

MATH 323 - EXERCISES 5- SOLUTIONS

- 1 Need to show f non-negative, and sums to one over the range of X . Sum of geometric progression gives result, that is,

$$\sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} \frac{1}{1+\lambda} \left(\frac{\lambda}{1+\lambda} \right)^x = \frac{1}{1+\lambda} \left(1 - \frac{\lambda}{1+\lambda} \right)^{-1} = 1$$

Distribution function F given, for $x = 0, 1, \dots$ by

$$F(x) = \sum_{i=0}^x f(i) = \sum_{i=0}^x \frac{1}{1+\lambda} \left(\frac{\lambda}{1+\lambda} \right)^i = (1-\theta) \sum_{i=0}^x \theta^i = (1-\theta) \frac{1-\theta^{x+1}}{1-\theta} = 1 - \theta^{x+1}$$

where $\theta = \frac{\lambda}{1+\lambda}$.

- 2 $Y = X - n \implies \mathbb{Y} = \{0, 1, 2, \dots\}$ and

$$f_Y(y) = P[Y = y] = P[X - n = y] = P[X = y + n] = f(y + n) = \binom{n+y-1}{n-1} \theta^n (1-\theta)^y$$

- 3 $X \sim \text{Hypergeometric}(N, R, n)$, so for $x \in \{\text{Max}(0, n - N + R), \dots, \text{Min}(n, R)\}$,

$$\begin{aligned} f(x) &= \frac{\binom{R}{x} \binom{N-R}{n-x}}{\binom{N}{n}} = \frac{R!}{x!(R-x)!} \frac{(N-R)!}{(n-x)!(N-R-n+x)!} \frac{n!(N-n)!}{N!} \\ &= \binom{n}{x} \frac{R!}{(R-x)!} \frac{(N-R)!}{(N-R-n+x)!} \frac{(N-n)!}{N!} \\ &\approx \binom{n}{x} R^x (N-R)^{n-x} \frac{1}{N^n} = \binom{n}{x} \theta^x (1-\theta)^{n-x} \end{aligned}$$

where $\theta = R/N$. Hence for N and R large, with $\theta = R/N$ fixed, the hypergeometric distribution tends to the binomial distribution, and thus sampling without replacement from a large population is approximately equivalent to sampling with replacement.

- 4 Let $U = \text{"Number of Heads"}$, then $X = U - (n - U) = 2U - n$. Thus

$$\mathbb{X} = \{-n, -n+2, -n+4, \dots, n-4, n-2, n\}$$

and

$$f(x) = P[X = x] = P[2U - n = x] = P[U = (x+n)/2] = f_U((x+n)/2)$$

where $U \sim \text{Binomial}(n, 1/2)$.

5 If $X \sim \text{Geometric}(\theta)$, then

$$f(x) = (1 - \theta)^{x-1}\theta \quad F(x) = 1 - (1 - \theta)^x$$

for $x \in \{1, 2, 3, \dots\}$. Thus $P[X > n] = (1 - \theta)^n$, and hence

$$\begin{aligned} P[X = n + k | X > n] &= \frac{P[X = n + k, X > n]}{P[X > n]} = \frac{P[X = n + k]}{P[X > n]} \\ &= \frac{(1 - \theta)^{n+k-1}\theta}{(1 - \theta)^n} = (1 - \theta)^{k-1}\theta = P[X = k] \end{aligned}$$

6 Need $f(x)$ non-negative and convergent;

- (i) always non-negative and convergent; need $k = 1$
- (ii) always non-negative and convergent if $\alpha < -1$; no closed form for k .

7 (i) $X = \text{Min}\{X_1, \dots, X_n\}$, so $\mathbb{X} = \{0, 1\}$.

$$P[X = 1] = P[\text{Min}\{X_1, \dots, X_n\} = 1] = P[X_1 = 1, X_2 = 1, \dots, X_n = 1] = \theta^n$$

$$P[X = 0] = 1 - \theta^n$$

Hence

$$f(x) = \begin{cases} 1 - \theta^n & x = 0 \\ \theta^n & x = 1 \end{cases}$$

(ii) $X = \text{Max}\{X_1, \dots, X_n\}$, so $\mathbb{X} = \{0, 1\}$.

