

MATH 323: PROBABILITY

CONVOLUTION CALCULATIONS

The convolution formula can be used to compute the distribution of a random variable Y that is the sum of two independent random variables Y_1 and Y_2 .

- In the **discrete** case, we have that

$$p_Y(y) = \sum_{y_1=-\infty}^{\infty} p_{Y_1}(y_1)p_{Y_2}(y-y_1) = \sum_{y_1=-\infty}^{\infty} P(Y_1=y_1)P(Y_2=y-y_1) \quad y \in \mathbb{Z}$$

where $p_{Y_1}(\cdot)$ and $p_{Y_2}(\cdot)$ are the pmfs of Y_1 and Y_2 , and the range of the summation may be restricted if either of the mass functions is computed to be zero for a particular argument.

EXAMPLE: If $Y_1 \sim \text{Binomial}(n_1, p)$ and $Y_2 \sim \text{Binomial}(n_2, p)$, then $Y = Y_1 + Y_2$ can take values on the set $\mathcal{Y} = \{0, 1, 2, \dots, n_1 + n_2\}$, and for y in this set

$$\begin{aligned} f_Y(y) &= \sum_{y_1=-\infty}^{\infty} f_{Y_1}(y_1)f_{Y_2}(y-y_1) \\ &= \sum_{y_1=0}^{\min\{n_1, y\}} \binom{n_1}{y_1} p^{y_1} (1-p)^{n_1-y_1} \binom{n_2}{y-y_1} p^{y-y_1} (1-p)^{n_2-y+y_1} \\ &= p^y (1-p)^{n_1+n_2-y} \sum_{y_1=0}^{\min\{n_1, y\}} \binom{n_1}{y_1} \binom{n_2}{y-y_1} \\ &= \binom{n_1+n_2}{y} p^y (1-p)^{n_1+n_2-y} \end{aligned}$$

by an earlier combinatorial identity. Therefore $Y \sim \text{Binomial}(n_1 + n_2, p)$.

EXAMPLE: If $Y_1 \sim \text{Geometric}(p)$ and $Y_2 \sim \text{Geometric}(p)$, then $Y = Y_1 + Y_2$ can take values on the set $\mathcal{Y} = \{2, 3, \dots\}$, and for y in this set

$$\begin{aligned} f_Y(y) &= \sum_{y_1=-\infty}^{\infty} f_{Y_1}(y_1)f_{Y_2}(y-y_1) \\ &= \sum_{y_1=1}^{y-1} (1-p)^{y_1-1} p (1-p)^{y-y_1-1} p \\ &= p^2 (1-p)^{y-2} \sum_{y_1=1}^{y-1} 1 \\ &= (y-1)p^2 (1-p)^{y-2} = \binom{y-1}{r-1} p^r (1-p)^{y-r} \end{aligned}$$

with $r = 2$. Therefore $Y \sim \text{NegBinomial}(2, p)$.

- In the **continuous** case, we have that

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y_1}(y_1) f_{Y_2}(y - y_1) dy_1 \quad y \in \mathbb{R}$$

where $f_{Y_1}(\cdot)$ and $f_{Y_2}(\cdot)$ are the pdfs of Y_1 and Y_2 , and the range of integration may be restricted if either of the pdfs is computed to be zero for a particular argument.

EXAMPLE: If $Y_1 \sim \text{Gamma}(\alpha_1, \beta)$ and $Y_2 \sim \text{Gamma}(\alpha_2, \beta)$, then $Y = Y_1 + Y_2$ can take values on $\mathbb{R}^+$, and for $y > 0$

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{Y_1}(y_1) f_{Y_2}(y - y_1) dy_1 \\ &= \int_0^y \frac{1}{\beta^{\alpha_1} \Gamma(\alpha_1)} y_1^{\alpha_1-1} e^{-y_1/\beta} \frac{1}{\beta^{\alpha_2} \Gamma(\alpha_2)} (y - y_1)^{\alpha_2-1} e^{-(y-y_1)/\beta} dy_1 \\ &= \frac{1}{\beta^{\alpha_1} \Gamma(\alpha_1)} \frac{1}{\beta^{\alpha_2} \Gamma(\alpha_2)} e^{-y/\beta} \int_0^y y_1^{\alpha_1-1} (y - y_1)^{\alpha_2-1} dy_1 \\ &= \frac{1}{\beta^{\alpha_1+\alpha_2} \Gamma(\alpha_1) \Gamma(\alpha_2)} y^{\alpha_1+\alpha_2-1} e^{-y/\beta} \int_0^1 t^{\alpha_1-1} (1-t)^{\alpha_2-1} dt \quad (t = y_1/y) \\ &= \frac{1}{\beta^{\alpha_1+\alpha_2} \Gamma(\alpha_1) \Gamma(\alpha_2)} y^{\alpha_1+\alpha_2-1} e^{-y/\beta} \frac{\Gamma(\alpha_1) \Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)} = \frac{1}{\beta^{\alpha_1+\alpha_2} \Gamma(\alpha_1 + \alpha_2)} y^{\alpha_1+\alpha_2-1} e^{-y/\beta} \end{aligned}$$

