MATH 323 - ASSIGNMENT 2– SOLUTIONS

Please submit your assignment by 11.59 pm on Friday 8th October by uploading a pdf to myCourses Assignment 2.

Note: this assignment will be hand-graded, so any legible numerical answer format is acceptable.

1. Suppose that A and B are events in sample space S. Using the stated information, compute the requested conditional probability. Show your working. If the requested conditional probability cannot be computed due to lack of information or due to contradictory information, explain why. If the requested conditional probability is not defined, explain why.

   (a) $P(A) = 0.423$, $P(B) = 0.190$, $P(A \cap B) = 0.115$. Compute $P(B|A)$ — 1 MARK

   Solution: By definition

   $$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.115}{0.423} = 0.2719$$

   (b) $P(A \cup B) = 0.157$, $P(A \cap B') = 0.659$, $P(A' \cap B) = 0.044$. Compute $P(A|B)$. — 2 MARKS

   Solution: Here we have

   $$P(A \cup B) < P(A \cap B') + P(A' \cap B)$$

   *which is a **contradiction** of the rules of probability.*

   (c) $P(A' \cap B) = 0.420$, $P(A \cap B') = 0.580$. Compute $P(A|B)$. — 2 MARKS

   Solution: Here we have

   $$P(A' \cap B) + P(A \cap B') = 1$$

   so therefore we must have that A and B partition S. Therefore

   $$P(A|B) = \frac{P(A \cap B)}{P(B)} = 0$$

   as $P(A \cap B) = 0$.

   (d) $P(A) = 0.050$, $P(B) = 0.891$, $P(A \cap B) = 0.046$. Compute $P(B'|A)$. — 2 MARKS

   Solution: By the Axioms and the definition of conditional probability

   $$P(B'|A) = 1 - P(B|A) = 1 - \frac{P(A \cap B)}{P(A)} = 1 - \frac{0.046}{0.050} = 0.080$$

   (e) $P(A) = 0.635$, $P(B) = 0.433$, $P(A \cap B) = 0.210$. Compute $P(A'|B')$. — 2 MARKS

   Solution: By definition of conditional probability, de Morgan's Law and the Axioms,

   $$P(A'|B') = \frac{P(A' \cap B')}{P(B')} = \frac{1 - P(A \cup B)}{1 - P(B)}$$

   as $A' \cap B' = (A \cup B)'$. By the general addition rule

   $$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.635 + 0.433 - 0.210 = 0.858$$

   so therefore

   $$P(A'|B') = \frac{1 - 0.858}{1 - 0.433} = 0.2504.$$

2. The operating lifetime (in hours) of a phone battery can be described using a continuous sample space $S = \mathbb{R}^+$ (ie the positive real numbers) by considering events of the type

$$A_x = \text{“Lifetime of battery exceeds (ie is greater than or equal to) } x \text{ hours”}$$

and by specifying probabilities of events of this type using the formula

$$P(A_x) = \exp\{-(x/\lambda)^2\} \qquad x > 0$$

where λ is a positive constant value. For the purpose of the question, $\lambda = 2000$.

(a) Compute the probability that the lifetime exceeds 1000 hours. 1 MARK

Solution: This is

$$P(A_{1000}) = \exp\{-(1000/2000)^2\} = e^{-1/4} = 0.7788.$$

(b) Compute the probability that the lifetime exceeds 1000 hours if it is **known** that the lifetime exceeds 500 hours. 2 MARKS

Solution: This is

$$P(A_{1000}|A_{500}) = \frac{P(A_{500} \cap A_{1000})}{P(A_{500})} \equiv \frac{P(A_{1000})}{P(A_{500})} = \exp\{-(1/2)^2 + (1/4)^2\} = \exp\{-3/16\} = 0.8290.$$

(c) If five identical phone batteries were used under identical operating conditions in five different phones, so that any events concerning the battery lifetimes could be considered mutually independent, compute that probability that **one or more** of the battery lifetimes did **not** exceed 500 hours. 2 MARKS

Solution: The event that any given battery lifetime exceeds 500 hours is A_{500}, and we have that

$$P(A_{500}) = \exp\{-(1/4)^2\} = 0.9394.$$

*By mutual independence, the probability that **all** of the battery lifetimes exceed 500 hours is*

$$\{P(A_{500})\}^5 = 0.7316$$

and hence the required probability is the complement probability

$$1 - \{P(A_{500})\}^5 = 0.2684.$$

3. A bag contains five coins: two are standard (Head/Tail), but two are double-headed (Head/Head) and one is double-tailed (Tail/Tail). When any of the coins is tossed, each side is equally likely to land face-up.