$$P[X = 0] = P[\text{Max}\{X_1, \dots, X_n\} = 0] = P[X_1 = 0, X_2 = 0, \dots, X_n = 0] = (1 - \theta)^n$$

$$P[X = 1] = 1 - (1 - \theta)^n$$

Hence

$$f(x) = \begin{cases} (1 - \theta)^n & x = 0 \\ 1 - (1 - \theta)^n & x = 1 \end{cases}$$

8 $I_E : S \longrightarrow \{0, 1\}$ and

$$P[I_E(\omega) = 1] \equiv P[\omega \in E] = P(E) = \theta$$

and hence I_E is a Bernoulli random variable with parameter θ . If X is discrete, can write $\mathbb{X} = \{x_1, x_2, \dots\}$, and hence

$$X = \sum_{i=1}^{\infty} I_{E_i} x_i$$

where

$$E_i = \{\omega : X(\omega) = x_i\}$$

- 9 (i) The Poisson mass function is a valid specification as, first, it is non-negative for all $x \in \mathbb{R}$, and secondly as

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} e^{\lambda} = 1$$

by a standard power series result.

- (ii) For $z = 1, 2, \dots$, write $f_Z(z) = P[Z = z] = cP[X = z]$. Then

$$\sum_{z=1}^{\infty} f_Z(z) = \sum_{z=1}^{\infty} c P[X = z] = c \sum_{z=1}^{\infty} f(z) = c P[X > 0] = c (1 - P[X = 0]) = c(1 - e^{-\lambda})$$

Hence $c = 1/(1 - e^{-\lambda})$, and so

$$f_Z(z) = \frac{e^{-\lambda} \lambda^z}{(1 - e^{-\lambda}) z!} \quad z = 1, 2, \dots$$

and zero otherwise.

- (iii) Using the partition, and/or the Theorem of Total Probability, for $y = 0, 1, 2, \dots$,

$$\begin{aligned} f_Y(y) = P[Y = y] &= \sum_{x_1=0}^{\infty} P[(X_1 = x_1) \cap (X_2 = y - x_1)] = \sum_{x_1=0}^y P[X_1 = x_1] P[X_2 = y - x_1] \\ &= \sum_{x_1=0}^y \frac{e^{-\lambda_1} \lambda_1^{x_1}}{x_1!} \frac{e^{-\lambda_2} \lambda_2^{y-x_1}}{(y-x_1)!} = e^{-(\lambda_1+\lambda_2)} \sum_{x_1=0}^y \frac{1}{x_1!(y-x_1)!} \lambda_1^{x_1} \lambda_2^{y-x_1} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{y!} \sum_{x_1=0}^y \frac{y!}{x_1!(y-x_1)!} \lambda_1^{x_1} \lambda_2^{y-x_1} = \frac{e^{-(\lambda_1+\lambda_2)}}{y!} (\lambda_1 + \lambda_2)^y \end{aligned}$$

where the first line follows from independence of X_1 and X_2 . Hence $Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

- 10 (i) If $X \sim \text{Poisson}(\lambda)$, then

$$G_X(t) = \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!} t^x = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda t)^x}{x!} = e^{-\lambda} e^{\lambda t} = e^{\lambda(t-1)}$$

- (ii) If X_1 and X_2 are independent, and $Y = X_1 + X_2$, then by using the argument from (iii) above,

$$\begin{aligned} G_Y(t) &= \sum_{y=-\infty}^{\infty} f_Y(y) t^y = \sum_{y=-\infty}^{\infty} \left\{ \sum_{x_1=-\infty}^{\infty} f_{X_1}(x_1) f_{X_2}(y - x_1) \right\} t^y \\ &= \sum_{x_1=-\infty}^{\infty} \sum_{x_2=-\infty}^{\infty} f_{X_1}(x_1) f_{X_2}(x_2) t^{x_1+x_2} \quad \text{changing variables to } x_2 = y - x_1 \\ &= \left\{ \sum_{x_1=-\infty}^{\infty} f_{X_1}(x_1) t^{x_1} \right\} \left\{ \sum_{x_2=-\infty}^{\infty} f_{X_2}(x_2) t^{x_2} \right\} = G_{X_1}(t) G_{X_2}(t) \end{aligned}$$