McGill

Fall 2021

Practice Final Examination

Version Number:

PROBABILITY

MATH 323 SECTION: 001 & 002

DATE: TIME:

STUDENT NAME:		**McGILL ID:**	

INSTRUCTIONS

EXAM:	CLOSED BOOK ☒ OPEN BOOK ☐ SINGLE SIDED ☐ PRINTED ON BOTH SIDES ☒ MULTIPLE CHOICE ANSWER SHEETS ☐ NOTE: The Examination Security Monitor Program detects pairs of students with unusually similar answer patterns on multiple-choice exams. Data generated by this program can be used as admissible evidence, either to initiate or corroborate an investigation or a charge of cheating under Section 16 of the Code of Student Conduct and Disciplinary Procedures. ANSWER IN BOOKLET ☐ EXTRA BOOKLETS PERMITTED ☐ ANSWER ON EXAM ☒
	SHOULD THE EXAM BE: RETURNED ☒ KEPT BY STUDENT ☐
CRIB SHEETS:	NOT PERMITTED ☒ PERMITTED ☐ Specifications: No crib sheets allowed.
DICTIONARIES:	TRANSLATION ONLY ☒ REGULAR ☒ NONE ☐
CALCULATORS:	NOT PERMITTED ☐ PERMITTED (Non-Programmable) ☒
ANY SPECIAL INSTRUCTIONS:	Special Instructions on next page:

Note: the real exam will be a 'write-on-the-exam-paper' exam

Please write your answers on the exam in the boxes provided; only answers written inside the boxes will be marked. Please write clearly.

This paper contains five questions. Each question carries 20 marks. Credit will be given for all questions attempted. The total mark available is 100 but rescaling of the final mark may occur.

A formula sheet, a distribution sheet, a sheet of useful summation results, and a table of the standard Normal distribution are provided.

Only non-programmable calculators may be used. Dictionaries and Translation dictionaries are permitted.

The usual abbreviations are used: pmf indicates the probability mass function; pdf indicates the probability density function; cdf indicates the cumulative distribution function; mgf indicates the moment-generating function.

If the intention of a question is unclear to you, please state your interpretation and provide an answer accordingly. Your comment will be taken into account in marking.

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1. (a) For events A, B and C in sample space S, suppose that

$$P(A) = 0.4 \qquad P(B) = 0.3 \qquad P(C) = 0.1.$$

Write down

(i) an upper bound on $P(A \cup B \cup C)$, and 2 MARKS

(ii) an upper bound on $P(A \cap B \cap C)$ 2 MARKS

which is, in each case, less than 1. Justify your answer. If it is not possible to find a bound lower than 1, explain why.

(b) Suppose that two events E and F partition S. Show that

$$P(E \cap F) + P(E \cup F) = 1.$$

2 MARKS

(c) A soccer squad comprises 6 defenders, 6 midfielders, and 4 attackers as its outfield players. The team manager decides to select the outfield team with 4 defenders, 3 midfielders, and 3 attackers. How many possible teams with this configuration could be selected ? 4 MARKS

(d) In a multiple choice test, each question has three possible answers: A,B,C. How many possible answer patterns are there, if it is known that each answer letter must appear precisely five times each ? 5 MARKS

(e) A bag contains 10 balls: three are Red, three are Blue, three are Green and one is White. Balls are to be selected from the bag, with all balls equally likely to be selected from those that are still in the bag at every selection. What is the probability that

(i) the White ball is not selected on any of the first ten selections, if selections are made *with replacement* at each stage ?; 2 MARKS

(ii) two Red balls are selected during the first five selections, if selections are made *without replacement* at each stage ?. 3 MARKS

2. (a) Two eggs are needed for a cake recipe, and they are selected from a box of twelve eggs. Unfortunately, the box contains two bad eggs that will affect the taste of the cake:

- ▶ if no bad eggs are selected, then the cake will have an acceptable taste with probability 0.95;
- ▶ if precisely one bad egg is selected, then the cake will have an acceptable taste with probability 0.60;
- ▶ if two bad eggs are selected, then the cake will have an acceptable taste with probability 0.10.

