MATH 323 - ASSIGNMENT 4

Please submit your assignment by 11.59 pm on Friday 19th November by uploading a pdf to myCourses.

1. Find the pmfs corresponding to the following mgfs defined for a discrete random variable. Write your solutions in the form

$$p(y) = \begin{cases} p_1 & y = y_1 \\ p_2 & y = y_2 \\ \vdots & \vdots \\ p_k & y = y_k \\ 0 & \text{otherwise} \end{cases}$$

or in the form

$$p(y) = \begin{cases} \text{some formula} & y \in \mathcal{Y} \\ 0 & \text{otherwise} \end{cases}$$

for a set $\mathcal{Y}$ to be identified, or identify the distribution by name with any parameters identified. If the function is NOT the mgf of a discrete pmf, state this and explain why.

(a) $m(t) = (e^t + e^{-t})/2$, defined for $t \in \mathbb{R}$. 2 MARKS

(b) $m(t) = 2e^t$, defined for $t \in \mathbb{R}$. 2 MARKS

(c) For $t \in \mathbb{R}$

$$m(t) = \frac{(1 + 2e^t)^3}{27}$$

2 MARKS

(d) For $t \in \mathbb{R}$

$$m(t) = \exp\left\{t + \frac{t^2}{2} + \cdots + \frac{t^j}{j!} + \cdots\right\}$$

2 MARKS

(e) For parameter p, $0 < p < 1$.

$$m(t) = \frac{\ln(1 - pe^t)}{\ln(1 - p)} \qquad \text{for } |pe^t| < 1$$

4 MARKS

2. The *Poisson process* is a model for events that occur in continuous time, at a **constant rate** $\lambda > 0$ per unit time, with events occurring **independently** of each other. Specifically, if $X(t)$ is the discrete random variable recording the number of events that are observed to occur in the interval $[0, t)$, then we have that $X(t) \sim Poisson(\lambda t)$, that is

$$p(x) = P(X(t) = x) = e^{-\lambda t}\frac{(\lambda t)^x}{x!} \quad x = 0, 1, 2, \ldots$$

and zero otherwise. Also, the counts of events in **disjoint** time intervals are probabilistically **independent**: for example, for intervals $[0, t)$ and $[t, t + s)$, the numbers of events in the two intervals, X_1 and X_2 say, have the property

$$P(X_1 = x_1 \cap X_2 = x_2) = P(X_1 = x_1)P(X_2 = x_2)$$

with

$$X_1 \sim Poisson(\lambda t) \qquad X_2 \sim Poisson(\lambda s).$$

With this information, answer the following questions based on the Poisson process model and its relationship with the Poisson distribution.

(a) Radioactive particles are detected by a counter according to a Poisson process with rate parameter $\lambda = 0.5$ particles per second. What is the probability that two particles are detected in any given one second interval ? 1 MARK

(b) Page visits to a particular website occur according to a Poisson process with rate parameter $\lambda = 20$ per minute. What is the **expected** number of visits to the website in any given one hour period ? 1 MARK

(c) What is the probability that the time of the first event that is observed to occur in a Poisson process with rate λ per unit time, after initiation at $t = 0$, occurs **later than** time $t = t_0$, for fixed value t_0 ? Justify your answer. 2 MARKS

3. The mgf of the $Poisson(\lambda)$ distribution takes the form

$$m(t) = \exp\{\lambda(e^t - 1)\} \quad t \in \mathbb{R}.$$

Also, if Y is a random variable with mgf $m_Y(t)$, and $X = aY + b$ is a transformed version of Y, then the mgf for X takes the form

$$m_X(t) = e^{bt}m_Y(at).$$

(a) If $Y \sim Poisson(\lambda)$, write down the mgf of the random variable

$$X = \frac{Y - \lambda}{\sqrt{\lambda}} = \frac{Y}{\sqrt{\lambda}} - \sqrt{\lambda}.$$

1 MARK

(b) Find the form of the mgf of X as $\lambda \longrightarrow \infty$. 3 MARKS