjoint events.

2. (a) We have

$$\begin{aligned}
P(A' \cap B) &= P(B) - P(A \cap B) = P(B) - P(A)P(B) = (1 - P(A))P(B) = P(A')P(B) \\
P(A' \cap B') &= 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B) \\
&= 1 - P(A) - P(B) + P(A)P(B) = (1 - P(A))(1 - P(B)) = P(A')P(B')
\end{aligned}$$

   (b) By definition of disjoint events

$$P(A \cap B) = 0 \quad \Longleftrightarrow \quad P(A)P(B) = 0 \quad \Longleftrightarrow \quad \text{at least one of } P(A), P(B) \text{ is zero}$$

3. (a) Using the general addition rule

$$P(A) = P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) = 0.95 + 0.9 - 0.95 \times 0.9 = 0.995$$

   (b) By the partition of B,

$$\begin{aligned}
P(B) &= P(B_1 \cap B_2 \cap B_3') + P(B_1 \cap B_2' \cap B_3) + P(B_1' \cap B_2 \cap B_3) + P(B_1 \cap B_2 \cap B_3) \\
&= (0.8 \times 0.8 \times 0.2) + (0.8 \times 0.2 \times 0.8) + (0.2 \times 0.8 \times 0.8) + (0.8 \times 0.8 \times 0.8) \\
&= 0.896
\end{aligned}$$

   (c) By the general multiplication rule

$$\begin{aligned}
P(\text{System Functions}) &= P(A \cap B \cap C \cap D \cap E) = P(A)P(B)P(C)P(D)P(E) \\
&= 0.995 \times 0.896 \times 0.95^3 = 0.764
\end{aligned}$$

4. For the first part: given $P(A) = P(B) = P(C) = 1/3$, and $P(G_{AB}|A) = 1/2$, $P(G_{AB}|B) = 0$ and $P(G_{AB}|C) = 1$, and hence by Bayes theorem we can compute that $P(A|G_{AB}) = 1/3$ and hence governor is correct – no information has been given away.

   (ii) For the second part: we still have $P(G_{WB}|A) = 1/2$, $P(G_{WB}|B) = 0$ but $P(G_{WB}|C) = 1/2$, so by Bayes theorem $P(C|G_{WB}) = 1/2$, and hence C is right to feel happier.

5. Given $P(G) = p$, $P(A|G) = 1$, $P(A|G') = \pi$. Then by Bayes Theorem

$$P(G|A) = \frac{P(A|G)P(G)}{P(A|G)P(G) + P(A|G')P(G')} = \frac{1 \times p}{1 \times p + \pi \times (1-p)}$$

so

$$\frac{P(G|A)}{P(G'|A)} = \frac{p}{\pi \times (1-p)} = \frac{1}{\pi}\frac{P(G)}{P(G')}$$

6. Let $T \equiv$ "Test positive", $D \equiv$ "Disease Sufferer". Then $P(T|D) = 0.95$, $P(T|D') = 0.10$, $P(D) = 0.005$. Hence

(a) $P(T) \quad = \quad = P(T|D)P(D) + P(T|D')P(D')(0.95 \times 0.005) + (0.1 \times 0.995) = 0.10425$

(b) $$P(D|T) = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D')P(D')} = \frac{0.95 \times 0.005}{(0.95 \times 0.005) + (0.1 \times 0.995)} = 0.0455$$

(c) $$P(D'|T') = \frac{P(T'|D')P(D')}{P(T')} = \frac{0.9 \times 0.995}{1 - 0.10425} = 0.9997$$

(d) $P(M) = P(T \cap D') + P(T' \cap D) = P(T|D')P(D') + P(T'|D)P(D) = 0.09975$

7. Let $T_1 \equiv$ "first test positive", $T_2 \equiv$ "second test positive", $C \equiv$ "drugs present in sample". Then given that

$$P(T_1|C) = P(T_2|C) = 0.995 \qquad P(T_1^{'}|C') = P(T_2^{'}|C') = 0.98.$$

(a) By the Theorem of Total Probability

$$P(T_1) = P(T_1|C)P(C) + P(T_1|C')P(C') = 0.995 \times 0.001 + (1 - 0.98) \times 0.999 = 0.021.$$

(b) By Bayes Theorem

$$P(C|T_1) = \frac{P(T_1|C)P(C)}{P(T_1|C)P(C) + P(T_1|C')P(C')} = \frac{0.995 \times 0.001}{0.995 \times 0.001 + (1 - 0.98) \times 0.999} = 0.047.$$

(c) By the Theorem of Total Probability and conditional independence

$$\begin{aligned} P(T_1 \cap T_2) &= P(T_1 \cap T_2|C)P(C) + P(T_1 \cap T_2|C')P(C') \\ &= P(T_1|C)P(T_2|C)P(C) + P(T_1|C')P(T_2|C')P(C') \\ &= 0.995^2 \times 0.001 + (1 - 0.98)^2 \times 0.999 = 0.0014. \end{aligned}$$

(d) By Bayes Theorem

$$\begin{aligned} P(C|T_1 \cap T_2) &= \frac{P(T_1 \cap T_2|C)P(C)}{P(T_1 \cap T_2|C)P(C) + P(T_1 \cap T_2|C')P(C')} \\ &= \frac{P(T_1|C)P(T_2|C)P(C)}{P(T_1|C)P(T_2|C)P(C) + P(T_1|C')P(T_2|C')P(C')} \\ &= \frac{0.995^2 \times 0.001}{0.995^2 \times 0.001 + (1 - 0.98)^2 \times 0.999} = 0.712 \end{aligned}$$