What is the probability that

(i) the cake does not have an acceptable taste ?; 5 MARKS

(ii) given that the cake does not have an acceptable taste, what is the probability that precisely one bad egg was used ? 5 MARKS

Assume that all selections of two eggs from twelve are equally likely.

(b) A bag contains 10 balls which are known to be Red or White. Four balls are selected from the bag, with all such selections deemed to be equally likely.

(i) If the number of Red balls in the bag **known** to be five, compute the probability of obtaining three Red balls in the selection. 4 MARKS

(ii) If the number of Red balls in the bag is **unknown**, but it is believed *before* the selection that all possible numbers of Red balls in the bag are equally likely, derive an expression for the probability that number of Red balls is r, given that three of the balls in the sample are observed to be Red, for all suitable values of r.

6 MARKS

3. (a) Suppose that a discrete random variable Y has mgf $m(t)$ given by

$$m(t) = c(1 + 2e^t + 3e^{2t})$$

for $t \in \mathbb{R}$ for some c.

(i) Find the value of c. 2 MARKS

(ii) Find $\mathbb{E}[Y^3]$. 3 MARKS

(iii) Find $\mathbb{E}[Y(Y-1)]$. 3 MARKS

(iv) Suppose that Y_1 and Y_2 are independent random variables having the same distribution as Y. Find the pmf of

$$X = Y_1 + Y_2.$$

6 MARKS

(b) Suppose that Z_1 and Z_2 are independent Poisson random variables each with parameter λ. Let $S = Z_1 + Z_2$. Find the conditional probability that $Z_1 = z_1$, given that $S = s$ for s such that $P(S = s) > 0$. 6 MARKS

Hint: if $P(S = s) > 0$ then

$$P(Z_1 = z_1 | S = s) = \frac{P(Z_1 = z_1, S = s)}{P(S = s)}$$

for each relevant z_1.

4. (a) For each of the following transformation problems, compute the distribution (that is, compute the pmf, pdf or cdf, or identify by name) of the variable U for the stated initial variable Y.

   (i) $U = 2Y$, if $Y \sim Geometric(p)$. 2 MARKS

   (ii) $U = 2Y$, if $Y \sim Exponential(1)$. 2 MARKS

   (iii) $U = \exp\{-Y/2\}$, if $Y \sim Exponential(2)$. 2 MARKS

   (iv) $U = (Y - \mu)^2/4$, if $Y \sim Normal(\mu, 2^2)$. 2 MARKS

   (v) $U = \Phi(Y)$, if $Y \sim Normal(0, 1)$, and $\Phi(.)$ is the standard Normal cdf. 2 MARKS

   (b) Suppose that Y_1 and Y_2 are independent random variables with $Y_1 \sim Exponential(\beta_1)$ and $Y_2 \sim Exponential(\beta_2)$. Find expressions for

   (i) $P(Y_1 > 2|Y_1 > 1)$; 1 MARK

   (ii) $P(Y_1 > 2|Y_2 > 1)$; 1 MARK

   (iii) $P(Y_1 + Y_2 > 3)$, assuming that $\beta_1 = \beta_2 = \beta$, say; 6 MARKS

   (iv) $P(Y_1 > 3, Y_2 > 2)$. 2 MARKS

5. This question concerns sequences $Y_1, Y_2, \ldots, Y_n$ for $n \geq 1$ which are independent and identically distributed random variables. For any $n \geq 1$, define random variables

$$S_n = \sum_{i=1}^{n} Y_i \qquad \overline{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i.$$

(a) Suppose that $Y_1, Y_2, \ldots, Y_r$ each has a $Geometric(p)$ distribution, for parameter $0 < p < 1$, where $r \geq 2$ is a fixed value.

