

MATH 323: PROBABILITY

CALCULATIONS FOR MULTIVARIATE EXPECTATIONS

We consider a function $g(y_1, \dots, y_n)$ of n random variables. The expectation of this function is defined in the discrete case by

$$\mathbb{E}[g(Y_1, \dots, Y_n)] = \sum_{y_1=-\infty}^{\infty} \cdots \sum_{y_n=-\infty}^{\infty} g(y_1, \dots, y_n) p_{Y_1, \dots, Y_n}(y_1, \dots, y_n).$$

In the continuous case we define

$$\mathbb{E}[g(Y_1, \dots, Y_n)] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(y_1, \dots, y_n) f_{Y_1, \dots, Y_n}(y_1, \dots, y_n) dy_1 \dots dy_n.$$

These definitions hold when the sum/integral is finite. For two variables, this becomes

$$\mathbb{E}[g(Y_1, Y_2)] = \sum_{y_1=-\infty}^{\infty} \sum_{y_2=-\infty}^{\infty} g(y_1, y_2) p_{Y_1, Y_2}(y_1, y_2).$$

or

$$\mathbb{E}[g(Y_1, Y_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(y_1, y_2) f_{Y_1, Y_2}(y_1, y_2) dy_1 dy_2.$$

If we consider $g(Y_1, Y_2) \equiv g(Y_1)$, that is, the function is a function of Y_1 only, we have a simplification. Recall that

$$f_{Y_1, Y_2}(y_1, y_2) = f_{Y_1}(y_1) f_{Y_2|Y_1}(y_2|y_1).$$

Then in the continuous case (the discrete case proceeds identically) that

$$\begin{aligned} \mathbb{E}[g(Y_1)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(y_1) f_{Y_1, Y_2}(y_1, y_2) dy_2 dy_1 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(y_1) f_{Y_1}(y_1) f_{Y_2|Y_1}(y_2|y_1) dy_2 dy_1 \\ &= \int_{-\infty}^{\infty} g(y_1) f_{Y_1}(y_1) \left\{ \int_{-\infty}^{\infty} f_{Y_2|Y_1}(y_2|y_1) dy_2 \right\} dy_1 \end{aligned}$$

computing the double integral in the usual way. However for the inner integral

$$\int_{-\infty}^{\infty} f_{Y_2|Y_1}(y_2|y_1) dy_2 = 1$$

as $f_{Y_2|Y_1}(y_2|y_1)$ is a conditional pdf for all y_1 . Thus

$$\mathbb{E}[g(Y_1)] = \int_{-\infty}^{\infty} g(y_1) f_{Y_1}(y_1) dy_1$$

which is the usual formula for the expectation $\mathbb{E}[g(Y_1)]$. Therefore it does not matter if we compute the expectation with respect to the joint pdf of Y_1 and Y_2 or the marginal pdf for Y_1 – the result is the same.

COVARIANCE

Discrete Example: A bag contains ten balls: five are red, three are yellow, two are white. Four balls are selected, with all such selections being equally likely. Let

- Y_1 denote the number of red balls selected;
- Y_2 denote the number of yellow balls selected.

Then using combinatorial arguments, we see that the joint pmf of Y_1 and Y_2 is given by

$$p_{Y_1, Y_2}(y_1, y_2) = \frac{\binom{5}{y_1} \binom{3}{y_2} \binom{2}{4 - y_1 - y_2}}{\binom{10}{4}}$$

for (y_1, y_2) such that the combinatorial terms are defined, and zero when the terms are not. We need (y_1, y_2) simultaneously to satisfy $0 \leq y_1 \leq 5$, $0 \leq y_2 \leq 3$, and $0 \leq 4 - y_1 - y_2 \leq 2$ in order to have a non-zero probability. We obtain, for the joint pmf table

	Y_2			
	0	1	2	3
Y_1	0	0	$\frac{\binom{5}{0} \binom{3}{2} \binom{2}{2}}{\binom{10}{4}}$	$\frac{\binom{5}{0} \binom{3}{3} \binom{2}{1}}{\binom{10}{4}}$
	1	0	$\frac{\binom{5}{1} \binom{3}{1} \binom{2}{2}}{\binom{10}{4}}$	$\frac{\binom{5}{1} \binom{3}{3} \binom{2}{0}}{\binom{10}{4}}$
	2	$\frac{\binom{5}{2} \binom{3}{0} \binom{2}{2}}{\binom{10}{4}}$	$\frac{\binom{5}{2} \binom{3}{1} \binom{2}{1}}{\binom{10}{4}}$	0
	3	$\frac{\binom{5}{3} \binom{3}{0} \binom{2}{1}}{\binom{10}{4}}$	0	0
	4	$\frac{\binom{5}{4} \binom{3}{0} \binom{2}{0}}{\binom{10}{4}}$	0	0
5	0	0	0	0

that is, numerically,

	Y_2			
	0	1	2	3
Y_1	0	0.0000	0.0143	0.0095
	1	0.0000	0.0714	0.0238
	2	0.0476	0.2857	0.0000
	3	0.0952	0.1429	0.0000
	4	0.0238	0.0000	0.0000
5	0.0000	0.0000	0.0000	0.0000

