MATH 323 - ASSIGNMENT 4 : SOLUTIONS

1. (a) $m(t) = (e^t + e^{-t})/2$, defined for $t \in \mathbb{R}$. 2 MARKS

   Solution: this is the mgf of the pmf

$$p(y) = \begin{cases} \frac{1}{2} & y = -1 \\ \frac{1}{2} & y = 1 \\ 0 & \text{otherwise} \end{cases}$$

   (b) $m(t) = 2e^t$, defined for $t \in \mathbb{R}$. 2 MARKS

   Solution: this is not an mgf, as we require $m(0) = 1$.

   (c) For $t \in \mathbb{R}$

$$m(t) = \frac{(1 + 2e^t)^3}{27}$$

   2 MARKS

   Solution: by inspection of the formula sheet, we recognize this as the mgf of the $Binomial(3, 2/3)$ distribution

   (d) For $t \in \mathbb{R}$

$$m(t) = \exp\left\{t + \frac{t^2}{2} + \cdots + \frac{t^j}{j!} + \cdots\right\}$$

   2 MARKS

   Solution: We note that the term in the exponent is in fact the expansion of $e^t - 1$, and by inspection recognize this as the mgf of the $Poisson(1)$ distribution.

   (e) For parameter p, $0 < p < 1$.

$$m(t) = \frac{\ln(1 - pe^t)}{\ln(1 - p)} \qquad \text{for } |pe^t| < 1$$

   4 MARKS

   Solution: we have the series expansion for the numerator

$$-\ln(1 - pe^t) = \sum_{y=1}^{\infty} \frac{(pe^t)^y}{y} = \sum_{y=1}^{\infty} e^{ty} \frac{p^y}{y}$$

   so therefore we must have that

$$p(y) = -\frac{1}{\ln(1 - p)} \frac{(p)^y}{y} \qquad y = 1, 2, \ldots$$

   and zero otherwise.

2. (a) Radioactive particles are detected by a counter according to a Poisson process with rate parameter $\lambda = 0.5$ particles per second. What is the probability that two particles are detected in any given one second interval ? 1 MARK

Solution: this is given by the $Poisson(0.5)$ *distribution as*

$$P(X(1) = 2) = \frac{e^{-0.5} 0.5^2}{2!} = 0.0758.$$

(b) Page visits to a particular website occur according to a Poisson process with rate parameter $\lambda = 20$ per minute. What is the **expected** number of visits to the website in any given one hour period ? 1 MARK

Solution: this is given by the expectation of the $Poisson(20 \times 60)$ *distribution as 1200*

(c) What is the probability that the time of the first event that is observed to occur in a Poisson process with rate λ per unit time, after initiation at $t = 0$, occurs **later than** time $t = t_0$, for fixed value t_0 ? Justify your answer. 2 MARKS

Solution: 'First event occurs later than t_0*' is exactly equivalent to 'No events in* $[0, t_0)$*', which is given by the* $Poisson(\lambda t_0)$ *distribution as*

$$P(X(t_0) = 0) = e^{-\lambda t_0}.$$

Note that

$$P(X(t) > 0) = 1 - e^{-\lambda t} \longrightarrow 1 \qquad \text{as } t \longrightarrow \infty.$$

so the probability that an event ***ever*** *occurs is 1.*

3. The mgf of the $Poisson(\lambda)$ distribution takes the form

$$m(t) = \exp\{\lambda(e^t - 1)\} \quad t \in \mathbb{R}.$$

Also, if Y is a random variable with mgf $m_Y(t)$, and $X = aY + b$ is a transformed version of Y, then the mgf for X takes the form

$$m_X(t) = e^{bt} m_Y(at).$$

(a) If $Y \sim Poisson(\lambda)$, write down the mgf of the random variable 1 MARK

$$X = \frac{Y - \lambda}{\sqrt{\lambda}} = \frac{Y}{\sqrt{\lambda}} - \sqrt{\lambda}.$$

Solution: We have directly that

$$m_X(t) = e^{-\sqrt{\lambda} t} m_Y(t/\sqrt{\lambda}) = \exp\left\{-\sqrt{\lambda} t + \lambda(e^{t/\sqrt{\lambda}} - 1)\right\}$$

(b) Find the form of the mgf of X as $\lambda \longrightarrow \infty$. 3 MARKS

Solution: using the series expansion for the exponent term, we have that

$$m_X(t) = \exp\left\{-\sqrt{\lambda} t + \lambda\left(\frac{t}{\sqrt{\lambda}} + \frac{t^2}{2\lambda} + \frac{t^3}{6\lambda^{3/2}} + \cdots\right)\right\} = \exp\left\{\frac{t^2}{2} + \frac{t^3}{6\lambda^{1/2}} + \cdots\right\} \longrightarrow \exp\left\{\frac{t^2}{2}\right\}$$

as $\lambda \longrightarrow \infty$.