(i) Write down the expectation and variance of $\overline{Y}_r$. 2 MARKS

(ii) Find the distribution of S_r (that is, identify the distribution by name, or find the pmf). 4 MARKS

(iii) Find an expression for the conditional probability

$$P(Y_1 > 2 | Y_1 > 1).$$

2 MARKS

(iv) Find an expression for the conditional probability

$$P(Y_2 > 2 | Y_1 > 1)$$

2 MARKS

(b) Suppose that $Y_1, Y_2, \ldots, Y_n$ each has a $Normal(0, 1)$ distribution. Find the probability

$$P\left(\overline{Y}_n \leq \frac{1}{2}\right)$$

if $n = 4$. 4 MARKS

(c) Suppose that $Y_1, Y_2, \ldots, Y_n$ each has a $Bernoulli(1/2)$ distribution. Using the Central Limit Theorem, find an approximation to the probability

$$P\left(\overline{Y}_n \leq \frac{1}{2}\right)$$

if n is large. 3 MARKS

(d) Suppose that $Y_1, Y_2, \ldots, Y_n$ each has a $Normal(0, 1)$ distribution. For each i, define random variable

$$U_i = Y_i^2.$$

Describe the behaviour of

$$\overline{U}_n = \frac{1}{n} \sum_{i=1}^{n} U_i$$

as $n \longrightarrow \infty$.

3 MARKS

MATH 323 Fall 2021: Formula Sheet

- **Basic Probability:** For events A and B in sample space S
  - *General Addition Rule:*
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
  - *de Morgan's Laws:*
$$P(A' \cap B') = P((A \cup B)') \qquad P(A' \cup B') = P((A \cap B)')$$

- **Combinatorics:**
  - *Multiplication principle:* A sequence of k operations, in which operation i can result in n_i possible outcomes, can result in $n_1 \times n_2 \times \cdots \times n_k$ possible sequences of outcomes.
  - *Permutations:* An ordered arrangement of r distinct objects is called a *permutation*. The number of ways of ordering r distinct objects taken from n without replacement is denoted by
$$P_r^n = \frac{n!}{(n-r)!}$$
  - *Multinomial coefficients:* The number of ways of partitioning n distinct objects into k disjoint subsets of sizes $n_1, n_2, \ldots, n_k$ where $n_1 + n_2 + \cdots + n_k = n$ is
$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! \times n_2! \ldots \times n_k!}.$$
  - *Combinations:* The number of *combinations* of n objects taken r at a time is the number of subsets of size r that can be formed. This number is denoted
$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$
  - *Binary sequences:* the number of binary sequences of length n containing r 1s is C_r^n.
  - *Hypergeometric probabilities:* Finite population of size N comprises R Type I objects and $N - R$ Type II objects, and a sample of size $n \leq N$ is obtained from the population. The probability that the sample contains r Type I objects and $n - r$ Type II objects is
$$\frac{\binom{n}{r}\binom{N-n}{R-r}}{\binom{N}{R}} = \frac{\binom{R}{r}\binom{N-R}{n-r}}{\binom{N}{n}}$$

- **Probability Theorems:** Suppose events $B_1, B_2, \ldots, B_n$ form a partition of sample space S, and let $A \subseteq S$. Provided all the conditional probabilities are well-defined, we have
  - *Theorem of Total Probability:*
$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i).$$
  - *Bayes Theorem:*
$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A)} = \frac{P(A|B_i)P(B_i)}{\sum_{j=1}^{n} P(A|B_j)P(B_j)}.$$

- **Probability Distributions & Properties:**
  - ▶ *Distributions:*
    - * *Discrete case:* A discrete probability distribution places probability a countable number of real values. For $y \in \mathbb{R}$, we the pmf is the function

$$p(y) = P(Y = y)$$

      The cdf is the function

$$F(y) = P(Y \leq y)$$

      In the discrete case, $F(y)$ is a step function.
    - * *Continuous case:* A continuous probability distribution is a distribution where the cdf

$$F(y) = P(Y \leq y)$$

      is continuous for all $y \in \mathbb{R}$. The pdf, $f(.)$, is the function for which