It is clear that Y_1 and Y_2 are **not independent** as the *support* of the joint pmf (where $p_{Y_1, Y_2}(y_1, y_2) > 0$) is not the Cartesian product of the sets $\{0, 1, 2, 3, 4\}$ and $\{0, 1, 2, 3\}$ which are the supports of $p_{Y_1}(y_1)$ and $p_{Y_2}(y_2)$ respectively.

It is not too difficult to compute the means, variances and covariances analytically in this case, however, we can more easily compute these quantities numerically.

Marginal for Y_1 : sum across the rows of the table to obtain

y_1	0	1	2	3	4	5
$p_{Y_1}(y_1)$	0.0238	0.2381	0.4762	0.2381	0.0238	0.0000

so therefore

$$\mathbb{E}[Y_1] = \sum_{y_1=0}^5 y_1 p_{Y_1}(y_1) = 2 \quad \mathbb{E}[Y_1^2] = \sum_{y_1=0}^5 y_1^2 p_{Y_1}(y_1) = 4.6666 \quad \mathbb{V}[Y_1] = 0.6666$$

Marginal for Y_2 : sum down the columns of the table to obtain

y_2	0	1	2	3
$p_{Y_2}(y_2)$	0.1667	0.5000	0.3000	0.0333

so therefore

$$\mathbb{E}[Y_2] = \sum_{y_2=0}^3 y_2 p_{Y_2}(y_2) = 1.1999 \quad \mathbb{E}[Y_2^2] = \sum_{y_2=0}^3 y_2^2 p_{Y_2}(y_2) = 1.9997 \quad \mathbb{V}[Y_2] = 0.55994$$

Covariance: For the covariance

$$\mathbb{E}[Y_1 Y_2] = \sum_{y_1=0}^5 \sum_{y_2=0}^3 y_1 y_2 p_{Y_1, Y_2}(y_1, y_2) = 2$$

so therefore

$$\text{Cov}[Y_1, Y_2] = \mathbb{E}[Y_1 Y_2] - \mathbb{E}[Y_1] \mathbb{E}[Y_2] = 2 - 2 \times 1.1999 = -0.3998.$$

Thus the covariance is **negative**: this is the case as if Y_1 is large, then Y_2 must be small, and vice versa.

Continuous Example: Let Y_1 and Y_2 be continuous random variables with joint pdf

$$f_{Y_1, Y_2}(y_1, y_2) = c(y_1 + y_2 + y_1 y_2) \quad 0 < y_1 < 1, 0 < y_2 < 1$$

and zero otherwise, for some constant $c > 0$.

Note that Y_1 and Y_2 are **not independent**, as $f_{Y_1, Y_2}(y_1, y_2)$ cannot be factorized $f_{Y_1}(y_1)f_{Y_2}(y_2)$. The marginal pdf for Y_1 is computed by integrating out y_2 from the joint pdf:

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1, Y_2}(y_1, y_2) dy_2 = \int_0^1 c(y_1 + y_2 + y_1 y_2) dy_2$$

which yields that

$$f_{Y_1}(y_1) = c \left[y_1 y_2 + \frac{y_2^2}{2} + \frac{y_1 y_2^2}{2} \right]_0^1 = c \frac{(3y_1 + 1)}{2} \quad 0 < y_1 < 1.$$

The corresponding cdf for the range $(0, 1)$ is

$$F_{Y_1}(y_1) = \int_0^{y_1} f_{Y_1}(t) dt = c \frac{3y_1^2 + 2y_1}{4}$$

as $F_{Y_1}(1)$ must be 1, we can deduce that $c = 4/5$. Therefore

$$F_{Y_1}(y_1) = \frac{(3y_1^2 + 2y_1)}{5} \quad 0 < y_1 < 1.$$

The marginal pdf for Y_2 is identical in form, due to the symmetry of form of the joint pdf. Finally, the conditional pmf of Y_2 given that $Y_1 = y_1$ for $0 < y_1 < 1$ is

$$f_{Y_2|Y_1}(y_2|y_1) = \frac{f_{Y_1,Y_2}(y_1, y_2)}{f_{Y_1}(y_1)} = \frac{2(y_1 + y_2 + y_1 y_2)}{(3y_1 + 1)} \quad 0 < y_2 < 1.$$

Now consider the calculation of $\mathbb{E}[Y_1^r Y_2^s]$ for r and s non-negative integers.

