McGill

Fall 2021

Practice Final Examination

Practice Final

PROBABILITY
MATH 323
SOLUTIONS

1. (a) (i) Bonferroni's inequality: $P(A \cup B \cup C) \leq P(A) + P(B) + P(C) = 0.8$. 2 MARKS

(ii) As $A \cap B \cap C$ is a subset of A, of B and of C

$$P(A \cap B \cap C) \leq \min\{P(A), P(B), P(C)\} = 0.1$$

2 MARKS

(b) By definition of a partition, $E \cap F = \emptyset$ and $E \cup F = S$, so therefore by Axiom 1 and the rule for complements

$$P(E \cap F) + P(E \cup F) = P(\emptyset) + P(S) = 1.$$

2 MARKS

(c) This is a sequence of separate combinatorial selections, and using the multiplication principle, the solution is

$$\binom{6}{4} \times \binom{6}{3} \times \binom{4}{3} = 15 \times 20 \times 4 = 1200.$$

where the first term is the number of ways of choosing the defenders (4 from 6), midfielders (3 from 6) and attackers (3 from 4) respectively. 4 MARKS

(d) This is a multinomial/partition problem and the solution is

$$\frac{15!}{5! \times 5! \times 5!} = 756756.$$

5 MARKS

(e) (i) The probability of getting the white ball is $1/10$ for each of the selections as selections are made with replacement, so the probability is

$$\left(\frac{9}{10}\right)^{10} = 0.3487.$$

2 MARKS

(ii) This is given by the hypergeometric formula, with $N = 10$, $R = 3$, $n = 5$ and $y = 2$, that is

$$\frac{\binom{n}{y}\binom{N-n}{R-y}}{\binom{N}{R}} = \frac{\binom{5}{2}\binom{5}{1}}{\binom{10}{3}} = \frac{10 \times 5}{120} = \frac{50}{120} = 0.4167$$

or equivalently

$$\frac{\binom{R}{y}\binom{N-R}{n-y}}{\binom{N}{n}} = \frac{\binom{3}{2}\binom{7}{3}}{\binom{10}{5}} = \frac{3 \times 35}{252} = \frac{105}{252} = 0.4167.$$

3 MARKS

2. (a) Let E_i be the event that i bad eggs are selected for $i = 0, 1, 2$, and A be the event that the taste is acceptable. We have that

$$P(A|E_0) = 0.95 \qquad P(A|E_1) = 0.60 \qquad P(A|E_2) = 0.10$$

and can compute that

$$P(E_0) = \frac{10}{12} \times \frac{9}{11} = \frac{90}{132} \qquad P(E_2) = \frac{2}{12} \times \frac{1}{11} = \frac{2}{132}$$

so that $P(E_1) = 40/132$.

(i) Using the Theorem of Total Probability

$$P(A) = \sum_{i=0}^{2} P(A|E_i)P(E_i) = 0.95 \times \frac{90}{132} + 0.60 \times \frac{40}{132} + 0.10 \times \frac{2}{132} = 0.8311.$$

so $P(A') = 0.1689$. 5 MARKS

(ii) By Bayes Theorem

$$P(E_1|A') = \frac{P(A'|E_1)P(E_1)}{\sum_{i=0}^{2} P(A'|E_i)P(E_i)} = \frac{0.40 \times \frac{40}{132}}{0.1689} = 0.7175.$$

5 MARKS

(b) (i) If $R = 5$ is the number of Red balls in the bag of $N = 10$ balls, $n = 4$ is the sample size, and $y = 3$ is the number of Red balls in the sample, this is given by the hypergeometric formula as

$$\frac{\binom{R}{y}\binom{N-R}{n-y}}{\binom{N}{n}} = \frac{\binom{5}{3}\binom{5}{1}}{\binom{10}{4}} = \frac{10 \times 5}{210} = \frac{50}{210} = 0.2381.$$

