MATH 323 - SKILL TEST 2 –SOLUTIONS

Q1. In a diagnostic testing setting, let D denote the event that a person being tested for a disease is actually a sufferer from the disease, and let T be the event that the test, when carried out, is positive. A positive test implies that the person is a sufferer, but this in fact may not be true, as the test is not perfect. Suppose that

$$P(T|D) = 1 - a \qquad P(T|D') = b \qquad P(D) = p$$

Compute the probability that when a person is tested, the test is positive.

Note: I changed S to D to avoid confusion with the sample space notation.

Solution: the correct formula for $P(T)$ is

$$P(T) = P(T|D)P(D) + P(T|D')P(D') = (1-a)p + b(1-p)$$

as per the Theorem of Total Probability.

Q2. Set up as in Q1: suppose a person tests positive for the disease: **given** this information, what is the probability that the person is actually a sufferer ?

Solution: the correct formula for $P(D|T)$ is

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)} = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D')P(D')} = \frac{(1-a)p}{(1-a)p + b(1-p)}$$

as per Bayes Theorem.

Q3. Set up as in Q1: suppose a person tests negative for the disease: **given** this information, what is the probability that the person is actually a sufferer ?

Solution: the correct formula for $P(D|T')$ is

$$P(D|T') = \frac{P(T'|D)P(D)}{P(T')} = \frac{P(T'|D)P(D)}{P(T'|D)P(D) + P(T'|D')P(D')} = \frac{ap}{ap + (1-b)(1-p)}$$

as per Bayes Theorem.

Q4. Set up as in Q1: A person enters the testing centre: they take a test, and test positive. To check the result, another test is carried out; the second test is also positive. If it is believed that the two tests are conditionally independent given that the person is in fact a sufferer, and given the information that both tests are positive, what is the probability that the person is actually a sufferer ?

Solution: denoting the two test results by T_1 and T_2, the correct formula for $P(D|T_1 \cap T_2)$ is

$$P(D|T_1 \cap T_2) = \frac{P(T_1 \cap T_2|D)P(D)}{P(T_1 \cap T_2)} = \frac{P(T_1 \cap T_2|D)P(D)}{P(T_1 \cap T_2|D)P(D) + P(T_1 \cap T_2|D')P(D')}.$$

as per Bayes Theorem. Under conditional independence of the tests given D and D' – the latter statement was erroneously omitted – we have that

$$P(T_1 \cap T_2|D) = P(T_1|D) \times P(T_2|D) = (1-a) \times (1-a) = (1-a)^2$$

and

$$P(T_1 \cap T_2|D') = P(T_1|D') \times P(T_2|D') = b \times b = b^2$$

we have the formula

$$P(D|T_1 \cap T_2) = \frac{(1-a)^2 p}{(1-a)^2 p + b^2(1-p)}.$$

Q5. Three soccer teams are each made up of 11 players. Team 1 has n_1 right-footed players, Team 2 has n_2 right-footed players, and Team 3 has n_3 right-footed players. A team is selected, with all teams equally likely to be selected, and then a player is selected from the team, with all team members being equally likely to be selected. What is the probability that the player is right-footed?

Solution: letting T_1, T_2 and T_3 denote the selected team, and R denote the fact that the selected player is right-footed, the correct formula for $P(R)$ is

$$P(R) = P(R|T_1)P(T_1) + P(R|T_2)P(T_2) + P(R|T_3)P(T_3)$$

as per the Theorem of Total Probability. Under the equally likely outcomes assumption

$$P(T_1) = P(T_2) = P(T_3) = 1/3$$

and

$$P(R|T_1) = \frac{n_1}{11} \qquad P(R|T_2) = \frac{n_2}{11} \qquad P(R|T_3) = \frac{n_3}{11}.$$

Therefore

$$P(R) = \left(\frac{n_1}{11} \times \frac{1}{3}\right) + \left(\frac{n_2}{11} \times \frac{1}{3}\right) + \left(\frac{n_3}{11} \times \frac{1}{3}\right) = \frac{1}{3}\frac{n_1 + n_2 + n_3}{11}.$$

Q6. Setup as in Q5: If it is **known** that the player is right-footed, what is the probability that they were a member of Team 3.

Solution: the correct formula for $P(T_3|R)$ is

$$P(T_3|R) = \frac{P(R|T_3)P(T_3)}{P(R)} = \frac{P(R|T_3)P(T_3)}{P(R|T_1)P(T_1) + P(R|T_2)P(T_2) + P(R|T_3)P(T_3)}$$

as per Bayes Theorem. Due to equally likely selection of teams, and the fact that every team has 11 players from which to select, this simplifies to

$$P(T_3|R) = \frac{n_3}{n_1 + n_2 + n_3}.$$

Q7. When setting an exam, a professor can select from five question banks.