$$F(y) = \int_{-\infty}^{y} f(t)\, dt$$

      so that

$$f(y) = \frac{dF(y)}{dy}$$

      wherever the derivative exists.
  - ▶ *Expectations:* The expectation of a function $g(Y)$ of random variable Y is defined by

$$\mathbb{E}[g(Y)] = \begin{cases} \sum\limits_{y} g(y)p(y) & \text{Discrete case} \\ \int_{-\infty}^{\infty} g(y)f(y)\, dy & \text{Continuous case} \end{cases}$$

    whenever the sum or integral is well-defined.
  - ▶ *mgfs:* The mgf of random variable Y, $m(t)$, is defined by

$$m(t) = \sum_{j=0}^{\infty} \mu'_j \frac{t^j}{j!} = \mu'_0 + \mu'_1 t + \mu'_2 \frac{t^2}{2!} + \cdots$$

    where $\mu'_j = \mathbb{E}[Y^j]$ is the jth moment for $j = 0, 1, 2, \ldots$. We have

$$m(t) = \mathbb{E}[e^{tY}] = \begin{cases} \sum\limits_{y} e^{ty}p(y) & \text{Discrete case} \\ \int_{-\infty}^{\infty} e^{ty}f(y)\, dy & \text{Continuous case} \end{cases}$$

    provided the expectation exists for $t \in (-b, b)$ for some $b > 0$.
    - * $m(0) = 1$ and
    - * For $k = 1, 2, \ldots$,

$$m^{(k)}(0) = \left.\frac{d^k m(t)}{dt^k}\right|_{t=0} = \mathbb{E}[Y^k]$$

    - * If Y has mgf $m_Y(t)$, and $X = \mu + \sigma Y$, then $m_X(t) = e^{\mu t} m_Y(\sigma t)$.

- ▶ **Sums of independent random variables:**
  - ∗ *Convolution result:* If Y_1 and Y_2 are independent random variables, and $Y = Y_1 + Y_2$, we have that the distribution of Y is given by

$$p_Y(y) = \sum_{y_1=-\infty}^{\infty} p_{Y_1}(y_1) p_{Y_2}(y - y_1) \qquad \text{Discrete case}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y_1}(y_1) f_{Y_2}(y - y_1)\, dy_1 \qquad \text{Continuous case}$$

  - ∗ *mgfs:* If $Y_1, Y_2, \ldots, Y_n$ are independent random variables with mgfs $m_1(t), \ldots, m_n(t)$, and

$$Y = \sum_{i=1}^{n} Y_i$$

  then the mgf of Y, $m_Y(t)$, is computed as

$$m_Y(t) = m_1(t) \times m_2(t) \times \ldots \times m_n(t) = \prod_{i=1}^{n} m_i(t).$$

- **Inequalities:**
  - ▶ *Markov's inequality* If $g(Y)$ is a function of random variable Y such that $\mathbb{E}[|g(Y)|] < \infty$. For any $k > 0$,

$$P(|g(Y)| \geq k) \leq \frac{\mathbb{E}[|g(Y)|]}{k} \qquad \text{or} \qquad P(|g(Y)| < k) \geq 1 - \frac{\mathbb{E}[|g(Y)|]}{k}.$$

  - ▶ *Chebychev's Inequality* Suppose that Y has finite mean μ and variance σ^2.

$$P(|Y - \mu| < c\sigma) \geq 1 - \frac{1}{c^2} \qquad \text{or} \qquad P(|Y - \mu| \geq c\sigma) \leq \frac{1}{c^2}.$$