Therefore $Y \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$.

EXAMPLE: If $Y_1 \sim \text{Normal}(\mu_1, \sigma_1^2)$ and $Y_2 \sim \text{Normal}(\mu_2, \sigma_2^2)$, then $Y = Y_1 + Y_2$ can take values on $\mathbb{R}$, and

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{Y_1}(y_1) f_{Y_2}(y - y_1) dy_1 \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left\{-\frac{(y_1 - \mu_1)^2}{2\sigma_1^2}\right\} \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left\{-\frac{(y - y_1 - \mu_2)^2}{2\sigma_2^2}\right\} dy_1 \\ &= \frac{1}{\sqrt{2\pi\sigma_1^2}} \frac{1}{\sqrt{2\pi\sigma_2^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(y_1 - \mu_1)^2}{2\sigma_1^2}\right\} \exp\left\{-\frac{(y - y_1 - \mu_2)^2}{2\sigma_2^2}\right\} dy_1 \\ &= \frac{1}{\sqrt{2\pi\sigma_1^2}} \frac{1}{\sqrt{2\pi\sigma_2^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2} \left[\frac{1}{\sigma_1^2} (y_1 - \mu_1)^2 + \frac{1}{\sigma_2^2} (y - y_1 - \mu_2)^2 \right] \right\} dy_1 \end{aligned}$$

The term in square brackets can be rewritten using the ‘completing the square’ identity

$$A(y - a)^2 + B(y - b)^2 = (A + B)(y - \mu)^2 + \frac{AB}{A + B}(a - b)^2$$

where $\mu = (Aa + Bb)/(A + B)$: we have that the term in square brackets is

$$\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right) (y_1 - \mu)^2 + \frac{1}{\sigma_1^2 + \sigma_2^2} (y - \mu_1 - \mu_2)^2$$

where $\mu = (\sigma_2^2\mu_1 + \sigma_1^2\mu_2)/(\sigma_1^2 + \sigma_2^2)$. Thus

$$\begin{aligned}
f_Y(y) &= \frac{1}{\sqrt{2\pi\sigma_1^2}} \frac{1}{\sqrt{2\pi\sigma_2^2}} \int_{-\infty}^{\infty} \exp \left\{ -\frac{1}{2} \left[\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right) (y_1 - \mu)^2 + \frac{1}{\sigma_1^2 + \sigma_2^2} (y - \mu_1 - \mu_2)^2 \right] \right\} dy_1 \\
&= \frac{1}{\sqrt{2\pi\sigma_1^2}} \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp \left\{ -\frac{1}{2} \frac{1}{\sigma_1^2 + \sigma_2^2} (y - \mu_1 - \mu_2)^2 \right\} \int_{-\infty}^{\infty} \exp \left\{ -\frac{1}{2} \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right) (y_1 - \mu)^2 \right\} dy_1 \\
&= \frac{1}{\sqrt{2\pi\sigma_1^2}} \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp \left\{ -\frac{1}{2} \frac{1}{\sigma_1^2 + \sigma_2^2} (y - \mu_1 - \mu_2)^2 \right\} \sqrt{2\pi} \sqrt{\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}}
\end{aligned}$$

as, in the integral, the integrand is proportional to a Normal pdf with parameters μ and

$$\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right) = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2\sigma_2^2}.$$

Cancelling terms, we have that

$$f_Y(y) = \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} \exp \left\{ -\frac{1}{2} \frac{1}{\sigma_1^2 + \sigma_2^2} (y - \mu_1 - \mu_2)^2 \right\}$$

Therefore $Y \sim \text{Normal}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.