4 MARKS

(ii) We now need to use Bayes Theorem, with $P(R = r)$ constant across $r = 0, 1, 2, \ldots, 10$ before the sample is taken

$$P(R = r|y = 3, n = 4, N = 10) = \frac{P(y = 3|n = 4, R = r, N = 10)P(R = r)}{\sum_{j=0}^{N} P(y = 3|n = 4, R = j, N = 10)P(R = j)}$$

From the information given (that in a sample of 4 balls, three are Red and one is White), the only legitimate values of r to consider are $r = 3, 4, \ldots, 9$ as we know that there are at least 3 and at most 9 Red balls. Then

$$P(y = 3|n = 4, R = r, N = 10) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}} = \frac{\binom{r}{3}\binom{10-r}{1}}{\binom{10}{4}}$$

Cancelling $P(R = j)$ in numerator and denominator as it is constant, we have that

$$P(R = r|y = 3, n = 4, N = 10) = \frac{\binom{r}{3}\binom{10-r}{1}}{\sum_{j=3}^{9}\binom{j}{3}\binom{10-j}{1}}$$

6 MARKS

3. (a) The mgf for a discrete and non-negative random variable is defined by

$$m(t) = \sum_{y=0}^{\infty} e^{ty} p(y) = p(0) + e^t p(1) + e^{2t} p(2) + \cdots$$

and equating this to the given form demonstrates that we must have

$$p(0) = c \qquad p(1) = 2c \qquad p(2) = 3c.$$

with $p(y) = 0$ for all other values of y. Thus the corresponding random variable takes values on the set $\{0, 1, 2\}$ with positive probability.

(i) We need $c + 2c + 3c = 1$ so $c = 1/6$. 2 MARKS

(ii) By direct calculation

$$\mathbb{E}[Y^3] = \sum_{y=0}^{\infty} y^3 p(y) = \frac{1}{6}(0^3 + 1^3 \times 2 + 2^3 \times 3) = 4.333.$$

Alternatively, we can compute using the derivative result:

$$\mathbb{E}[Y^3] = \frac{d^3}{dt^3} \{m(t)\}_{t=0} = m^{(3)}(0)$$

Here we obtain $\mathbb{E}[Y^3] = \frac{1}{6}(2 + 3 \times 2^3) = 26/6 = 4.333$.

3 MARKS

(iii) By direct calculation

$$\mathbb{E}[Y(Y-1)] = \sum_{y=0}^{\infty} y(y-1) p(y) = \frac{1}{6}(3 \times 2) = 1.$$

Here we can also use the pgf

$$G(t) = m(\ln t) = \frac{1}{6}(1 + 2t + 3t^2)$$

and the derivative result

$$\mathbb{E}[Y(Y-1)] = \frac{d^2}{dt^2} \{G(t)\}_{t=1} = G^2(1)$$

and here we obtain $\mathbb{E}[Y(Y-1)] = 1$. 3 MARKS

(iv) Using the mgf result for the sum of independent random variables, we have

$$\begin{aligned} m_Y(t) = m_{Y_1}(t) m_{Y_2}(t) &= \frac{1}{36}(1 + 2e^t + 3e^{2t})^2 \\ &= \frac{1}{36}(1 + 4e^t + 10e^{2t} + 12e^{3t} + 9e^{4t}) \end{aligned}$$

and therefore, using the initial result we have that

$$p_Y(y) = \begin{cases} 1/36 & y = 0 \\ 4/36 & y = 1 \\ 10/36 & y = 2 \\ 12/36 & y = 3 \\ 9/36 & y = 4 \end{cases}$$

and $p(y) = 0$ for all other values of y. 6 MARKS

(b) We first compute the distribution of S; using the mgf result for the sum of independent random variables

$$m_S(t) = m_{Z_1}(t)m_{Z_2}(t) = \exp\{\lambda(e^t - 1)\}\exp\{\lambda(e^t - 1)\} = \exp\{2\lambda(e^t - 1)\}$$

so therefore $S \sim Poisson(2\lambda)$ by the uniqueness of mgfs. Now consider the joint probability $P(Z_1 = z, S = s)$. We have

$$P(Z_1 = z, S = s) \equiv P(Z_1 = z, Z_2 = s - z) = P(Z_1 = z)P(Z_2 = s - z)$$

and hence

$$P(Z_1 = z, S = s) = e^{-\lambda}\frac{\lambda^z}{z!}e^{-\lambda}\frac{\lambda^{s-z}}{(s-z)!}$$