- **Limit Theorems:** Suppose that $Y_1, Y_2, \ldots, Y_n$, are independent random variables that have the same distribution, and have expectation μ and variance σ^2.
  - ▶ *The Weak Law of Large Numbers (WLLN):* For each n, let

$$\overline{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i$$

  be the sample mean random variable. Then $\overline{Y}_n \xrightarrow{p} \mu$ as $n \longrightarrow \infty$, that is, the probability distribution of $\overline{Y}_n$ becomes **concentrated** at μ.
  - ▶ *The Central Limit Theorem (CLT):* The random variable

$$U_n = \frac{\sqrt{n}(\overline{Y}_n - \mu)}{\sigma}$$

  converges in distribution to a *standard Normal distribution*. When n is large, we can approximate the distributions of $\overline{Y}_n$, or of

$$S_n = \sum_{i=1}^{n} Y_i$$

  using a Normal distribution, irrespective of the actual distribution of $Y_1, \ldots, Y_n$. That is
    - ∗ $\overline{Y}_n$ is approximately distributed as

$$Normal(\mu, \sigma^2/n)$$

    - ∗ S_n is approximately distributed as

$$Normal(n\mu, n\sigma^2)$$

DISCRETE DISTRIBUTIONS

	$\mathcal{Y}$	PARAMETERS	$p(y)$	$F(y)$	$\mathbb{E}[Y]$	$\mathbb{V}[Y]$	$m(t)$
$Bernoulli(p)$	$\{0,1\}$	$p \in (0,1)$	$p^y(1-p)^{1-y}$		p	$p(1-p)$	$1-p+pe^t$
$Binomial(n,p)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, p \in (0,1)$	$\binom{n}{y}p^y(1-p)^{n-y}$		np	$np(1-p)$	$(1-p+pe^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\frac{e^{-\lambda}\lambda^y}{y!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(p)$	$\{1,2,...\}$	$p \in (0,1)$	$(1-p)^{y-1}p$	$1-(1-p)^y$	$\frac{1}{p}$	$\frac{(1-p)}{p^2}$	$\frac{pe^t}{1-e^t(1-p)}$
$NegBinomial(r,p)$	$\{r,r+1,...\}$	$r \in \mathbb{Z}^+, p \in (0,1)$	$\binom{y-1}{r-1}p^r(1-p)^{y-r}$		$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left(\frac{pe^t}{1-e^t(1-p)}\right)^r$
or	$\{0,1,2,...\}$	$r \in \mathbb{Z}^+, p \in (0,1)$	$\binom{r+y-1}{r-1}p^r(1-p)^y$		$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$	$\left(\frac{p}{1-e^t(1-p)}\right)^r$
$Hypergeom(N,r,n)$	$\{\max\{0,n-N+r\},\ldots$ $\ldots,\min\{n,r\}\}$	$N \geq r,n$	$\frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}$		$n\left(\frac{r}{N}\right)$	$n\left(\frac{r}{N}\right)\left(1-\frac{r}{N}\right)\left(\frac{N-n}{N-1}\right)$	
or			$\frac{\binom{n}{y}\binom{N-n}{r-y}}{\binom{N}{r}}$				

