

MATH 323: PROBABILITY

MULTIVARIATE 1-1 TRANSFORMATIONS

Given a collection of variables $(Y_1, \dots, Y_k)$ with range $\mathcal{Y}^{(k)}$ and joint pdf $f_{Y_1, \dots, Y_k}$ we can construct the pdf of a transformed set of variables $(U_1, \dots, U_k)$ using the following steps:

1. Write down the set of transformation functions $h_1, \dots, h_k$

$$\begin{aligned} U_1 &= h_1(Y_1, \dots, Y_k) \\ &\vdots \\ U_k &= h_k(Y_1, \dots, Y_k) \end{aligned} .$$

2. Write down the set of inverse transformation functions $h_1^{-1}, \dots, h_k^{-1}$

$$\begin{aligned} Y_1 &= h_1^{-1}(U_1, \dots, U_k) \\ &\vdots \\ Y_k &= h_k^{-1}(U_1, \dots, U_k) \end{aligned}$$

3. Consider the joint range of the new variables, $\mathcal{U}^{(k)}$.
4. Compute the Jacobian of the transformation: first form the matrix of partial derivatives

$$D_y = \begin{bmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} & \dots & \frac{\partial y_1}{\partial u_k} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} & \dots & \frac{\partial y_2}{\partial u_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_k}{\partial u_1} & \frac{\partial y_k}{\partial u_2} & \dots & \frac{\partial y_k}{\partial u_k} \end{bmatrix}$$

where, for each (i, j)

$$\frac{\partial y_i}{\partial u_j} = \frac{\partial}{\partial u_j} \{h_i^{-1}(u_1, \dots, u_k)\}$$

and then set $|J_y(u_1, \dots, u_k)| = |\det D_y|$

Note that

$$\det D_y = \det D_y^T$$

so that an alternative but equivalent Jacobian calculation can be carried out by forming D_y^T . Note also that

$$|J_y(u_1, \dots, u_k)| = \frac{1}{|J_u(y_1, \dots, y_k)|}$$

where $J_u(y_1, \dots, y_k)$ is the Jacobian of the transformation regarded in the reverse direction (that is, if we start with $(U_1, \dots, U_k)$ and transform to $(Y_1, \dots, Y_k)$)

5. Write down the joint pdf of $(U_1, \dots, U_k)$ as

$$f_{U_1, \dots, U_k}(u_1, \dots, u_k) = f_{Y_1, \dots, Y_k}(h_1^{-1}(u_1, \dots, u_k), \dots, h_k^{-1}(u_1, \dots, u_k)) \times |J_y(u_1, \dots, u_k)|$$

for $(u_1, \dots, u_k) \in \mathcal{U}^{(k)}$

EXAMPLE Suppose that Y_1 and Y_2 have joint pdf

$$f_{Y_1, Y_2}(y_1, y_2) = 2 \quad 0 < y_1 < y_2 < 1$$

and zero otherwise. Compute the joint pdf of random variables

$$U_1 = \frac{Y_1}{Y_2} \quad U_2 = Y_2$$

SOLUTION

1. Given that $\mathcal{Y}^{(2)} \equiv \{(y_1, y_2) : 0 < y_1 < y_2 < 1\}$ and

$$h_1(t_1, t_2) = \frac{t_1}{t_2} \quad h_2(t_1, t_2) = t_2$$

2. Inverse transformations:

$$\left. \begin{array}{l} U_1 = Y_1/Y_2 \\ U_2 = Y_2 \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{l} Y_1 = U_1 U_2 \\ Y_2 = U_2 \end{array} \right.$$

and thus

$$h_1^{-1}(t_1, t_2) = t_1 t_2 \quad h_2^{-1}(t_1, t_2) = t_2$$

3. Range: to find $\mathcal{U}^{(2)}$ consider point by point transformation from $\mathcal{Y}^{(2)}$ to $\mathcal{U}^{(2)}$ For a pair of points $(y_1, y_2) \in \mathcal{Y}^{(2)}$ and $(u_1, u_2) \in \mathcal{U}^{(2)}$ linked via the transformation, we have

$$0 < y_1 < y_2 < 1 \Longleftrightarrow 0 < u_1 u_2 < u_2 < 1$$

and hence we can extract the inequalities

$$0 < u_2 < 1 \text{ and } 0 < u_1 < 1 \quad \therefore \quad \mathcal{U}^{(2)} \equiv (0, 1) \times (0, 1)$$

4. The Jacobian for points $(u_1, u_2) \in \mathcal{U}^{(2)}$ is

$$D_y = \begin{bmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} \end{bmatrix} = \begin{bmatrix} u_2 & u_1 \\ 0 & 1 \end{bmatrix} \Rightarrow |J_y(u_1, u_2)| = |\det D_y| = |u_2| = u_2$$

Note that for points $(y_1, y_2) \in \mathcal{Y}^{(2)}$ the Jacobian for the reverse transformation is

$$D_u = \begin{bmatrix} \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial y_2} \\ \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{y_2} & \frac{y_1}{y_2^2} \\ 0 & 1 \end{bmatrix} \Rightarrow |J_u(y_1, y_2)| = |\det D_u| = \left| \frac{1}{y_2} \right| = \frac{1}{y_2}$$

so that

$$|J_y(u_1, u_2)| = \frac{1}{|J_u(y_1, y_2)|}$$

5. Finally, we have

$$f_{U_1, U_2}(u_1, u_2) = f_{Y_1, Y_2}(u_1 u_2, u_2) \times u_2 = 2u_2 \quad 0 < u_1 < 1, 0 < u_2 < 1$$

and zero otherwise

It is straightforward to check that this is a valid joint pdf.

