MATH 323 - PRACTICE MID TERM
SOLUTIONS

Q1. This information can be displayed as

	C: Yes		C: No	
	B: Yes	B: No	B: Yes	B: No
A: Yes	35	22	17	3
A: No	16	8	4	18

or

A	B	C	Count
N	N	N	18
N	N	Y	8
N	Y	N	4
Y	N	N	3
N	Y	Y	16
Y	N	Y	22
Y	Y	N	17
Y	Y	Y	35
Total			123

(a) *the selected person drives to work;*

This is $P(B) = (4 + 16 + 17 + 35)/123 = 72/123 = 0.5853$. 2 MARKS

(b) *the selected person drives to work and would use a public mass transportation system if it were available;*

This is $P(B \cap C) = (16 + 35)/123 = 51/123 = 0.4146$. 2 MARKS

(c) *the selected person lives more than 10km from the city centre, drives to work, but would not use a mass transportation system.*

This is $P(A \cap B \cap C') = 17/123 = 0.1382$. 2 MARKS

For the second city

(d) *The probability that a person lives more than 10km away from the city centre.*

This is $P(A) = P(A \cap B) + P(A \cap B') = 0.20 + 0.42 = 0.62$. 2 MARKS

(e) *The probability that a person lives more than 10km away from the city centre,* ***given*** *that the person regularly drives to work.*

This is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A \cap B) + P(A' \cap B)} = \frac{0.20}{0.20 + 0.15} = 0.5714$$

2 MARKS

Q2. (a) *Four letters are to be selected from the first eight letters of the alphabet, abcdefgh. How many distinct letter patterns can be formed if the selection is made **without replacement** ? How many patterns are possible if the selection is made **with replacement** ?*

Without replacement: This is $8 \times 7 \times 6 \times 5 = 8!/4! = P_4^8 = 1680$. 1 MARK

With replacement: This is $8 \times 8 \times 8 \times 8 = 8^4 = 4096$. 1 MARK

(b) *Compute the number of distinct committees of size $n_1 = 2$, $n_2 = 3$ and $n_3 = 5$ can be formed from a collection of $n = 10$ individuals.*

This is, by the multinomial formula,

$$\frac{10!}{2!3!5!} = \frac{3628800}{2 \times 6 \times 120} = 2520$$

2 MARKS

(c) *How many **distinct** letter patterns can be created by rearranging the letters TACTIC ?*

This is

$$\frac{6!}{2! \times 2! \times 1! \times 1!} = \frac{720}{4} = 180$$

There are $6!$ permutations of the letters, but we do not want to double count the multiple permutations where different copies of the same letter. For example, labelling the two Ts as T_1, T_2 and the two Cs as C_1, C_2, we want to avoid counting the sequences

$$\begin{array}{cccccc} T_1 & A & C_1 & T_2 & I & C_2 \\ T_1 & A & C_2 & T_2 & I & C_1 \\ T_2 & A & C_1 & T_1 & I & C_2 \\ T_2 & A & C_2 & T_1 & I & C_1 \end{array}$$

as being different. There are $2! \times 2! = 4$ such sequences that should be counted as identical, and this goes for every permutation of the 6 letters. 2 MARKS

(d) *In each month of a given year, a company selects one from six contractors to fulfil a contract in that month, with all selections being equally likely within a month, and with selections made in successive months being independent.*

*Compute the probability that a given contractor is selected **exactly twice** during the year.*

In any month, the probability that the contractor is selected is $1/6$. There are C_2^{12} sequences of selections where the contractor is selected precisely twice during the twelve months in the year, and the probability of being selected precisely twice **in a specified collection of months** is, by the multiplication principle

$$\frac{2^2 \times 10^{10}}{12^{12}} = \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{10}$$

Therefore the required probability is

$$\binom{12}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{10} = 0.2961.$$

2 MARKS

(e) *A bag contains 10 apples: 6 are Grade I (high quality) and 4 are Grade II (medium quality). A sample of 3 apples is selected without replacement from the bag. What is the probability that 2 of the selected apples are Grade I apples ?*

By the hypergeometric formula, the required probability is

$$\frac{\binom{6}{2}\binom{4}{1}}{\binom{10}{3}} = 0.5$$

2 MARKS

Q3. *In a certain population, a genetic marker at a given genomic locus takes one of three forms (or genotypes), AA, Aa, or aa, and the genotype for an individual is thought to determine their risk of a specific disease in later life.*

- *60% of genotype AA individuals develop the disease;*
- *35% of genotype Aa individuals develop the disease;*
- *5% of genotype aa individuals develop the disease.*

The proportions of type AA and aa individuals are 0.04 and 0.64 respectively. Let D denote the event that an individual develops the disease, and let G_1, G_2 and G_3 denote the event that an individual has the AA, Aa and aa genotypes respectively.

(a) *What proportion of the population develop the disease ?*

This is a Theorem of Total Probability question.

$$P(G_1) = 0.04 \qquad P(G_3) = 0.64 \qquad P(G_2) = 1 - P(G_1) - P(G_3) = 0.32$$

and hence

$$\begin{aligned} P(D) &= P(D|G_1)P(G_1) + P(D|G_2)P(G_2) + P(D|G_3)P(G_3) \\ &= (0.6 \times 0.04) + (0.35 \times 0.32) + (0.05 \times 0.64) = 0.168. \end{aligned}$$

2 MARKS

(b) *Given that a person develops the disease, what is the probability that their genotype is AA ?*

This is a Bayes Theorem (or conditional probability) question. We have

$$P(G_1|D) = \frac{P(D|G_1)P(G_1)}{P(D)} = \frac{(0.6 \times 0.04)}{0.168} = 0.1429.$$

2 MARKS

Suppose that new research determines that in fact, the disease can also develop due to non genetic reasons: each person may develop the disease with probability 0.1 ***irrespective*** *of their genotype, and only 70% of cases are determined by genetics.*

In light of the new research, what proportion of population develop the disease ?

This is also a Theorem of Total Probability question.

$$P(D) = P(D|G)P(G) + P(D|G')P(G')$$

with $P(G) = 0.70$. Now $P(D|G)$ is precisely what was computed in (a): formally, we have

$$\begin{aligned} P(D|G) &= P(D|G \cap G_1)P(G_1|G) + P(D|G \cap G_2)P(G_2|G) + P(D|G \cap G_3)P(G_3|G) \\ &= P(D|G \cap G_1)P(G_1) + P(D|G \cap G_2)P(G_2) + P(D|G \cap G_3)P(G_3) \end{aligned}$$

as G relates to the disease-causing mechanism, not the genotype. Thus as $P(D|G') = 0.1$,

$$P(D) = (0.168 \times 0.70) + (0.1 \times 0.30) = 0.1476.$$

3 MARKS

If a person does develop the disease, what is the probability that they developed it for a non genetic reason ?

The required probability is $P(G'|D)$, and by Bayes Theorem we have

$$P(G'|D) = \frac{P(D|G')P(G')}{P(D)} = \frac{(0.1 \times 0.30)}{(0.168 \times 0.70) + (0.1 \times 0.30)} = 0.2033.$$

3 MARKS