- The first bank has n_1 questions, h_1 of which are extremely hard;
- The second bank has n_2 questions, h_2 of which are extremely hard;
- The third bank has n_3 questions, h_3 of which are extremely hard;
- The fourth bank has n_4 questions, h_4 of which are extremely hard;
- The fifth bank has n_5 questions, h_5 of which are extremely hard

The professor, in a mutually independent series of selections, picks five questions for the test by picking a question bank, and then a question from that bank, with all banks and all questions equally likely to be selected each time. Selections are made with replacement, so any question may be selected more than once (the prof uses small numerical changes each time to make the questions different).

What is the probability that all five questions selected are extremely hard ?

Solution: let B_j denote that bank j is selected. Consider the first selected question, and let E denote the event that the question is extremely hard. We have that

$$P(E) = \sum_{j=1}^{5} P(E|B_j)P(B_j)$$

by the Theorem of Total Probability. The banks are equally likely to be selected, and the questions are equally likely to be selected from a bank. Therefore,

$$P(E) = \sum_{j=1}^{5} \frac{h_j}{n_j}\frac{1}{5} = \frac{1}{5}\left(\frac{h_1}{n_1} + \frac{h_2}{n_2} + \frac{h_3}{n_3} + \frac{h_4}{n_4} + \frac{h_5}{n_5}\right).$$

The same calculation holds for each question selection, and so by mutual independence, the required probability is $\{P(E)\}^5$.

Q8. Setup as in Q7: The professor picks a bank and a question from the bank, with all selections being equally likely at each time. What is the probability that the selected question came from question bank 1, given that it turned out to be extremely hard ?

Solution: The correct formula for $P(B_1|E)$ is

$$P(B_1|E) = fracP(E|B_1)P(B_1)\sum_{j=1}^{5} P(E|B_j)P(B_j)$$

by Bayes Theorem. The banks are equally likely to be selected, and the questions are equally likely to be selected from a bank. Therefore,

$$P(B_1|E) = \frac{\frac{1}{5}\frac{h_1}{n_1}}{\frac{1}{5}\left(\frac{h_1}{n_1} + \frac{h_2}{n_2} + \frac{h_3}{n_3} + \frac{h_4}{n_4} + \frac{h_5}{n_5}\right)} = \frac{\frac{h_1}{n_1}}{\left(\frac{h_1}{n_1} + \frac{h_2}{n_2} + \frac{h_3}{n_3} + \frac{h_4}{n_4} + \frac{h_5}{n_5}\right)}$$

Q9. One of two coins is selected and tossed three times (with the tosses mutually independent) and three Heads are obtained. One coin is fair (with probability of obtaining a Head equal to 1/2), but the other coin is biased, with the probability of obtaining a Head being equal to p. If at the start each coin was equally likely to be selected, what is the probability that the fair coin was selected ?

Solution: let E denote that a Head was obtained on all three tosses, and F denote that the coin was fair. The correct formula for $P(F|E)$ is

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)} = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F')P(F')}$$

as per Bayes Theorem. By mutual independence, we have

$$P(E|F) = \left(\frac{1}{2}\right)^3 \qquad P(E|F') = p^3$$

so, as the coins were equally likely to be selected giving $P(F) = 1/2$, we have

$$P(F|E) = \frac{\left(\frac{1}{2}\right)^3\frac{1}{2}}{\left(\frac{1}{2}\right)^3\frac{1}{2} + p^3\frac{1}{2}} = \frac{\left(\frac{1}{2}\right)^3}{\left(\frac{1}{2}\right)^3 + p^3}$$

Q10. A wireless mouse can be powered by one of two brands of battery. The first brand, Duracizer, is high quality - the probability that the mouse functions for more than x hours if powered by a Duracizer battery is given by the formula $\exp\{-x/100\}$. The second brand, Poundcell, is low quality - the probability that the mouse functions for more than x hours if powered by a Poundcell battery is given by the formula $\exp\{-x/20\}$.

If is observed that the mouse is still operational after 40 hours, and it is known that p percent of batteries available for installation are Duracizer brand, what is the probability that the battery installed is a Duracizer ?

Solution: Let D denote the event that the battery in the mouse is a Duracizer brand, and let A denote the event that the mouse functions for more than 40 hours. The correct formula for $P(D|A)$ is

$$P(D|A) = \frac{P(A|D)P(D)}{P(A)} = \frac{P(A|D)P(D)}{P(A|D)P(D) + P(A|D')P(D')}$$

as per Bayes Theorem. We have

$$P(A|D) = \exp\{-(40/100)\} \qquad P(A|D') = \exp\{-(40/20)\}$$

and $P(D) = (p/100)$ as p is measured as a percentage. Thus

$$P(D|A) = \frac{\exp\{-(40/100)\}(p/100)}{\exp\{-(40/100)\}(p/100) + \exp\{-(40/20)\}(1 - p/100)}$$