$$\begin{aligned} \mathbb{E}[Y_1^r Y_2^s] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_1^r y_2^s f_{Y_1,Y_2}(y_1, y_2) dy_2 dy_1 = \int_0^1 \int_0^1 y_1^r y_2^s c(y_1 + y_2 + y_1 y_2) dy_2 dy_1 \\ &= c \int_0^1 \int_0^1 (y_1^{r+1} y_2^s + y_1^r y_2^{s+1} + y_1^{r+1} y_2^{s+1}) dy_2 dy_1 \\ &= c \left(\frac{1}{(r+2)(s+1)} + \frac{1}{(r+1)(s+2)} + \frac{1}{(r+2)(s+2)} \right) \end{aligned}$$

- When $r = s = 0$, we must get 1 as the integral is then merely the integral of the joint pdf. Thus

$$1 = c \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{4} \right) = c \frac{5}{4}$$

so that $c = 4/5$ as before.

- When $r = 1, s = 0$, we have that

$$\mathbb{E}[Y_1] = \frac{4}{5} \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{6} \right) = \frac{3}{5}$$

and we get the same result when $r = 0, s = 1$, so $\mathbb{E}[Y_2] = 3/5$.

- Finally, when $r = s = 1$, we have that

$$\mathbb{E}[Y_1 Y_2] = \frac{4}{5} \left(\frac{1}{6} + \frac{1}{6} + \frac{1}{9} \right) = \frac{48}{135}.$$

Therefore

$$\text{Cov}[Y_1, Y_2] = \frac{48}{135} - \frac{3}{5} \frac{3}{5} = -\frac{15}{3375} = -\frac{1}{225} = -0.0044.$$

For the variances, we need to consider $r = 2, s = 0$, so that

$$\mathbb{E}[Y_1^2] = \frac{4}{5} \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{8} \right) = \frac{104}{240} = 0.4333$$

so

$$\mathbb{V}[Y_1] = \mathbb{E}[Y_1^2] - \{\mathbb{E}[Y_1]\}^2 = \frac{104}{240} - \left(\frac{3}{5} \right)^2 = 0.0733.$$

The variance of Y_2 is identical. Therefore, for the correlation,

$$\text{Corr}[Y_1, Y_2] = \frac{\text{Cov}[Y_1, Y_2]}{\sqrt{\mathbb{V}[Y_1] \mathbb{V}[Y_2]}} = -\frac{0.0044}{0.0733} = -0.0600.$$

Uncorrelated but not independent: Let Y_1 and Y_2 be continuous random variables with joint pdf

$$f_{Y_1, Y_2}(y_1, y_2) = \frac{1}{\pi} \quad (y_1, y_2) \in \mathcal{C}$$

and zero otherwise, where $\mathcal{C}$ is the disc of radius 1 centered at the origin. The marginal pdf for Y_1 is given for $-1 \leq y_1 \leq 1$ by

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1, Y_2}(y_1, y_2) dy_2 = \int_{-\sqrt{1-y_1^2}}^{\sqrt{1-y_1^2}} \frac{1}{\pi} dy_2 = \frac{2}{\pi} \sqrt{1-y_1^2}$$

and zero otherwise – the range of integration is determined as the line segment internal to $\mathcal{C}$ at y_1 . From this result, we can determine that

$$\mathbb{E}[Y_1] = \int_{-\infty}^{\infty} y_1 f_{Y_1}(y_1) dy_1 = \int_{-1}^1 y_1 \frac{2}{\pi} \sqrt{1-y_1^2} dy_1 = \left[\frac{4}{3\pi} (1-y_1^2)^{3/2} \right]_{-1}^1 = 0$$

– note that the integrand is an odd function around zero. By symmetry of form, we note that

$$f_{Y_2}(y_2) = \frac{2}{\pi} \sqrt{1-y_2^2} \quad -1 \leq y_2 \leq 1$$

and zero otherwise. Note also for the covariance

$$\begin{aligned} \mathbb{E}[Y_1 Y_2] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{Y_1, Y_2}(y_1, y_2) dy_2 dy_1 = \int_{-1}^1 \left\{ \int_{-\sqrt{1-y_1^2}}^{\sqrt{1-y_1^2}} y_1 y_2 dy_2 \right\} dy_1 \\ &= \int_{-1}^1 y_1 \left[\frac{y_2^2}{2} \right]_{-\sqrt{1-y_1^2}}^{\sqrt{1-y_1^2}} dy_1 \\ &= 0 \end{aligned}$$

Therefore the covariance is zero also, and Y_1 and Y_2 are **uncorrelated**. However, it is clear that

$$f_{Y_1, Y_2}(y_1, y_2) \neq f_{Y_1}(y_1) f_{Y_2}(y_2)$$

in general, so the two variables are **not independent**.