MATH 323 - EXERCISES 3 – SOLUTIONS

1. (a) Special case of the Binomial Expansion of $(a+b)^n$ with $a=1, b=-1$.

   (b) Take $a=x$ and $b=1$ for simplicity; then $(x+1)^{m+n}=(x+1)^m(x+1)^n$. We have

$$\text{L.H.S.} \quad : \quad (x+1)^{m+n} = \sum_{k=0}^{m+n}\binom{m+n}{k}x^k$$

$$\text{R.H.S.} \quad : \quad (x+1)^m(x+1)^n = \left\{\sum_{i=0}^{m}\binom{m}{i}x^i\right\}\left\{\sum_{j=0}^{n}\binom{n}{j}x^j\right\}$$

$$= \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}x^{i+j} = \sum_{k=0}^{m+n}\left\{\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}\right\}x^k$$

   putting $i+j=k$, and re-arranging summations. This holds for arbitrary x, so result follows by equating left and right hand sides and comparing coefficients of x^k.

2. $N=49$, $R=6$ (winning numbers), and $n=6$ (drawn numbers). To select precisely r winning numbers (for $r=0,1,2,3,4,5,6$), we would need to first pick r from R, but then also pick $n-r$ non winning numbers from the $N-R$ available. The number of ways of completing these selections is

$$\binom{R}{r}\times\binom{N-R}{n-r}$$

   whereas there are

$$\binom{N}{n}$$

   ways of drawing the numbers in total.

   (a) $r=6$

$$\text{Probability} = \frac{\binom{6}{6}\binom{43}{0}}{\binom{49}{6}} = \frac{1}{13983816} \approx 7.15\times 10^{-8}$$

   (b) $r=3$

$$\text{Probability} = \frac{\binom{6}{3}\binom{43}{3}}{\binom{49}{6}} \approx 0.0177$$

   (c) $r=0$

$$P(\text{No winning numbers}) = \frac{\binom{n}{r}\binom{N-n}{R-r}}{\binom{N}{R}} = \frac{\binom{43}{6}}{\binom{49}{6}} = 0.435965.$$

   (d) By considering the multiplication principle and noting that the result in one week does not affect the result in the next week, we have that

$$P(\text{No winning numbers for } x \text{ weeks}) = \left\{\frac{\binom{43}{6}}{\binom{49}{6}}\right\}^x$$

3. Similar 'hypergeometric probabilities' with $N = 80$, $R = 20$, $n = 5, 10, 15$, $r = 0$, for which probabilities are

$$\text{(a)}\ \frac{\binom{5}{0}\binom{75}{20}}{\binom{80}{20}} = \frac{\binom{20}{0}\binom{60}{5}}{\binom{80}{5}} \quad \text{(b)}\ \frac{\binom{10}{0}\binom{70}{20}}{\binom{80}{20}} = \frac{\binom{20}{0}\binom{60}{10}}{\binom{80}{10}} \quad \text{(c)}\ \frac{\binom{15}{0}\binom{65}{20}}{\binom{80}{20}} = \frac{\binom{20}{0}\binom{60}{15}}{\binom{80}{15}}$$

General formula is given by the hypergeometric formula.

4. This is a 'hypergeometric' problem with $N = 200$, $R = 120$, $n = 5$ and $r = 5$, thus the probability required is

$$\frac{\binom{5}{5}\binom{200-5}{120-5}}{\binom{200}{120}} = \frac{120 \times 119 \times 118 \times 117 \times 116}{200 \times 199 \times 198 \times 197 \times 196} = 0.075$$

which may also be calculated using a conditional probability argument.

Thus the probability that an all-Science committee is selected, if all selections of five from two hundred are equally likely, is 0.075. This is small, but conventionally we consider a probability of 0.05 as providing sufficient evidence against a hypothesized model, so perhaps this is not conclusive evidence of bias.

5. (a) Choose 13 cards from $52 \Longrightarrow n_S = \binom{52}{13}$

   (b) There are 36 cards that are not Aces, Kings, Queens or Jacks:

$$\text{Choose 13 cards from } 36 \Longrightarrow n_A = \binom{36}{13} \Longrightarrow P(A) = \frac{n_A}{n_S} \approx 0.003$$

6. Total number of allocations of birthdays to months is $n_S = 12^{30}$. For n_A, select 6 months from 12 as the "two-birthday" months, and then *partition* the 30 birthdays into 12 subgroups, six which contain 2 birthdays, and six which contain three. Hence, using the multinomial formula for partitions,

$$n_A = \binom{12}{6}\frac{30!}{(2!)^6(3!)^6} \quad \Longrightarrow \quad P(A) = \binom{12}{6}\frac{30!}{12^{30}(2!)^6(3!)^6}$$

7. Hypergeometric formula again: $2n$ Arts (TYPE I) and $2n$ Science (TYPE II), sample of size $2n$ without replacement from population of size $4n$. Hence probability of even split in each group is

$$\frac{\binom{2n}{n}\binom{2n}{n}}{\binom{4n}{2n}}$$

8. Multistage Hypergeometric: select the first hand with n then the second hand with m all without replacement. Hence by multiplication theorem, probability is

$$\frac{\binom{13}{n}\binom{39}{13-n}}{\binom{52}{13}} \times \frac{\binom{13-n}{m}\binom{26+n}{13-m}}{\binom{39}{13}}$$

By extension, need to select third hand in similar fashion, but once first three hands are selected, final hand is fixed. Hence probability is

$$\frac{\binom{13}{r_1}\binom{39}{13-r_1}}{\binom{52}{13}} \times \frac{\binom{13-r_1}{r_2}\binom{26+r_1}{13-r_2}}{\binom{39}{13}} \times \frac{\binom{13-r_1-r_2}{r_3}\binom{13+r_1+r_2}{13-r_3}}{\binom{26}{13}}$$

9. Total number of seating arrangements is $n!$. Of these, you and friend sit together in

$$(n-1) \times 2! \times (n-2)!$$

arrangements (choose a pair of adjacent seats for you and friend from $n-1$ pairs available, arrange yourselves in one of 2! ways, and then arrange remaining $n-2$ people in remaining seats). Hence probability is

$$\frac{(n-1) \times 2! \times (n-2)!}{n!} = \frac{2}{n}$$

10. Total number of possible allocations is $n \times n \times ... \times n = n^n$. For n_A, consider selecting two different boxes from n to be the empty box and the box containing two balls; after this selection has been made, can arrange the balls in the boxes in $n!$ ways; hence

$$n_A = \binom{n}{2} \times n!$$

and

$$P(A) = \frac{\binom{n}{2} \times n!}{n^n}$$

11. Total number of arrangements of $2n$ objects is $(2n)!$.

    (i) For n_A, there are n *pairs* of *matched* short and long parts, which can be arranged in $n!$ ways in the sequence; each pair can be arranged in 2 ways. Hence $n_A = 2^n n!$, and hence

    $$P(A) = \frac{2^n n!}{(2n)!}$$

    (ii) Here, for n_A, consider arranging a sequence of n short parts and a parallel sequence of n long parts to match with them; there are $n!$ ways of arranging the short parts and $n!$ ways of arranging the long parts. Once the parallel sequences have been arranged individually and paired, each pair can be arranged in 2 ways (long/short or short/long). Hence $n_A = 2^n n!n!$, and

    $$P(A) = \frac{2^n n!n!}{(2n)!} = \frac{2^n}{\binom{2n}{n}}$$