

MATH 323 - EXERCISES 7- SOLUTIONS

- 1 (a) Density function must integrate to 1 over $\mathcal{Y} = [0, 1]$, so

$$\int_0^1 f(y) dy = 1 \implies \int_0^1 cy^2 + y dy = 1 \implies \left[c \frac{y^3}{3} + \frac{y^2}{2} \right]_0^1 = 1 \implies c = \frac{3}{2}$$

- (b) Distribution function F given for $0 \leq y \leq 1$ by

$$F(y) = \int_0^y f(t) dt = \frac{y^3 + y^2}{2}$$

and $F(y) = 0$ for $y < 0$, and $F(y) = 1$ for $y > 1$.

- (c) $P(Y < 1/2) = F(1/2) = 3/16$.

- (d) From definition of conditional probability

$$P(Y > 1/2 | Y > 1/4) = \frac{P(Y > 1/2, Y > 1/4)}{P(Y > 1/4)} = \frac{P(Y > 1/2)}{P(Y > 1/4)} = \frac{1 - F(1/2)}{1 - F(1/4)} = \frac{104}{123}$$

- (e) If S is the sum random variable, then $S \sim \text{Binomial}(200, p)$ where

$$p = P(Y_i = 1) = P(Y < 1/6) = F(1/6) = \frac{7}{432}$$

Then by direct calculation

$$P(S \leq 3) = P(S = 0) + P(S = 1) + P(S = 2) + P(S = 3) = 0.593$$

- 2 Density function must integrate to 1 over $\mathcal{Y} = [-1, 1]$, so

$$\int_{-1}^1 f(y) dy = 1 \implies c = \frac{3}{4}$$

Distribution function F given for $-1 \leq y \leq 1$ by

$$F(y) = \int_{-1}^y f(t) dt = \left[\frac{3t}{4} - \frac{t^3}{4} \right]_{-1}^y = \frac{3}{4} \left(y - \frac{y^3}{3} \right) + \frac{2}{3}$$

and $F(y) = 0$ for $y < -1$, and $F(y) = 1$ for $y > 1$.

- 3 (a) Density function must integrate to 1 over $\mathcal{Y} = [0, \pi/2]$, so

$$\int_0^{\pi/2} f(y) dy = 1 \implies \int_0^{\pi/2} cy(\pi - y) dy = 1 \implies c = \frac{12}{\pi^3}$$

- (b) The area is $X = \frac{1}{2} \sin Y$, so $\mathcal{Y} = [0, \pi/2] \implies \mathcal{X} = [0, 1/2]$, and distribution function is F_X given by

$$F_X(x) = P(X \leq x) = P\left(\frac{1}{2} \sin Y \leq x\right) = P(Y \leq \sin^{-1}(2x)) = F(\sin^{-1}(2x))$$

as $\sin$ is monotone on $[0, \pi/2]$. Hence by differentiation with respect to x ,

$$f_X(x) = \frac{2}{\sqrt{1-4x^2}} f(\sin^{-1}(2x)) = \quad 0 \leq x \leq 1/2$$

and zero otherwise.

- 4 By differentiation, $f(y) = 2ye^{-y^2}$, $y > 0$, and zero otherwise

- 5 Need to consider ranges of integration carefully;

$$F(y) = \begin{cases} \int_0^y t \, dt & = y^2/2 & 0 \leq y \leq 1 \\ \int_0^1 t \, dt + \int_1^y (2-t) \, dt & = 2y - y^2/2 - 1 & 1 \leq y \leq 2 \end{cases}$$

and $F(y) = 0$ for $y < 0$, and $F(y) = 1$ for $y > 2$. Hence $P(0.8 < Y < 1.2) = F(1.2) - F(0.8) = 0.36$.

- 6 (a) If $\Phi(\cdot)$ denotes the standard Normal cdf, then

$$P(T > 3 | A) = 1 - P(T \leq 3 | A) = 1 - \Phi((3-2)/(3/4)) = 1 - \Phi(4/3) = 1 - 0.90878 = 0.0912.$$

- (b) By the Theorem of Total Probability

$$P(T > 3) = P(T > 3 | A)P(A) + P(T > 3 | B)P(B) = (1 - \Phi(4/3)) \times \frac{3}{10} + (1 - \Phi(-4/3)) \times \frac{7}{10}$$

so $P(T > 3) = 0.6635$.

- (c) By Bayes Theorem

$$P(A | T > 3) = \frac{P(T > 3 | A)P(A)}{P(T > 3)} = \frac{0.0912 \times 0.3}{0.6635} = 0.0412.$$

- 7 (a) $Y \sim \text{Exponential}(\beta) \implies F(y) = 1 - e^{-y/\beta}$ for $y > 0$, so $F(y) = 1/2 \implies y = \beta \ln 2$.

- (b) $\ln Y \sim \text{Normal}(\mu, \sigma^2)$, so

$$F(y) = 1/2 \implies P(Y \leq y) = 1/2 \implies P((\ln Y - \mu)/\sigma \leq (\ln y - \mu)/\sigma) = 1/2$$

$$\implies \Phi((\ln y - \mu)/\sigma) = 1/2 \implies (\ln y - \mu)/\sigma = 0 \implies \ln y = \mu \implies y = e^\mu$$