MATH 323 - EXERCISES 6

NOT FOR ASSESSMENT.

GENERATING FUNCTIONS

For discrete random variable Y with pmf $p(y)$ we define

- *moment-generating function* (mgf), $m(t)$: for $-b \leq t \leq b$, some $b > 0$,

$$m(t) = \mathbb{E}\left[e^{tY}\right] = \sum_{y} e^{ty} p(y)$$

- *probability-generating function* (pgf), $G(t)$: For a non-negative discrete random variable Y, and for $1 - b \leq t \leq 1 + b$, some $b > 0$,

$$G(t) = \mathbb{E}\left[t^{Y}\right] = \sum_{y=0}^{\infty} t^{y} p(y) = p(0) + tp(1) + t^{2}p(2) + \cdots$$

These definitions hold under the condition that the sum is convergent. Note if $m(t)$ and $G(t)$ are well-defined, we can compute *Taylor series expansions* around $t = 0$ for these two functions, that is

$$m(t) = \sum_{j=0}^{\infty} m^{(j)}(0) \frac{t^{j}}{j!} \qquad G(t) = \sum_{j=0}^{\infty} G^{(j)}(0) \frac{t^{j}}{j!}$$

where

$$m^{(j)}(t) = \frac{d^{j} m(t)}{dt^{j}} \qquad G^{(j)}(t) = \frac{d^{j} G(t)}{dt^{j}}.$$

are the jth derivatives.

1. Show how to compute $p(y)$ from $G(t)$. Hence show that if two pgfs $G_1(t)$ and $G_2(t)$ equal for all $|t| < 1$, then their corresponding pmfs, $p_1(y)$ and $p_2(y)$, are also equal for all y.

   *(Hint: differentiate, and evaluate at a particular values of t. Use the result that two real polynomials are equal everywhere on a bounded region **if and only if** their coefficients are equal).*

2. Identify the pmfs/distributions corresponding to the following mgfs and pgfs:

   (a) $m(t) = (1 + e^{t})/2$.
   (b) $m(t) = e^{t}(1 + e^{t} + e^{2t})/3$.
   (c) $m(t) = (2 + e^{t})^{3}/27$.
   (d) $G(t) = (2 + t)/3$.
   (e) $G(t) = e^{2t}e^{-2}$.
   (f) For some specific $c > 0$

$$G(t) = c(1 + t + t^{2} + t^{3} + \cdots + t^{N}) = \frac{c(1 - t^{N+1})}{1 - t}$$

   Hint: some answers can be found on the Distributions Formula Sheet.

3. Show that the function

$$G(t) = 1 - (1 - t^{2})^{1/2}$$

   a valid pgf for t in a range to be identified.

4. Suppose that discrete random variable Y has mgf $m_Y(t)$ and pgf $G_Y(t)$. Define random variable X by $X = aY + b$ for constants a and b. Using the properties of expectations, derive the mgf and pgf of X in terms of $m_Y(t)$ and $G_Y(t)$.

5. Suppose that
$$p(y) = \frac{1}{2}p^{|y|-1}(1-p) \qquad y = \pm 1, \pm 2, \pm 3, \ldots$$
for some p, $0 < p < 1$, and $p(y) = 0$ for all other values of y. Find the corresponding mgf $m(t)$.

6. Suppose Y is a non-negative discrete random variables with pmf $p(y) > 0$ for $y = 0, 1, 2, \ldots$. Show that
$$\sum_{y=0}^{\infty} P(Y < y)t^y = \frac{t}{1-t}G(t)$$
whenever the required sum is convergent.

Hint: first express $P(Y < y)$ as a sum involving the mass function.