MATH 323 - EXERCISES 2 – SOLUTIONS

1. (a) Partition according to disease status : $(D \cap T') \cup (D' \cap T)$

   (b) Partition into correctly classified sufferers/non-sufferers

(i)	Sufferers	:	$(D \cap (X \cup T) \cap A)$
(ii)	Non-sufferers	:	$(D' \cap (((X' \cap T') \cap A) \cup A'))$

   (i) follows as sufferers (D) must have the correct doctor's diagnosis (A), **and** at least one the tests indicating the disease ($X \cup T$).

   (ii) follows as non-sufferers (D') can **either** receive the incorrect – for them – doctor's diagnosis of being a sufferer (A), but be cleared as they receive (correct) negative X-ray and test results ($X' \cap T' = (X \cup T)'$), **or** merely receive a (correct) negative test diagnosis from the doctor A'.

2. Let $C =$ "exactly one occurs". Then $C = (A \cap B') \cup (A' \cap B)$, and by Axiom (III)

   $$P(C) = P(A \cap B') + P(A' \cap B).$$

   Also,

   $$A = (A \cap B) \cup (A \cap B') \implies P(A) = P(A \cap B) + P(A \cap B')$$
   $$B = (A \cap B) \cup (A' \cap B) \implies P(B) = P(A \cap B) + P(A' \cap B)$$

   so therefore

   $$P(C) = P(A) + P(B) - 2P(A \cap B).$$

3. (a) FALSE : (e.g. $A_1 = \emptyset$, $A_2 = S$).

   (b) FALSE : (e.g. coin toss, let $A_1 = A_2 = \{H\}$).

   (c) TRUE : $A_1 \subseteq A_2 \implies P(A_1) \leq P(A_2) = 1 - P(A_2') = 1 - P(A_1) \implies 2P(A_1) \leq 1$.

   (d) TRUE : $P(A_1 \cup A_2) = P(A_1) + P(A_1) - P(A_1 \cap A_2) \geq (1 - x_1) + (1 - x_2) - 1 = 1 - x_1 - x_2$.

4. For general events A and B,

   (a) $B \equiv (A \cap B) \cup (A' \cap B)$, so by Axiom (III)

   $$P(B) = P(A \cap B) + P(A' \cap B) \implies P(A' \cap B) = P(B) - P(A \cap B)$$

   (b) $A \cup B \equiv A \cup (A' \cap B)$, so by Axiom (III)

   $$P(A \cup B) = P(A) + P(A' \cap B) = P(A) + P(B) - P(A \cap B)$$

   (c) $A \subseteq B \implies B \equiv A \cup (A' \cap B)$, so by Axiom (III), as $P(A' \cap B) \geq 0$,

   $$P(B) = P(A) + P(A' \cap B) \geq P(A)$$

   (d) *Bonferroni Inequality*: From (ii), as $P(A \cup B) \leq 1$,

   $$P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1$$

5. (a) $A' \cup B' = (A \cap B)' \implies P(A' \cup B') = 1 - P(A_1 \cap A_2) = 1 - z$.

   (b) $B = (A \cap B) \cup (A' \cap B)$, so $P(B) = P(A \cap B) + P(A' \cap B)$, so $P(A' \cap B) = y - z$.

   (c) $A' \cup B = A' \cup (A \cap B) \implies P(A' \cup B) = P(A') + P(A \cap B) = 1 - x + z$.

   (d) $A' \cap B' = (A \cup B)' \implies P(A' \cap B') = 1 - P(A \cup B) = 1 - x - y + z$.

6. Suppose $P(A) \leq 1$ for arbitrary A, then we must have that $P(A') \leq 1$ also, as A' is an event in its own right. But $A \cup A' = S$, so by Axiom (III) $P(A) + P(A') = 1$. Thus $P(A) \geq 0$. Hence we may take Axiom (I) or Axiom (I*) as equivalent: one holds if and only if the other holds.

7. (a) True for $n = 2$ as $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$. So assume true for $n = k$; then by the inductive hypothesis

$$P(A_1 \cup \ldots \cup A_k \cup A_{k+1}) \leq P(A_1 \cup \ldots \cup A_k) + P(A_{k+1}) = \sum_{i=1}^{k+1} P(A_i)$$

where we have applied the result for $n = 2$ to the two events

$$(A_1 \cup \ldots \cup A_k) \qquad \text{and} \qquad A_{k+1}.$$

Hence the result is true for $n = k + 1$.

(b) As for (a) except

$$P(A_1 \cup \ldots \cup A_k \cup A_{k+1}) = P(A_1 \cup \ldots \cup A_k) + P(A_{k+1}) - P\left((A_1 \cup \ldots \cup A_k) \cap A_{k+1}\right). \qquad (1)$$

Result follows by substituting required form for $P(A_1 \cup \ldots \cup A_k)$, and re-writing

$$P\left((A_1 \cup \ldots \cup A_k) \cap A_{k+1}\right) = P\left((A_1 \cap A_{k+1}) \cup \ldots \cup (A_k \cap A_{k+1})\right)$$

(using distributivity) which is the union of k events, and hence can use the inductive hypothesis to re-write this final expression in the required form. Specifically, let $B_i = A_i \cap A_{k+1}$; then we have

$$\begin{aligned} P(B_1 \cup B_2 \cup \ldots \cup B_k) &= \sum_i P(B_i) - \sum_i \sum_j P(B_i \cap B_j) + \ldots (-1)^{k-1} P(B_1 \cap B_2 \cap \ldots \cap B_k) \\ &= \sum_i P(A_i \cap A_{k+1}) - \sum_i \sum_j P(A_i \cap A_j \cap A_{k+1}) + \ldots \\ &\qquad (-1)^{k-1} P(A_1 \cap A_2 \cap \ldots \cap A_k \cap A_{k+1}) \end{aligned}$$

as, for example,

$$B_i \cap B_j = (A_i \cap A_{k+1}) \cap (A_j \cap A_{k+1}) = A_i \cap A_j \cap A_{k+1}$$

Hence, using the inductive hypothesis to express

$$P(A_1 \cup \ldots \cup A_k)$$

in (1) and adding in the final term, we complete the proof.

You do not need to know how to derive these results for this course, but the results themselves will be useful.