

MATH 323 - EXERCISES 2

ELEMENTARY PROBABILITY

- 1 A person selected from a population and subjected to screening-for-a particular-disease is either a sufferer (D) or not (D^c). The screening will either result in a positive test (T), or a negative (T^c). If the result of the test is used to classify the person, write down the event “person misclassified”.

Suppose now that an X-ray is also taken and either gives a positive (X), or a negative (X^c) indication, and also a doctor carries out an examination resulting in a positive (E), or negative (E^c) assessment. If a person is classified as a sufferer if and only if the doctor and at least one of the tests points in this direction, represent the event “correct classification made”.

- 2 Given two events $E, F \subseteq S$, prove that the probability of *one and only one* of them occurring is

$$P(E) + P(F) - 2P(E \cap F).$$

- 3 Consider the following statements, which are claimed to be true for events E_1, E_2 in a sample space S :

- (a) $P(E_1) = 0 \implies P(E_1 \cup E_2) = 0$
- (b) $P(E_1) = P(E_2^c) \implies E_1^c = E_2$
- (c) $E_1 \subseteq E_2$ and $P(E_1) = P(E_2^c) \implies P(E_1) \leq 1/2$
- (d) $P(E_1^c) = x_1, P(E_2^c) = x_2 \implies P(E_1 \cup E_2) \geq 1 - x_1 - x_2$

In each case, either prove that the statement is true for all S, E_1, E_2 , or provide a specific counter-example to show that there exists S, E_1, E_2 for which it is false

- 4 Given two events $E, F \subseteq S$, prove that

- (a) $P(E^c \cap F) = P(F) - P(E \cap F)$
- (b) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
- (c) $E \subseteq F \implies P(E) \leq P(F)$
- (d) $P(E \cap F) \geq P(E) + P(F) - 1$

(d) is known as *Bonferroni's Inequality*.

- 5 Suppose that E and F are events such that $P(E) = x, P(F) = y$ and $P(E \cap F) = z$. Express the following terms of x, y and z :

- (a) $P(E^c \cup F^c)$
- (b) $P(E^c \cap F)$
- (c) $P(E^c \cup F)$
- (d) $P(E^c \cap F^c)$.

- 6 Show that the first probability axiom can be relaxed to:

$$(I)^* : P(E) \geq 0 \text{ for all } E \subseteq S,$$

that is, given that (I)*, (II) and (III) hold, prove that $P(E) \leq 1$ for all $E \subseteq S$.

7 Use mathematical induction to prove that, for any events $E_1, E_2, \dots, E_n$ in S :

(a) *Boole's Inequality*

$$P(E_1 \cup E_2 \cup \dots \cup E_n) \leq P(E_1) + \dots + P(E_n)$$

(b) *General Addition Rule*

$$\begin{aligned} P(E_1 \cup E_2 \cup \dots \cup E_n) = & \sum_i P(E_i) - \sum_i \sum_j P(E_i \cap E_j) + \sum_i \sum_j \sum_k P(E_i \cap E_j \cap E_k) - \\ & \dots + (-1)^{n-1} P(E_1 \cap E_2 \cap \dots \cap E_n) \end{aligned}$$

In (b), summations are over *distinct* sets of subscripts, each counted once only. For example, if $n = 3$, the result would correspond to

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3).$$