MATH 323 - ASSIGNMENT 3– SOLUTIONS

1. Suppose that the number of eggs produced by a particular species of fly in a given batch is a discrete random variable Y with pmf

$$p(y) = \begin{cases} c\dfrac{p^y}{y} & y \in \{1, 2, 3, \dots\} \\ 0 & \text{otherwise} \end{cases}$$

for parameter p, $0 < p < 1$, and some constant c.

(a) By finding an appropriate value of constant c, show that this is a valid pmf. 2 MARKS

Solution: we recognize that the series with terms of the form p^y/y is the logarithmic series, that is

$$-\log(1-t) = t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots = \sum_{y=1}^{\infty} \frac{t^y}{y}$$

when $|t| < 1$, so we may deduce directly that

$$c = -\frac{1}{\log(1-p)}.$$

(b) Find the probability $P(Y > 4)$ if $p = 0.2$. 2 MARKS

Solution: we have that

$$\begin{aligned} P(Y > 4) = 1 - P(Y \le 4) &= 1 - P(Y = 1) - P(Y = 2) - P(Y = 3) - P(Y = 4) \\ &= 1 + \frac{1}{\log(1-p)}\left[p + \frac{p^2}{2} + \frac{p^3}{3} + \frac{p^4}{4}\right] \end{aligned}$$

which, for $p = 0.2$, yields that

$$P(Y > 4) = 0.00034.$$

(c) For this distribution, which is bigger, $\mathbb{E}[Y]$ or $\mathbb{E}[Y^2]$? Justify your answer. 4 MARKS

Solution: For this distribution, we can in fact compute $\mathbb{E}[Y]$ and $\mathbb{E}[Y^2]$ directly:

$$\begin{aligned} \mathbb{E}[Y] = \sum_{y=1}^{\infty} yp(y) &= \sum_{y=1}^{\infty} -\frac{y}{\log(1-p)}\frac{p^y}{y} \\ &= -\frac{1}{\log(1-p)}\sum_{y=1}^{\infty} p^y \\ &= -\frac{1}{\log(1-p)}\left(\frac{1}{1-p} - 1\right) \qquad \textit{Geometric series result} \\ &= -\frac{1}{\log(1-p)}\frac{p}{1-p} \end{aligned}$$

$$\mathbb{E}[Y^2] = \sum_{y=1}^{\infty} y^2 p(y) = \sum_{y=1}^{\infty} -\frac{y^2}{\log(1-p)} \frac{p^y}{y}$$

$$= -\frac{1}{\log(1-p)} \sum_{y=1}^{\infty} y p^y$$

$$= -\frac{p}{\log(1-p)} \frac{1}{(1-p)^2}$$

$$= -\frac{1}{\log(1-p)} \frac{p}{(1-p)^2}$$

so therefore as $p < 1$

$$\mathbb{E}[Y^2] > \mathbb{E}[Y].$$

However, we can also address this directly by noting that for $y > 1$, $y^2 > y$, so therefore

$$\mathbb{E}[Y^2] = \sum_{y=1}^{\infty} y^2 p(y) = p(1) + \sum_{y=2}^{\infty} y^2 p(y) > p(1) + \sum_{y=2}^{\infty} y p(y) = \mathbb{E}[Y].$$

2. Suppose that two random variables Y_1 and Y_2 are *independent*, that is, for all values of y_1 and y_2,

$$P((Y_1 = y_1) \cap (Y_2 = y_2)) = P(Y_1 = y_1)P(Y_2 = y_2)$$

that is, the events $(Y_1 = y_1)$ and $(Y_2 = y_2)$ are independent. Suppose also that Y_1 and Y_2 have the same Geometric distribution, that is $Y_1 \sim Geometric(p)$ and $Y_2 \sim Geometric(p)$. Define a third random variable Y as the sum of Y_1 and Y_2, that is, $Y = Y_1 + Y_2$.

By considering the Theorem of Total Probability and the partitioning result

$$(Y = y) \equiv \bigcup_{t=1}^{y-1} ((Y_1 = t) \cap (Y_2 = y - t))$$

for $y = 2, 3, \ldots$, derive the pmf of Y. 6 MARKS

Hint: apply Axiom III to the above partition, and use the pmfs for Y_1 and Y_2, and the assumed independence, to compute the right hand side..

Solution: We have by Axiom III applied to the partition, for $y = 2, 3, \ldots$

$$P(Y = y) = \sum_{t=1}^{y-1} P\left((Y_1 = t) \cap (Y_2 = y - t)\right)$$

$$= \sum_{t=1}^{y-1} P(Y_1 = t)P(Y_2 = y - t) \qquad \textit{independence}$$

$$= \sum_{t=1}^{y-1} (q^{t-1}p) \times (q^{y-t-1}p)$$

$$= p^2 q^{y-2} \sum_{t=1}^{y-1} 1 = (y-1)p^2 q^{y-2} = \binom{y-1}{1} q^{y-2} p^2$$

and we spot that this is the Negative Binomial distribution with parameters $r = 2$ and p.

3. Suppose that $Y \sim Binomial(n, p)$ (that is, discrete random variable Y has a Binomial distribution with parameters n and p, where n is a (strictly) positive integer, and $0 < p < 1$).

   (a) Verify using general properties of expectations that

$$\mathbb{E}[Y^3] = \mathbb{E}[Y(Y-1)(Y-2)] + 3\mathbb{E}[Y(Y-1)] + \mathbb{E}[Y]$$

2 MARKS

Solution: expanding the terms

$$\mathbb{E}[Y^3 - Y^2 - 2Y^2 + 2Y] + 3\mathbb{E}[Y^2 - Y] + \mathbb{E}[Y] = \mathbb{E}[Y^3]$$

   (b) Find an expression for $\mathbb{E}[Y^3]$ for $n \geq 3$. Justify your answer. 4 MARKS

Solution: From lectures, we know that

$$\mathbb{E}[Y(Y-1)] = n(n-1)p^2 \qquad \mathbb{E}[Y] = np$$

and by using the same approach we can compute

$$\begin{aligned}
\mathbb{E}[Y(Y-1)(Y-2)] &= \sum_{y=0}^{n} y(y-1)(y-2)p(y) \\
&= \sum_{y=0}^{n} y(y-1)(y-2)\binom{n}{y}p^y q^{n-y} \\
&= \sum_{y=3}^{n} y(y-1)(y-2)\frac{n!}{y!(n-y)!}p^y q^{n-y} \\
&= n(n-1)(n-2)\sum_{y=3}^{n} \frac{(n-3)!}{(y-3)!(n-y)!}p^y q^{n-y} \\
&= n(n-1)(n-2)p^3 \sum_{y=3}^{n} \frac{(n-3)!}{(y-3)!((n-3)-(y-3))!}p^{y-3} q^{(n-3)-(y-3)} \\
&= n(n-1)(n-2)p^3 \sum_{j=0}^{n-3} \frac{(n-3)!}{j!((n-3)-j)!}p^{j} q^{(n-3)-j} \qquad j = y-3 \\
&= n(n-1)(n-2)p^3(p+q)^{n-3} \\
&= n(n-1)(n-2)p^3
\end{aligned}$$

so therefore

$$\mathbb{E}[Y^3] = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np.$$