

MATH 323 - EXERCISES 6- SOLUTIONS

- 1 (i) Density function must integrate to 1 over $\mathbb{X} = [0, 1]$, so

$$\int_0^1 f(x) dx = 1 \implies \int_0^1 cx^2 + x dx = 1 \implies \left[c\frac{x^3}{3} + \frac{x^2}{2} \right]_0^1 = 1 \implies c = \frac{3}{2}$$

- (ii) Distribution function F given for $0 \leq x \leq 1$ by

$$F(x) = \int_0^x f(t) dt = \frac{x^3 + x^2}{2}$$

and $F(x) = 0$ for $x < 0$, and $F(x) = 1$ for $x > 1$.

- (iii) $P[X < 1/2] = F(1/2) = 3/16$.

- (iv) From definition of conditional probability

$$P[X > 1/2 \mid X > 1/4] = \frac{P[X > 1/2, X > 1/4]}{P[X > 1/4]} = \frac{P[X > 1/2]}{P[X > 1/4]} = \frac{1 - F(1/2)}{1 - F(1/4)} = \frac{104}{123}$$

- (v) $Y \sim \text{Binomial}(200, \theta)$ where

$$\theta = P[Y_i = 1] = P[X < 1/6] = F(1/6) = \frac{7}{432}$$

Then

$$P[Y \leq 3] = P[Y = 0] + P[Y = 1] + P[Y = 2] + P[Y = 3] = 0.593$$

- 2 Density function must integrate to 1 over $\mathbb{X} = [-1, 1]$, so

$$\int_{-1}^1 f(x) dx = 1 \implies c = \frac{3}{4}$$

Distribution function F given for $-1 \leq x \leq 1$ by

$$F(x) = \int_{-1}^x f(t) dt = \left[\frac{3x}{4} - \frac{x^3}{4} \right]_{-1}^1 = \frac{3}{4} \left(x - \frac{x^3}{3} \right) + \frac{2}{3}$$

and $F(x) = 0$ for $x < -1$, and $F(x) = 1$ for $x > 1$.

- 3 (i) Density function must integrate to 1 over $\mathbb{X} = [0, \pi/2]$, so

$$\int_0^{\pi/2} f(x) dx = 1 \implies \int_0^{\pi/2} cx(\pi - x) dx = 1 \implies c = \frac{12}{\pi^3}$$

- (ii) $\mathbb{X} = [0, \pi/2] \implies \mathbb{Y} = [0, 1/2]$, and distribution function F_Y given by

$$F_Y(y) = P[Y \leq y] = P[\sin X \leq 2y] = P[X \leq \sin^{-1}(2y)] = F(\sin^{-1}(2y))$$

as $\sin$ is monotone on $[0, \pi/2]$. Hence by differentiation with respect to y ,

$$f_Y(y) = \frac{2}{\sqrt{1 - 4y^2}} f(\sin^{-1}(2y)) \quad 0 \leq y \leq 1/2$$

and zero otherwise.

4 By differentiation, $f(x) = 2xe^{-x^2}$, $x > 0$, and zero otherwise

5 Need to consider ranges of integration carefully;

$$F(x) = \begin{cases} \int_0^x t \, dt & = x^2/2 & 0 \leq x \leq 1 \\ \int_0^1 t \, dt + \int_1^x (2-t) \, dt & = 2x - x^2/2 - 1 & 1 \leq x \leq 2 \end{cases}$$

and $F(x) = 0$ for $x < 0$, and $F(x) = 1$ for $x > 2$. Hence $P[0.8 < X < 1.2] = F(1.2) - F(0.8) = 0.36$.

6 (i) $P[T > 3 | A] = 1 - P[T \leq 3 | A] = 1 - \Phi((3-2)/(3/4)) = 1 - \Phi(4/3)$

(ii) By the Theorem of Total Probability

$$P[T > 3] = P[T > 3 | A]P(A) + P[T > 3 | B]P(B) = (1 - \Phi(4/3)) \times \frac{3}{10} + (1 - \Phi(-4/3)) \times \frac{7}{10}$$

(iii) By Bayes Theorem

$$P[A | T > 3] = \frac{P[T > 3 | A]P(A)}{P[T > 3]}$$

so obtain final answer by evaluating $\Phi(4/3)$ via statistical tables, and noting that $\Phi(-4/3) = 1 - \Phi(4/3)$.

7 (i) $X \sim \text{Exponential}(1) \implies F(x) = 1 - e^{-x}$ for $x > 0$, so $F(x) = 1/2 \implies x = \log 2$.

(ii) $\log X \sim \text{Normal}(\mu, \sigma^2)$, so

$$\begin{aligned} F(x) = 1/2 &\implies P[X \leq x] = 1/2 \implies P[(\log X - \mu)/\sigma \leq (\log x - \mu)/\sigma] = 1/2 \\ &\implies \Phi((\log x - \mu)/\sigma) = 1/2 \implies (\log x - \mu)/\sigma = 0 \implies \log x = \mu \implies x = e^\mu \end{aligned}$$