MATH 323 - EXERCISES 2
NOT FOR ASSESSMENT.

ELEMENTARY PROBABILITY

1. A person selected from a population and subjected to screening for a particular disease is either a sufferer (D) or not (D'). The screening will either result in a positive test (T), or a negative (T'). If the result of the test is used to classify the person, write down the event "person misclassified".

   Suppose now that an X-ray is also taken and either gives a positive (X), or a negative (X') indication, and also a doctor carries out an examination resulting in a positive (A), or negative (A') assessment. If a person is classified as a sufferer if and only if the doctor and at least one of the tests points in this direction, represent the event "correct classification made".

2. Given two events $A, B \subseteq S$, prove that the probability of *one and only one* of them occurring is

$$P(A) + P(B) - 2P(A \cap B).$$

3. Consider the following statements, which are claimed to be true for events A_1, A_2 in a sample space S:

$$\begin{array}{lll}
\text{(a)} & P(A_1) = 0 & \Longrightarrow \; P(A_1 \cup A_2) = 0 \\
\text{(b)} & P(A_1) = P(A_2') & \Longrightarrow \; A_1' = A_2 \\
\text{(c)} & A_1 \subseteq A_2 \text{ and } P(A_1) = P(A_2') & \Longrightarrow \; P(A_1) \leq 1/2 \\
\text{(d)} & P(A_1') = x_1, P(A_2') = x_2 & \Longrightarrow \; P(A_1 \cup A_2) \geq 1 - x_1 - x_2
\end{array}$$

   In each case, either prove that the statement is true for all S, A_1, A_2, or provide a specific counter-example to show that there exists S, A_1, A_2 for which it is false

4. Given two events $A, B \subseteq S$, prove that

$$\begin{array}{ll}
\text{(a)} & P(A' \cap B) = P(B) - P(A \cap B) \\
\text{(b)} & P(A \cup B) = P(A) + P(B) - P(A \cap B) \\
\text{(c)} & A \subseteq B \Longrightarrow P(A) \leq P(B) \\
\text{(d)} & P(A \cap B) \geq P(A) + P(B) - 1
\end{array}$$

   (d) is known as *Bonferroni's Inequality*.

5. Suppose that A and B are events such that $P(A) = x, P(B) = y$ and $P(A \cap B) = z$. Express the following terms of x, y and z:

   (a) $P(A' \cup B')$
   (b) $P(A' \cap B)$
   (c) $P(A' \cup B)$
   (d) $P(A' \cap B')$.

6. Probability Axiom I can be equivalently stated as:

$$\text{(I)}^* : P(A) \leq 1 \text{ for all } A \subseteq S,$$

   Assuming that Axioms (I)*, (II) and (III) hold, prove that $P(A) \geq 0$ for all $A \subseteq S$.

7. Use *mathematical induction* to prove that, for any events $A_1, A_2, \ldots, A_n$ in S:

   (a) *Boole's Inequality*

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) \leq P(A_1) \ + \ \ldots \ + \ P(A_n)$$

   (b) *General Addition Rule*

$$\begin{aligned} P(A_1 \cup A_2 \cup \ldots \cup A_n) \ &= \ \sum_i P(A_i) \ - \sum\sum_{i \neq j} P(A_i \cap A_j) \\ &+ \sum\sum\sum_{i \neq j \neq k} P(A_i \cap A_j \cap A_k) \ - \\ &\cdots \ + (-1)^{n-1} \ P(A_1 \cap A_2 \cap \ldots \cap A_n) \end{aligned}$$

In (b), summations are over *distinct* sets of subscripts. For example, if $n = 3$, the result would correspond to

$$P(A_1 \cup A_2 \cup A_3) \ = \ P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3).$$

If you have not met 'mathematical induction' formally, follow this strategy (which essentially summarizes the method):

(i) *Establish the result is true for the special cases $n = 1$ and $n = 2$.*

(ii) *For $k \geq 1$, **assume** the result is true for $n = k$*; in part (a), this means assuming that

$$P(A_1 \cup A_2 \cup \ldots \cup A_k) \leq P(A_1) \ + \ \ldots \ + \ P(A_k)$$

(iii) *Using the assumption in (ii), deduce that the result is also true for $n = k + 1$.*

Hint for (a) and (iii):

$$P(A_1 \cup A_2 \cup \ldots \cup A_k \cup A_{k+1}) \equiv P(B \cup A_{k+1})$$

where

$$B = A_1 \cup A_2 \cup \ldots \cup A_k.$$