

MATH 323 - EXERCISES 3- SOLUTIONS

1 For events $E_1, E_2, F \subseteq S$ with $P(F) > 0$;

$$(I) \quad P(E_1|F) = \frac{P(E_1 \cap F)}{P(F)} \geq 0, \text{ and } E_1 \cap F \subseteq F \implies P(E_1|F) \leq 1.$$

$$(II) \quad P(S|F) = \frac{P(S \cap F)}{P(F)} = \frac{P(F)}{P(F)} = 1$$

$$(III) \quad P(E_1 \cup E_2|F) = \frac{P((E_1 \cup E_2) \cap F)}{P(F)} = \frac{P(E_1 \cap F)}{P(F)} + \frac{P(E_2 \cap F)}{P(F)} = P(E_1|F) + P(E_2|F)$$

as $(E_1 \cup E_2) \cap F = (E_1 \cap F) \cup (E_2 \cap F)$ which are disjoint events.

2 (a) We have

$$P(E^c \cap F) = P(F) - P(E \cap F) = P(F) - P(E)P(F) = (1 - P(E))P(F) = P(E^c)P(F)$$

$$\begin{aligned} P(E^c \cap F^c) &= 1 - P(E \cup F) = 1 - P(E) - P(F) + P(E \cap F) \\ &= 1 - P(E) - P(F) + P(E)P(F) = (1 - P(E))(1 - P(F)) = P(E^c)P(F^c) \end{aligned}$$

(b) By definition of disjoint events

$$P(E \cap F) = 0 \iff P(E)P(F) = 0 \iff \text{at least one of } P(E), P(F) \text{ is zero}$$

3 (a) Using the general addition rule

$$P(A) = P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) = 0.95 + 0.9 - 0.95 \times 0.9 = 0.995$$

(b) By the partition of B ,

$$\begin{aligned} P(B) &= P(B_1 \cap B_2 \cap B_3^c) + P(B_1 \cap B_2^c \cap B_3) + P(B_1^c \cap B_2 \cap B_3) + P(B_1 \cap B_2 \cap B_3) \\ &= (0.8 \times 0.8 \times 0.2) + (0.8 \times 0.2 \times 0.8) + (0.2 \times 0.8 \times 0.8) + (0.8 \times 0.8 \times 0.8) \\ &= 0.896 \end{aligned}$$

(c) By the general multiplication rule

$$\begin{aligned} P(\text{System Functions}) &= P(A \cap B \cap C \cap D \cap E) = P(A)P(B)P(C)P(D)P(E) \\ &= 0.995 \times 0.896 \times 0.95^3 = 0.764 \end{aligned}$$

4 For the first part: given $P(A) = P(B) = P(C) = 1/3$, and $P(G_{AB}|A) = 1/2$, $P(G_{AB}|B) = 0$ and $P(G_{AB}|C) = 1$, and hence by Bayes theorem using an identical calculation to above, we have $P(A|G_{AB}) = 1/3$ and hence governor is correct.

(ii) For the second part: now $P(G_{WB}|A) = 1/2$, $P(G_{WB}|B) = 0$ but $P(G_{WB}|C) = 1/2$, so by Bayes theorem $P(C|G_{WB}) = 1/2$, and hence C is right to feel happier.

5 Given $P(G) = p$, $P(A|G) = 1$, $P(A|G^c) = \pi$. Then by Bayes Theorem

$$P(G|A) = \frac{P(A|G)P(G)}{P(A|G)P(G) + P(A|G^c)P(G^c)} = \frac{1 \times p}{1 \times p + \pi \times (1 - p)}$$

so

$$\frac{P(G|A)}{P(G^c|A)} = \frac{p}{\pi \times (1 - p)} = \frac{1}{\pi} \frac{P(G)}{P(G^c)}$$

6 Let $T \equiv$ "Test positive", $D \equiv$ "Disease Sufferer". Then $P(T|D) = 0.95$, $P(T|D^c) = 0.10$, $P(D) = 0.005$. Hence

$$(a) \quad P(T) = P(T|D)P(D) + P(T|D^c)P(D^c) = (0.95 \times 0.005) + (0.1 \times 0.995) = 0.10425$$

$$(b) \quad P(D|T) = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)} = \frac{0.95 \times 0.005}{(0.95 \times 0.005) + (0.1 \times 0.995)} = 0.0455$$

$$(c) \quad P(D^c|T^c) = \frac{P(T^c|D^c)P(D^c)}{P(T^c)} = \frac{0.9 \times 0.995}{1 - 0.10425} = 0.9997$$

$$(d) \quad P(M) = P(T \cap D^c) + P(T^c \cap D) = P(T|D^c)P(D^c) + P(T^c|D)P(D) = 0.09975$$

7 Let $T_1 \equiv$ "first test positive", $T_2 \equiv$ "second test positive", $C \equiv$ "drugs present in sample". Then given that

$$P(T_1|C) = P(T_2|C) = 0.995 \quad P(T_1^c|C^c) = P(T_2^c|C^c) = 0.98.$$

(a) By the Theorem of Total Probability

$$P(T_1) = P(T_1|C)P(C) + P(T_1|C^c)P(C^c) = 0.995 \times 0.001 + (1 - 0.98) \times 0.999 = 0.021.$$

(b) By Bayes Theorem

$$P(C|T_1) = \frac{P(T_1|C)P(C)}{P(T_1|C)P(C) + P(T_1|C^c)P(C^c)} = \frac{0.995 \times 0.001}{0.995 \times 0.001 + (1 - 0.98) \times 0.999} = 0.047.$$

(c) By the Theorem of Total Probability and conditional independence

$$\begin{aligned} P(T_1 \cap T_2) &= P(T_1 \cap T_2|C)P(C) + P(T_1 \cap T_2|C^c)P(C^c) \\ &= P(T_1|C)P(T_2|C)P(C) + P(T_1|C^c)P(T_2|C^c)P(C^c) \\ &= 0.995^2 \times 0.001 + (1 - 0.98)^2 \times 0.999 = 0.014. \end{aligned}$$

(d) By Bayes Theorem

$$\begin{aligned} P(C|T_1 \cap T_2) &= \frac{P(T_1 \cap T_2|C)P(C)}{P(T_1 \cap T_2|C)P(C) + P(T_1 \cap T_2|C^c)P(C^c)} \\ &= \frac{P(T_1|C)P(T_2|C)P(C)}{P(T_1|C)P(T_2|C)P(C) + P(T_1|C^c)P(T_2|C^c)P(C^c)} \\ &= \frac{0.995^2 \times 0.001}{0.995^2 \times 0.001 + (1 - 0.98)^2 \times 0.999} = 0.712 \end{aligned}$$