

MATH 323 - EXERCISES 3- SOLUTIONS

- 1 (a) By the Binomial distribution, counting the number of “successes” (individuals with the correct birthday)

$$p(x) = \binom{500}{x} \left(\frac{1}{365}\right)^x \left(1 - \frac{1}{365}\right)^{500-x} \quad x = 0, 1, 2, \dots, 500$$

- (b) If $\lambda = 500/365$, then

$$\begin{aligned} p(x) &= \frac{500!}{x!(500-x)!} \left(\frac{1}{365}\right)^x \left(1 - \frac{1}{365}\right)^{500-x} \\ &= \frac{1}{x!} \frac{500!}{(500-x)!} \left(\frac{1/365}{1-1/365}\right)^x \left(1 - \frac{500}{365 \cdot 500}\right)^{500} \\ &\approx \frac{1}{x!} 500^x \left(\frac{1}{365}\right)^x \left(1 - \frac{500}{365 \cdot 500}\right)^{500} = \frac{1}{x!} \lambda^x \left(1 - \frac{\lambda}{500}\right)^{500} \approx \frac{\lambda^x e^{-\lambda}}{x!} \end{aligned}$$

Numerical verification:

x	0	1	2	3	4	5	6
EXACT	0.2537	0.3484	0.2388	0.1089	0.0372	0.0101	0.0023
APPROXIMATE	0.2541	0.3481	0.2385	0.1089	0.0373	0.0102	0.0023

- 2 For both variables, range is $\{0, 1, 2\}$, and distribution is given by combinatorial arguments as

$$p(x) = \frac{\binom{3}{x} \binom{3}{2-x}}{\binom{6}{2}} \quad x = 0, 1, 2$$

and zero otherwise, with an identical result for $p(y)$.

- 3 (i) Range $\mathbb{X} = \{2, 3, 4, 5\}$. Now $p(x) = P[X = x] = n_E/n_S$, say, and

$$\begin{aligned} n_E &= \text{“number of ways of choosing two from five with largest equal to } x\text{”} = x - 1 \\ n_S &= \text{“number of ways of choosing two from five”} = \binom{5}{2} = 10 \end{aligned}$$

so $p(x) = P[X = x] = (x - 1)/10$.

- (ii) Range $\mathbb{Y} = \{3, 4, 5, 6, 7, 8, 9\}$ As above, define $p(y) = P[Y = y] = n_E/n_S$, say, and again $n_S = 10$. Enumeration of n_E achieved by considering distinguishable partitions of y into the sum of two integers in the range $\{1, 2, 3, 4, 5\}$. Hence if $y = 3, 4, 8, 9$, $n_E = 1$, but if $y = 5, 6, 7$, $n_E = 2$, so

$$p(y) = \begin{cases} 1/10 & y = 3, 4, 8, 9 \\ 2/10 & y = 5, 6, 7 \end{cases}$$

- 4 Consider a binary sequence of length $n = 5$ corresponding to the results of the procedures (1=Success, 0=Failure). All such sequences containing x 1s and $n - x$ have probability

$$\theta^x(1 - \theta)^{n-x}$$

by the multiplication rule for independent events. Thus the probability that $X = x$ is

$$p(x) = \binom{n}{x} \theta^x (1 - \theta)^{n-x} \quad x = 0, 1, 2, \dots, n$$

and zero otherwise, as there are $\binom{n}{x}$ such sequences. Hence $X \sim \text{Binomial}(n, \theta)$, so

$$(i) \theta = 0.8, P[X = 5] = 0.3227$$

$$(ii) \theta = 0.6, P[X = 4] = 0.2592$$

$$(iii) \theta = 0.3, P[X < 2] = P[X = 0] + P[X = 1] = 0.5282$$

- 5 Experiment : sequence of independent and identical binary trials until first success; for $X = x$, need $x - 1$ failures, then a success, and so as all successive tests are independent, we have

$$p(x) = (1 - \theta)^{x-1} \theta \quad x = 1, 2, 3, \dots$$

$$F(x) = 1 - (1 - \theta)^x \quad x = 1, 2, 3, \dots$$

(so that $X \sim \text{Geometric}(\theta)$) and hence

$$(i) \theta = 0.25, P[X \leq 3] = F(3) = 1 - (1 - 0.25)^3 = 0.5781.$$

$$(ii) \theta = 0.7, P[X > 5] = 1 - P[X \leq 5] = 1 - F(5) = (1 - 0.7)^5 = 0.00243$$

- 6 Experiment : sequence of independent and identical binary trials until third success. For $X = x$, we require that we have $n - 1$ successes in the first $x - 1$ trials, for which we can calculate the probability using the $\text{Binomial}(x - 1, \theta)$ formula, and then a success on the x th trial. Hence

$$P[X = x] = \binom{x-1}{n-1} \theta^{n-1} (1 - \theta)^{(x-1)-(n-1)} \times \theta = \binom{x-1}{n-1} \theta^n (1 - \theta)^{x-n}$$

for $x = n, n + 1, n + 2, \dots$. Hence we have

$$p(x) = \binom{x-1}{n-1} \theta^n (1 - \theta)^{x-n} = \binom{x-1}{2} (0.5)^3 (0.5)^{x-3} \quad x = 3, 4, 5, \dots$$

and zero otherwise.