EXAMPLE Suppose that Y_1 and Y_2 are **independent** and **identically distributed** random variables defined on $\mathbb{R}^+$ each with pdf of the form

$$f_X(x) = \sqrt{\frac{1}{2\pi x}} \exp\left\{-\frac{x}{2}\right\} \quad x > 0$$

and zero otherwise. Compute the joint pdf of random variables $U_1 = Y_1$ and $U_2 = Y_1 + Y_2$

SOLUTION

1. Given that $\mathcal{Y}^{(2)} \equiv \{(y_1, y_2) : 0 < y_1, 0 < y_2\}$ and

$$h_1(t_1, t_2) = t_1 \quad h_2(t_1, t_2) = t_1 + t_2$$

2. Inverse transformations:

$$\left. \begin{array}{l} U_1 = Y_1 \\ U_2 = Y_1 + Y_2 \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{l} Y_1 = U_1 \\ Y_2 = U_2 - U_1 \end{array} \right.$$

and thus

$$h_1^{-1}(t_1, t_2) = t_1 \quad h_2^{-1}(t_1, t_2) = t_2 - t_1$$

3. Range: to find $\mathcal{U}^{(2)}$ consider point by point transformation from $\mathcal{Y}^{(2)}$ to $\mathcal{U}^{(2)}$ For a pair of points $(y_1, y_2) \in \mathcal{Y}^{(2)}$ and $(u_1, u_2) \in \mathcal{U}^{(2)}$ linked via the transformation; as both original variables are strictly positive, we can extract the inequalities

$$0 < u_1 < u_2 < \infty$$

4. The Jacobian for points $(u_1, u_2) \in \mathcal{U}^{(2)}$ is

$$D_y = \begin{bmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \Rightarrow |J_y(u_1, u_2)| = |\det D_y| = |1| = 1$$

Note, here, $J_u(y_1, y_2) = |\det D_u| = 1$ also so that again

$$|J_y(u_1, u_2)| = \frac{1}{|J_u(y_1, y_2)|}$$

5. Finally, we have for $0 < u_1 < u_2 < \infty$

$$\begin{aligned} f_{U_1, U_2}(u_1, u_2) &= f_{Y_1, Y_2}(u_1, u_2 - u_1) \times 1 = f_{Y_1}(u_1) \times f_{Y_2}(u_2 - u_1) \quad \text{by independence} \\ &= \sqrt{\frac{1}{2\pi u_1}} \exp\left\{-\frac{u_1}{2}\right\} \sqrt{\frac{1}{2\pi(u_2 - u_1)}} \exp\left\{-\frac{(u_2 - u_1)}{2}\right\} \\ &= \frac{1}{2\pi} \frac{1}{\sqrt{u_1(u_2 - u_1)}} \exp\left\{-\frac{u_2}{2}\right\} \end{aligned}$$

and zero otherwise

Here, for $u_2 > 0$

$$\begin{aligned}
f_{U_2}(u_2) &= \int_{-\infty}^{\infty} f_{U_1, U_2}(u_1, u_2) du_1 = \int_0^{u_2} \frac{1}{2\pi} \frac{1}{\sqrt{u_1(u_2 - u_1)}} \exp\left\{-\frac{u_2}{2}\right\} du_1 \\
&= \frac{1}{2\pi} \exp\left\{-\frac{u_2}{2}\right\} \int_0^{u_2} \frac{1}{\sqrt{u_1(u_2 - u_1)}} du_1 \\
&= \frac{1}{2\pi} \exp\left\{-\frac{u_2}{2}\right\} \int_0^1 \frac{1}{\sqrt{tu_2(u_2 - tu_2)}} u_2 dt \quad \text{setting } u_1 = tu_2 \\
&= \frac{1}{2\pi} \exp\left\{-\frac{u_2}{2}\right\} \int_0^1 \frac{1}{\sqrt{t(1-t)}} dt \\
&= \frac{1}{2} \exp\left\{-\frac{u_2}{2}\right\}
\end{aligned}$$

as, comparing with the $Beta(1/2, 1/2)$ pdf

$$\int_0^1 \frac{1}{\sqrt{t(1-t)}} dt = \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1/2 + 1/2)} = \frac{\sqrt{\pi}\sqrt{\pi}}{1} = \pi.$$

Recall that we obtain $\Gamma(1/2) = \sqrt{\pi}$ by considering the standard Normal pdf: as

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = 2 \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = 1$$

we substitute $t = y^2$ in the integral to obtain that

$$\int_0^{\infty} \frac{1}{\sqrt{\pi}} t^{-1/2} e^{-t/2} dt = 1.$$

But the integrand is just the $Gamma(1/2, 2)$ pdf, and the normalizing constant must be the reciprocal of $2^{1/2}\Gamma(1/2)$. Hence $\sqrt{\pi} = \Gamma(1/2)$.