   (a) A coin is selected from the bag and tossed, with all coins equally likely to be selected. Compute the probability that the side that is face-down is a Head. 2 MARKS

   *Solution: First we establish the sample space, S comprising the possible **downward** faces. We can list the elements of S as*

   $$S = \{H_1, T_1, H_2, T_2, H_{11}, H_{12}, H_{21}, H_{22}, T_{11}, T_{12}\}$$

   where

   - *H_1 and T_1 are the faces of the first standard coin (coin C_1 say)*
   - *H_2 and T_2 are the faces of the second standard coin (coin C_2)*
   - *H_{11} and H_{12} are the faces of the first double-headed coin (coin C_3)*
   - *H_{21} and H_{22} are the faces of the second double-headed coin (coin C_4)*
   - *T_{11} and T_{12} are the faces of the double-tailed coin (coin C_5)*

   The experiment involves picking a coin with all coins equally likely, and then a face with both faces equally likely. Therefore each of the 10 outcomes in S is also equally likely. The event of interest is A_1 that the downward face is a Head, so

   $$A_1 = \{H_1, H_2, H_{11}, H_{12}, H_{21}, H_{22}\}$$

   so we must have $P(A_1) = 6/10 = 3/5$.

   (b) After the first toss, it is observed that the visible (face-up) side is a Head. Given this information, compute the probability that the side that is face-down is a Head. 2 MARKS

   Solution: We have observed the event B that the upward face is a Head. Therefore we can write with respect to sample space S that

   $$B = \{T_1, T_2, H_{11}, H_{12}, H_{21}, H_{22}\}$$

   so that $A_1 \cap B = \{H_{11}, H_{12}, H_{21}, H_{22}\}$, and hence

   $$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{4/10}{6/10} = 2/3.$$

   (c) The same selected coin is then tossed again. Compute the probability that the side that is face-down after the second toss is a Head, given the information gained after the first toss.

   2 MARKS

   *Solution: We know first that the double-tailed coin, C_5, is excluded from consideration. Secondly, we know that the coins for the second toss are **not equally likely** due to the information gathered in (b); given B, we have*

   $$P(C_1|B) = \frac{P(C_1 \cap B)}{P(B)} = \frac{1/10}{6/10} = \frac{1}{6} = P(C_2|B)$$

   but

   $$P(C_3|B) = \frac{P(C_3 \cap B)}{P(B)} = \frac{2/10}{6/10} = \frac{2}{6} = P(C_4|B)$$

   Now we wish to compute the probability of event A_2, that is, the probability that the downward face is a Head on the second toss, given that B occurs, that is $P(A_2|B)$. We have that

   $$P(A_2|B) = \frac{P(A_2 \cap B)}{P(B)}.$$

We may partition the event $A_2 \cap B$ via the coin choices, and write

$$P(A_2 \cap B) = \sum_{i=1}^{5} P(A_2 \cap B \cap C_i)$$

or using the conditioning on B from the start, write

$$P(A_2|B) = \sum_{i=1}^{5} P(A_2 \cap C_i|B).$$

Now, by the above logic, we have $P(A_2 \cap C_5|B) = 0$ (as given B, coin C_5 cannot be selected). Therefore, using the definition of conditional probability, we have

$$P(A_2|B) = \sum_{i=1}^{4} P(A_2|B \cap C_i)P(C_i|B) \equiv \sum_{i=1}^{4} P(A_2|C_i)P(C_i|B)$$

as, given that we know which coin is selected, knowledge of B is irrelevant. Thus as

$$P(A_2|C_1) = P(A_2|C_2) = \frac{1}{2} \qquad\qquad P(A_2|C_3) = P(A_2|C_4) = 1$$

we have that

$$P(A_2|B) = \left(\frac{1}{2} \times \frac{1}{6}\right) + \left(\frac{1}{2} \times \frac{1}{6}\right) + \left(1 \times \frac{2}{6}\right) + \left(1 \times \frac{2}{6}\right) = \frac{5}{6}$$