MATH 323 - EXERCISES 3

NOT FOR ASSESSMENT.

COMBINATORIAL PROBABILITY

Note: hints provided do not necessarily indicate the only way to solve the problem.

1. The *binomial expansion* is used to express $(a+b)^n$ in power series form:

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}.$$

   Use the binomial expansion to deduce the following identities: for non-negative integers m and n, and $0 \le k \le m+n$,

   (a)
$$1 - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0$$

   Hint: think about a specific choice of a and b.

   (b)
$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i}$$

   Hint: think about multiplying two brackets with different exponents together.

2. A lottery machine selects six numbers, without replacement, from the integers {1,2,...,49} to represent the winning set of numbers.

   (a) If you have one entry ticket with one choice of comprising 6 numbers, what is your chance of matching the winning set ?
   (b) A prize of $ 10 is won if exactly 3 of your numbers are in the winning sequence. What is the probability of such a win if you have 1 entry ticket?
   (c) What is the probability that, on any week, the ticket contains no winning numbers ?
   (d) Assuming that draws on successive weeks do not affect each other, find an expression in x for the probability that the tickets purchased contain no winning numbers for x successive weeks, where $x = 1, 2, 3 \ldots$..

3. The game of Keno is a lottery-type game which consists of cards numbered 1,2,...,80, twenty of which are to be drawn as winning cards. A player attempts to win at the game by nominating n cards before the draw is made, where n is any number not greater than fifteen. The amount won depends on the number of nominated cards that are subsequently drawn as winning cards.

   What is the probability that no nominated cards are drawn as winning cards if

   (a) $n = 5$ (b) $n = 10$ (c) $n = 15$

   What is the general formula for the probability of obtaining r winning cards ?

4. A committee of $n = 5$ students is to be selected, supposedly at random, from a class of $N = 200$ comprising $R = 120$ Science students and $N - R = 80$ Arts students.

   Comment (using probability reasoning) on the evidence for bias in the selection process if the committee turns out to comprise $r = 5$ Science students, and no Arts students.

   Hint: evaluate the probability of the observed event assuming that the selection process is truly random, that is, that all selections of $n = 5$ from $N = 200$ are equally likely.

5. A bridge hand consists of 13 cards from a standard deck of 52 cards.

   (a) How many different bridge hands are there?

   (b) What is the probability that a randomly chosen bridge hand contains no aces, kings, queens or jacks?
   Hint: hypergeometric formula.

6. A class comprises 30 pupils. What is the probability that, among the twelve months of the calendar, six contain two of the birthdays, and six contain three of the birthdays of the pupils ?

   Hint: partitions and the multinomial formula.

7. A group of $2n$ Arts students and $2n$ Science students is randomly divided into two equally-sized subgroups. What is the probability that each subgroup contains n Arts students and n Science students ?

   Hint: hypergeometric formula.

8. Two hands of thirteen cards are dealt (without replacement) from an ordinary pack. What is the probability that one hand contains exactly n Hearts, and the other contains exactly m Hearts ?

   Hint: multiplication rule and multinomial coefficients or hypergeometric formula.

   Now suppose that four hands of thirteen cards are dealt (without replacement) from an ordinary pack. What is the probability that the four hands contain r_1, r_2, r_3, r_4 Hearts respectively, for non-negative integers satisfying $r_1 + r_2 + r_3 + r_4 = 13$?

   Hint: multiplication rule and multinomial coefficients or hypergeometric formula.

9. n people, including yourself and a friend, are seated at random in a row of n chairs. What is the probability that you sit next to your friend?

   Hint: binary sequences.

10. If n balls are placed at random into n boxes, what is the probability that precisely one box remains empty ?

    Hint: partitions and the multinomial formula.

11. Each of n sticks is broken into two parts, long and short, and a new set of n sticks formed by pairing and joining the $2n$ parts at random. What is the probability that

    (a) each stick is paired and rejoined into its original form that is, there is a match between the rejoined long and short parts for all n sticks ?;

    (b) each of the n long parts are rejoined with a short part ?