Therefore for $s > 0$ and $0 \leq z \leq s$

$$P(Z_1 = z|S = s) = \frac{P(Z_1 = z, S = s)}{P(S = s)} = \frac{\dfrac{e^{-2\lambda}\lambda^s}{z!(s-z)!}}{\dfrac{e^{-2\lambda}(2\lambda)^s}{s!}}$$

$$= \frac{s!}{z!(s-z)!}\left(\frac{1}{2}\right)^s$$

and therefore given that $S = s$, Z_1 has a conditional $Binomial(s, 1/2)$ distribution.

6 MARKS

4. (a) (i) By discrete transformation theory U takes values on $\{2, 4, 6, \ldots\}$, and for u in this set

$$p_U(u) = P(U = u) = P(2Y = u) = P(Y = u/2) = (1-p)^{u/2-1}p$$

with $p_U(u)$ zero for all other values of u. 2 MARKS

(ii) From first principles, for $u > 0$

$$F_U(u) = P(U \leq u) = P(2Y \leq u) = P(Y \leq u/2) = F_Y(u/2) = 1 - e^{-u/2}$$

and so $U \sim Exponential(2)$. 2 MARKS

(iii) From first principles, for $0 < u < 1$

$$\begin{aligned} F_U(u) = P(U \leq u) &= P(\exp\{-Y/2\} \leq u) \\ &= P(Y \geq -2\ln u) \\ &= 1 - F_Y(-2\ln u) \\ &= 1 - (1 - \exp\{-(-2\ln u)/2\}) \\ &= u \end{aligned}$$

and so $U \sim Uniform(0, 1)$.

2 MARKS

(iv) We have that if $Z = (Y - \mu)/2$, then by the result for linear transformations of Normal random variables, $Z \sim Normal(0, 1)$, and $U = Z^2$. Thus

$$F_U(u) = P(U \leq u) = P(Z^2 \leq u) = P(-\sqrt{u} \leq Z \leq \sqrt{u}) = F_Z(\sqrt{u}) - F_Z(-\sqrt{u}).$$

Differentiating we have

$$\begin{aligned} f_U(u) &= \frac{1}{2\sqrt{u}} f_Z(\sqrt{u}) + \frac{1}{2\sqrt{u}} f_Z(-\sqrt{u}) \\ &= \frac{1}{\sqrt{u}} f_Z(\sqrt{u}) && \text{as } f_Z(z) \text{ is symmetric} \\ &= \frac{1}{\sqrt{2\pi}} u^{1/2-1} \exp\left\{-\frac{u}{2}\right\} \end{aligned}$$

so therefore $U \sim Gamma(1/2, 2) \equiv Chisquared(1)$. 2 MARKS

(v) Note that $\Phi(.)$ is a monotonic increasing transform mapping $\mathbb{R}$ to $(0, 1)$. Thus, from first princples, for $0 < u < 1$,

$$F_U(u) = P(U \leq u) = P(\Phi(Y) \leq u) = P(Y \leq \Phi^{-1}(u))$$

Thus

$$F_U(u) = \Phi(\Phi^{-1}(u)) = u$$

so $U \sim Uniform(0, 1)$. 2 MARKS

(b) (i) By conditional probability

$$P(Y_1 > 2|Y_1 > 1) = \frac{P(Y_1 > 2, Y_2 > 1)}{P(Y_2 > 1)} = \frac{P(Y_1 > 2)}{P(Y_2 > 1)} = \frac{e^{-2/\beta_1}}{e^{-1/\beta_1}} = e^{-1/\beta_1}.$$

