MATH 323 - EXERCISES 9

NOT FOR ASSESSMENT.

SUMS OF INDEPENDENT RANDOM VARIABLES

If Y_1 and Y_2 are independent random variables, it is possible to compute the distribution of the sum random variable

$$Y = Y_1 + Y_2$$

using the **convolution** approach

- **discrete case**

$$p_Y(y) = \sum_{y_1=-\infty}^{\infty} p_{Y_1}(y_1) p_{Y_2}(y - y_1) \quad y \in \mathbb{Z}$$

- **continuous case**

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y_1}(y_1) f_{Y_2}(y - y_1)\, dy_1 \quad y \in \mathbb{Z}$$

which are results based on the probability axioms and partitioning.

However we may also use moment generating functions (mgfs): if Y_1 and Y_2 are independent random variables with mgfs $m_{Y_1}(t)$ and $m_{Y_2}(t)$, then the mgf of Y is given by the product

$$m_Y(t) = m_{Y_1}(t)\, m_{Y_2}(t)$$

This result follows in a straightforward way using expectations of products of independent random variables. We have

$$m_Y(t) = \mathbb{E}[e^{tY}] = \mathbb{E}[e^{t(Y_1+Y_2)}] = \mathbb{E}[e^{tY_1}e^{tY_2}]] = \mathbb{E}[e^{tY_1}]\mathbb{E}[e^{tY_2}] = m_{Y_1}(t)m_{Y_2}(t).$$

Using these results, show that the following results hold when Y_1 and Y_2 are independent.

1. Binomial:

$$Y_1 \sim Binomial(n_1, p) \quad Y_2 \sim Binomial(n_2, p) \qquad \Longrightarrow \qquad Y \sim Binomial(n_1 + n_2, p)$$

2. Poisson:

$$Y_1 \sim Poisson(\lambda_1) \quad Y_2 \sim Poisson(\lambda_2) \qquad \Longrightarrow \qquad Y \sim Poisson(\lambda_1 + \lambda_2)$$

3. Geometric:

$$Y_1 \sim Geometric(p) \quad Y_2 \sim Geometric(p) \qquad \Longrightarrow \qquad Y \sim NegativeBinomial(2, p)$$

4. Exponential:

$$Y_1 \sim Exponential(\beta) \quad Y_2 \sim Exponential(\beta) \qquad \Longrightarrow \qquad Y \sim Gamma(2, \beta)$$

5. Gamma:

$$Y_1 \sim Gamma(\alpha_1, \beta) \quad Y_2 \sim Gamma(\alpha_2, \beta) \qquad \Longrightarrow \qquad Y \sim Gamma(\alpha_1 + \alpha_2, \beta)$$

6. Normal

$$Y_1 \sim Normal(\mu_1, \sigma_1^2) \quad Y_2 \sim Normal(\mu_2, \sigma_2^2) \qquad \Longrightarrow \qquad Y \sim Normal(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$