

MATH 323 - EXERCISES 3 – SOLUTIONS

- 1 (a) Special case of the Binomial Expansion of $(a + b)^n$ with $a = 1, b = -1$.
 (b) $(x + 1)^{m+n} = (x + 1)^m(x + 1)^n$, and

$$L.H.S. : (x + 1)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} x^k$$

$$\begin{aligned} R.H.S. : (x + 1)^m(x + 1)^n &= \left\{ \sum_{i=0}^m \binom{m}{i} x^i \right\} \left\{ \sum_{j=0}^n \binom{n}{j} x^j \right\} \\ &= \sum_{i=0}^m \sum_{j=0}^n \binom{m}{i} \binom{n}{j} x^{i+j} = \sum_{k=0}^{m+n} \left\{ \sum_{i=0}^k \binom{m}{i} \binom{n}{k-i} \right\} x^k \end{aligned}$$

putting $i + j = k$, and re-arranging summations. Result follows by equating left and right hand sides and comparing coefficients of x_k .

- 2 $N = 49, R = 6$ (winning numbers), and $n = 6$ (drawn numbers). To select precisely r winning numbers (for $r = 0, 1, 2, 3, 4, 5, 6$), we would need to first pick r from R , but then also pick $n - r$ non winning numbers from the $N - R$ available. The number of ways of completing these selections is

$$\binom{R}{r} \times \binom{N-R}{n-r}$$

whereas there are

$$\binom{N}{n}$$

ways of drawing the numbers in total.

- (a) $r = 6$

$$\text{Probability} = \frac{\binom{6}{6} \binom{43}{0}}{\binom{49}{6}} = \frac{1}{13983816} \approx 7.15 \times 10^{-8}$$

- (b) $r = 3$

$$\text{Probability} = \frac{\binom{6}{3} \binom{43}{3}}{\binom{49}{6}} \approx 0.0177$$

- (c) $r = 0$

$$\text{P(No winning numbers)} = \frac{\binom{n}{r} \binom{N-n}{R-r}}{\binom{N}{R}} = \frac{\binom{43}{6}}{\binom{49}{6}}$$

- (d)

$$\text{P(No winning numbers for } x \text{ weeks)} = \left\{ \frac{\binom{43}{6}}{\binom{49}{6}} \right\}^x \quad \text{by independence}$$

- 3 Similar 'hypergeometric probabilities' with $N = 80$, $R = 20$, $n = 5, 10, 15$, $r = 0$, for which probabilities are

$$(a) \frac{\binom{5}{0} \binom{75}{20}}{\binom{80}{20}} = \frac{\binom{20}{0} \binom{60}{5}}{\binom{80}{5}} \quad (b) \frac{\binom{10}{0} \binom{70}{20}}{\binom{80}{20}} = \frac{\binom{20}{0} \binom{60}{10}}{\binom{80}{10}} \quad (c) \frac{\binom{15}{0} \binom{65}{20}}{\binom{80}{20}} = \frac{\binom{20}{0} \binom{60}{15}}{\binom{80}{15}}$$

General formula is just the hypergeometric formula.

- 4 This is a 'hypergeometric' problem with $N = 200$, $R = 120$, $n = 5$ and $r = 5$, thus the probability required is

$$\frac{\binom{5}{5} \binom{200-5}{120-5}}{\binom{200}{120}} = \frac{120 \times 119 \times 118 \times 117 \times 116}{200 \times 199 \times 198 \times 197 \times 196} = 0.075$$

which may also be calculated using a conditional probability argument.

Thus the probability that an all-Science committee is selected, if all selections of five from two hundred are equally likely, is 0.075. This is small, but conventionally we consider a probability of 0.05 as providing sufficient evidence against a hypothesized model, so perhaps this is not conclusive evidence of bias.

- 5 (a) Choose 13 cards from 52 $\Rightarrow n = \binom{52}{13}$

(b) There are 36 cards that are not Aces, Kings, Queens or Jacks:

$$\text{Choose 13 cards from 36} \Rightarrow n_A = \binom{36}{13} \Rightarrow P(A) = \frac{n_A}{\binom{52}{13}} \approx 0.003$$

- 6 Total number of allocations of birthdays to months is 12^{30} . For n_A , select 6 months from 12 as the "two-birthday" months, and then *partition* the 30 birthdays into 12 subgroups, six which contain 2 birthdays, and six which contain three. Hence, using the multinomial formula for partitions,

$$n_A = \binom{12}{6} \frac{30!}{(2!)^6 (3!)^6} \Rightarrow P(A) = \frac{\binom{12}{6} 30!}{12^{30} (2!)^6 (3!)^6}$$

- 7 Hypergeometric formula again: $2n$ Arts (TYPE I) and $2n$ Science (TYPE II), sample of size $2n$ without replacement from population of size $4n$. Hence probability of even split in each group is

$$\frac{\binom{2n}{n} \binom{2n}{n}}{\binom{4n}{2n}}$$

- 8 Multistage Hypergeometric: select the first hand with n then the second hand with m all without replacement. Hence by multiplication theorem, probability is

$$\frac{\binom{13}{n} \binom{39}{13-n}}{\binom{52}{13}} \times \frac{\binom{13-n}{m} \binom{26+n}{13-m}}{\binom{39}{13}}$$

By extension, need to select third hand in similar fashion, but once first three hands are selected, final hand is fixed. Hence probability is

$$\frac{\binom{13}{r_1} \binom{39}{13-r_1}}{\binom{52}{13}} \times \frac{\binom{13-r_1}{r_2} \binom{26+r_1}{13-r_2}}{\binom{39}{13}} \times \frac{\binom{13-r_1-r_2}{r_3} \binom{13+r_1+r_2}{13-r_3}}{\binom{26}{13}}$$

- 9 Total number of seating arrangements is $n!$. Of these, you and friend sit together in

$$(n-1) \times 2! \times (n-2)!$$

arrangements (choose a pair of adjacent seats for you and friend from $n-1$ pairs available, arrange yourselves in one of $2!$ ways, and then arrange remaining $n-2$ people in remaining seats). Hence probability is

$$\frac{(n-1) \times 2! \times (n-2)!}{n!} = \frac{2}{n}$$

- 10 Total number of possible allocations is $n \times n \times \dots \times n = n^n$. For n_A , consider selecting two different boxes from n to be the empty box and the box containing two balls; after this selection has been made, can arrange the balls in the boxes in $n!$ ways; hence

$$n_A = \binom{n}{2} \times n!$$

and

$$P(A) = \frac{\binom{n}{2} \times n!}{n^n}$$

- 11 Total number of arrangements of $2n$ objects is $(2n)!$. For n_A , there are n pairs of *matched* short and long parts, which can be arranged in $n!$ ways in the sequence; each pair can be arranged in 2 ways. Hence $n_A = 2^n n!$, and hence

$$P(A) = \frac{2^n n!}{(2n)!}$$

(ii) Here, for n_A , consider arranging a sequence of n short parts and a parallel sequence of n long parts to match with them; there are $n!$ ways of arranging the short parts and $n!$ ways of arranging the long parts. Once the parallel sequences have been arranged individually and paired, each pair can be arranged in 2 ways (long/short or short/long). Hence $n_A = 2^n n! n!$, and

$$P(A) = \frac{2^n n! n!}{(2n)!} = \frac{2^n}{\binom{2n}{n}}$$