1 MARK

(ii) Y_1 and Y_2 are independent, so

$$P(Y_1 > 2|Y_2 > 1) \equiv P(Y_1 > 2) = e^{-2/\beta_1}.$$

1 MARK

(iii) If $\beta_1 = \beta_2 = \beta$ say, then using mgfs we have that

$$Y = Y_1 + Y_2 \sim Gamma(2, \beta)$$

so if $\lambda = 1/\beta$

$$P(Y > 3) = 1 - P(Y \leq 3) = 1 - \int_0^3 \lambda^2 y \exp\{-\lambda y\}\, dy$$

and integrating by parts we have that

$$\begin{aligned} P(Y > 3) &= 1 - [-\lambda y \exp\{-\lambda y\}]_0^3 - \lambda \int_0^3 \exp\{-\lambda y\}\, dy \\ &= 1 + 3\lambda \exp\{-3\lambda\} - (1 - \exp\{-3\lambda\}) \\ &= \exp\{-3\lambda\}(1 + 3\lambda) \end{aligned}$$

6 MARKS

(iv) By independence, using the Exponential cdf,

$$P(Y_1 > 3, Y_2 > 2) = P(Y_1 > 3)P(Y_2 > 2) = \exp\left\{-\left[\frac{3}{\beta_1} + \frac{2}{\beta_2}\right]\right\}.$$

6 MARKS

5. (a) (i) By results for independent random variables, and using the formula sheet

$$\mathbb{E}[\overline{Y}_r] = \frac{1}{p} \qquad \mathbb{V}[\overline{Y}_r] = \frac{(1-p)}{rp^2}.$$

2 MARKS

(ii) Using mgfs, we can identify that $S_r \sim NegativeBinomial(r, p)$. 4 MARKS

(iii) By definition, and using the formula sheet

$$P(Y_1 > 2|Y_1 > 1) = \frac{P(Y_1 > 2, Y_1 > 1)}{P(Y_1 > 1)} \equiv \frac{P(Y_1 > 2)}{P(Y_1 > 1)} = \frac{(1-p)^2}{(1-p)} = 1 - p$$

2 MARKS

(iv) By statement, Y_1 and Y_2 are independent, so

$$P(Y_2 > 2|Y_1 > 1) \equiv P(Y_2 > 2) = (1-p)^2.$$

2 MARKS

(b) For all $n \geq 1$, it follows using mgfs that $S_n \sim Normal(0, n)$, and using transformation methods (or mgfs)

$$\overline{Y}_n \sim Normal(0, 1/n).$$

Therefore

$$P\left(\overline{Y}_n \leq \frac{1}{2}\right) = P\left(\sqrt{n}\overline{Y}_n \leq \frac{\sqrt{n}}{2}\right) = \Phi(1) = 0.8413$$

using the table of the Normal distribution. 4 MARKS

(c) We have that by properties of the Bernoulli distribution,

$$\mathbb{E}[Y_i] = \frac{1}{2} \qquad \mathbb{V}[Y_i] = \frac{1}{4}.$$

By the CLT, we have then that

$$P\left(\overline{Y}_n \leq \frac{1}{2}\right) \doteq \frac{1}{2}$$

as the Normal approximation is symmetric around $1/2$. 3 MARKS

(d) We have that

$$\mathbb{E}[U_i] = \mathbb{E}[Y_i^2] = \mathbb{V}[Y_i] + \{\mathbb{E}[Y_i]\}^2 = 1.$$

so, as the variance of U_i is finite, by the weak law of large numbers

$$\overline{U}_n \longrightarrow \mathbb{E}[U_1] = 1$$

in probability as $n \longrightarrow \infty$. 3 MARKS