

MATH 323 - EXERCISES 2- SOLUTIONS

1 (a) Partition according to disease status : $(D \cap T^c) \cup (D^c \cap T)$

(b) Partition first into correctly classified sufferers/non-sufferers

- (i) Sufferers : $(D \cap (X \cup T) \cap D)$
- (ii) Non-sufferers : $((D \cap (X \cup T)^c) \cup D^c) \cap D^c$

(i) follows as sufferers must have the correct doctor's diagnosis, and at least one the tests indicating the disease. (ii) follows as non-sufferers can either receive the incorrect doctor's diagnosis (and be declared as sufferers), but be classified correctly as they receive (correct) negative X-ray and test results, or merely receive a (correct) negative test diagnosis from the doctor.

2 Let G ="exactly one occurs". Then $G = (E \cap F^c) \cup (E^c \cap F)$, and by Axiom (III)

$$P(G) = P(E \cap F^c) + P(E^c \cap F).$$

Also,

$$\begin{aligned} E &= (E \cap F) \cup (E \cap F^c) \implies P(E) = P(E \cap F) + P(E \cap F^c) \\ F &= (E \cap F) \cup (E^c \cap F) \implies P(F) = P(E \cap F) + P(E^c \cap F) \end{aligned}$$

so therefore

$$P(G) = P(E) + P(F) - 2P(E \cap F).$$

3 (a) FALSE : (e.g. $E_1 = \emptyset, E_2 = S$).

(b) FALSE : (e.g. coin toss, let $E_1 = E_2 = \{H\}$).

(c) TRUE : $E_1 \subseteq E_2 \implies P(E_1) \leq P(E_2) = 1 - P(E_2^c) = 1 - P(E_1) \implies 2P(E_1) \leq 1$.

(d) TRUE : $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \geq (1 - x_1) + (1 - x_2) - 1 = 1 - x_1 - x_2$.

4 For general events E and F ,

(a) $F \equiv (E \cap F) \cup (E^c \cap F)$, so by Axiom (III)

$$P(F) = P(E \cap F) + P(E^c \cap F) \implies P(E^c \cap F) = P(F) - P(E \cap F)$$

(b) $E \cup F \equiv E \cup (E^c \cap F)$, so by Axiom (III)

$$P(E \cup F) = P(E) + P(E^c \cap F) = P(E) + P(F) - P(E \cap F)$$

(c) $E \subseteq F \implies F \equiv E \cup (E^c \cap F)$, so by Axiom (III), as $P(E^c \cap F) \geq 0$,

$$P(F) = P(E) + P(E^c \cap F) \geq P(E)$$

(d) *Bonferroni Inequality*: From (ii), as $P(E \cup F) \leq 1$,

$$P(E \cap F) = P(E) + P(F) - P(E \cup F) \geq P(E) + P(F) - 1$$

5 (a) $E^c \cup F^c = (E \cap F)^c \implies P(E^c \cup F^c) = 1 - P(E \cap F) = 1 - z.$

(b) $F = (E \cap F) \cup (E^c \cap F)$, so $P(F) = P(E \cap F) + P(E^c \cap F)$, so $P(E^c \cap F) = y - z.$

(c) $E^c \cup F = E^c \cup (E \cap F) \implies P(E^c \cup F) = P(E^c) + P(E \cap F) = 1 - x + z.$

(d) $E^c \cap F^c = (E \cup F)^c \implies P(E^c \cap F^c) = 1 - P(E \cup F) = 1 - x - y + z.$

6 $E \cup E^c = S$, so by Axioms (II) and (III), $P(E) + P(E^c) = P(S) = 1$. Hence $P(E^c) = 1 - P(E)$. But, by Axiom (I)*, $P(E^c) \geq 0$, hence $P(E) \leq 1$.

7 (a) True for $n = 2$ as $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$. So assume true for $n = k$; then

$$P(E_1 \cup \dots \cup E_k \cup E_{k+1}) \leq P(E_1 \cup \dots \cup E_k) + P(E_{k+1}) = \sum_{i=1}^{k+1} P(E_i)$$

and hence true for $n = k + 1$.

(b) As for (a) except

$$P(E_1 \cup \dots \cup E_k \cup E_{k+1}) = P(E_1 \cup \dots \cup E_k) + P(E_{k+1}) - P((E_1 \cup \dots \cup E_k) \cap E_{k+1}). \quad (1)$$

Result follows by substituting required form for $P(E_1 \cup \dots \cup E_k)$, and re-writing

$$P((E_1 \cup \dots \cup E_k) \cap E_{k+1}) = P((E_1 \cap E_{k+1}) \cup \dots \cup (E_k \cap E_{k+1}))$$

(using distributivity) which is the union of k events, and hence can use the inductive hypothesis to re-write this final expression in the required form. Specifically, let $F_i = E_i \cap E_{k+1}$; then we have

$$\begin{aligned} P(F_1 \cup F_2 \cup \dots \cup F_k) &= \sum_i P(F_i) - \sum_i \sum_j P(F_i \cap F_j) + \dots (-1)^{k-1} P(F_1 \cap F_2 \cap \dots \cap F_k) \\ &= \sum_i P(E_i \cap E_{k+1}) - \sum_i \sum_j P(E_i \cap E_j \cap E_{k+1}) + \dots \\ &\quad (-1)^{k-1} P(E_1 \cap E_2 \cap \dots \cap E_k \cap E_{k+1}) \end{aligned}$$

as, for example,

$$F_i \cap F_j = (E_i \cap E_{k+1}) \cap (E_j \cap E_{k+1}) = E_i \cap E_j \cap E_{k+1}$$

Hence, using the inductive hypothesis to express

$$P(E_1 \cup \dots \cup E_k)$$

in (1) and adding in the final term, we complete the proof.