CONTINUOUS DISTRIBUTIONS

	$\mathcal{Y}$	PARAMETERS	$f(y)$	$F(y)$	$\mathbb{E}[Y]$	$\mathbb{V}[Y]$	$m(t)$
$Uniform(\theta_1, \theta_2)$ (standard model $\theta_1 = 0, \theta_2 = 1$)	(θ_1, θ_2)	$\theta_1 < \theta_2 \in \mathbb{R}$	$\frac{1}{\theta_2 - \theta_1}$	$\frac{y - \theta_1}{\theta_2 - \theta_1}$	$\frac{(\theta_1 + \theta_2)}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{\theta_2 t} - e^{\theta_1 t}}{t\,(\theta_2 - \theta_1)}$
$Exponential(\beta)$ (standard model $\beta = 1$)	$\mathbb{R}^+$	$\beta \in \mathbb{R}^+$	$\frac{1}{\beta} e^{-y/\beta}$	$1 - e^{-y/\beta}$	β	β^2	$\left(\frac{1}{1 - \beta t}\right)$
$Gamma(\alpha, \beta)$ (standard model $\beta = 1$)	$\mathbb{R}^+$	$\alpha, \beta \in \mathbb{R}^+$	$\frac{1}{\beta^\alpha \Gamma(\alpha)} y^{\alpha-1} e^{-y/\beta}$ where $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy$		$\alpha\beta$	$\alpha\beta^2$	$\left(\frac{1}{1 - \beta t}\right)^\alpha$
$Normal(\mu, \sigma^2)$ (standard model $\mu = 0, \sigma = 1$)	$\mathbb{R}$	$\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$	$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$\exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$
$Beta(\alpha, \beta)$	$(0, 1)$	$\alpha, \beta \in \mathbb{R}^+$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}$		$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
$Weibull(\alpha, \beta)$ (standard model $\beta = 1$)	$\mathbb{R}^+$	$\alpha, \beta \in \mathbb{R}^+$	$\alpha\beta y^{\alpha-1} e^{-\beta y^\alpha}$	$1 - e^{-\beta y^\alpha}$	$\frac{\Gamma(1 + 1/\alpha)}{\beta^{1/\alpha}}$	$\frac{\Gamma(1 + 2/\alpha) - \Gamma(1 + 1/\alpha)^2}{\beta^{2/\alpha}}$	
$Student(\nu)$	$\mathbb{R}$	$\nu \in \mathbb{R}^+$	$\frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)\sqrt{\pi\nu}\{1 + y^2/\nu\}^{(\nu+1)/2}}$		0 (if $\nu > 1$)	$\frac{\nu}{\nu - 2}$ (if $\nu > 2$)	
$Pareto(\theta, \alpha)$	$\mathbb{R}^+$	$\theta, \alpha \in \mathbb{R}^+$	$\frac{\alpha\theta^\alpha}{(\theta + y)^{\alpha+1}}$	$1 - \left(\frac{\theta}{\theta + y}\right)^\alpha$	$\frac{\theta}{\alpha - 1}$ (if $\alpha > 1$)	$\frac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha > 2$)	

For linear transformation $X = \mu + \sigma Y$

$$f_X(x) = f_Y\left(\frac{x-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_X(x) = F_Y\left(\frac{x-\mu}{\sigma}\right) \qquad m_X(t) = e^{\mu t} m_Y(\sigma t) \qquad \mathbb{E}[X] = \mu + \sigma\mathbb{E}[Y] \qquad \mathbb{V}[X] = \sigma^2\mathbb{V}[Y]$$

Useful Series Summations

GEOMETRIC: For $|t| < 1$,

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots = \sum_{k=0}^{\infty} t^k$$

For the finite series, for $n \geq 1$

$$\frac{1-t^{n+1}}{1-t} = 1 + t + \cdots + t^n$$

EXPONENTIAL:

$$e^t = 1 + t + \frac{t^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{t^k}{k!}$$

$$e^t = \lim_{n \longrightarrow \infty} \left(1 + \frac{t}{n}\right)^n = \lim_{n \longrightarrow \infty} \left(1 - \frac{t}{n}\right)^{-n}$$

$$e^{-t} = \lim_{n \longrightarrow \infty} \left(1 - \frac{t}{n}\right)^n = \lim_{n \longrightarrow \infty} \left(1 + \frac{t}{n}\right)^{-n}$$

BINOMIAL: For $n > 0$,

$$(1+t)^n = 1 + nt + \frac{n(n-1)}{2!}t^2 + \cdots + nt^{n-1} + t^n = \sum_{k=0}^{n} \binom{n}{k} t^k$$

NEGATIVE BINOMIAL: for $n > 0$ and $|t| < 1$,

$$\frac{1}{(1-t)^{n+1}} = 1 + (n+1)\,t + \frac{(n+1)\,(n+2)}{2!}t^2 + \cdots = \sum_{k=0}^{\infty} \binom{n+k}{k} t^k$$

LOGARITHMIC: for $|t| < 1$,

$$-\log(1-t) = t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots = \sum_{k=1}^{\infty} \frac{t^k}{k}$$

$$\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{t^k}{k}$$

Table of the standard Normal distribution

Entries in table are $P(Y \leq y)$

y	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990