

MATH 323 - EXERCISES 2 – SOLUTIONS

1. (a) Partition according to disease status : $(D \cap T') \cup (D' \cap T)$

(b) Partition first into correctly classified sufferers/non-sufferers

- (i) Sufferers : $(D \cap (X \cup T) \cap D)$
- (ii) Non-sufferers : $((D \cap (X \cup T)') \cup D') \cap D'$

(i) follows as sufferers must have the correct doctor's diagnosis, and at least one the tests indicating the disease. (ii) follows as non-sufferers can either receive the incorrect doctor's diagnosis (and be declared as sufferers), but be classified correctly as they receive (correct) negative X-ray and test results, or merely receive a (correct) negative test diagnosis from the doctor.

2. Let $C = \text{"exactly one occurs"}$. Then $C = (A \cap B') \cup (A' \cap B)$, and by Axiom (III)

$$P(C) = P(A \cap B') + P(A' \cap B).$$

Also,

$$A = (A \cap B) \cup (A \cap B') \implies P(A) = P(A \cap B) + P(A \cap B')$$

$$B = (A \cap B) \cup (A' \cap B) \implies P(B) = P(A \cap B) + P(A' \cap B)$$

so therefore

$$P(C) = P(A) + P(B) - 2P(A \cap B).$$

3. (a) FALSE : (e.g. $A_1 = \emptyset$, $A_2 = S$).

(b) FALSE : (e.g. coin toss, let $A_1 = A_2 = \{H\}$).

(c) TRUE : $A_1 \subseteq A_2 \implies P(A_1) \leq P(A_2) = 1 - P(A_2') = 1 - P(A_1) \implies 2P(A_1) \leq 1$.

(d) TRUE : $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \geq (1 - x_1) + (1 - x_2) - 1 = 1 - x_1 - x_2$.

4. For general events A and B ,

(a) $B \equiv (A \cap B) \cup (A' \cap B)$, so by Axiom (III)

$$P(B) = P(A \cap B) + P(A' \cap B) \implies P(A' \cap B) = P(B) - P(A \cap B)$$

(b) $A \cup B \equiv A \cup (A' \cap B)$, so by Axiom (III)

$$P(A \cup B) = P(A) + P(A' \cap B) = P(A) + P(B) - P(A \cap B)$$

(c) $A \subseteq B \implies B \equiv A \cup (A' \cap B)$, so by Axiom (III), as $P(A' \cap B) \geq 0$,

$$P(B) = P(A) + P(A' \cap B) \geq P(A)$$

(d) *Bonferroni Inequality*: From (ii), as $P(A \cup B) \leq 1$,

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1$$

5. (a) $A' \cup B' = (A \cap B)' \implies P(A' \cup B') = 1 - P(A \cap B) = 1 - z$.
- (b) $B = (A \cap B) \cup (A' \cap B)$, so $P(B) = P(A \cap B) + P(A' \cap B)$, so $P(A' \cap B) = y - z$.
- (c) $A' \cup B = A' \cup (A \cap B) \implies P(A' \cup B) = P(A') + P(A \cap B) = 1 - x + z$.
- (d) $A' \cap B' = (A \cup B)' \implies P(A' \cap B') = 1 - P(A \cup B) = 1 - x - y + z$.
6. This question was poorly phrased: the intention was to show that the two sets of three Axioms carried identical assumptions merely stated differently; of course if $0 \leq P(A) \leq 1$ then $P(A) \geq 0$. Apologies for the poor wording. The purpose of the exercise is really to show that the statement in Axiom (I)* that $P(A) \leq 1$ is redundant given the other axioms.
- The condition that $P(A) \leq 1$ is redundant in the new Axiom as A is arbitrary; if $P(A) \geq 0$ for arbitrary A , then we must have that $P(A') \geq 0$ also, as A' is an event in its own right. But $A \cup A' = S$, so by Axiom (III) $P(A) + P(A') = 1$. Thus $P(A) \leq 1$ automatically.
7. (a) True for $n = 2$ as $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$. So assume true for $n = k$; then by the inductive hypothesis

$$P(A_1 \cup \dots \cup A_k \cup A_{k+1}) \leq P(A_1 \cup \dots \cup A_k) + P(A_{k+1}) = \sum_{i=1}^{k+1} P(A_i)$$

where we have applied the result for $n = 2$ to the two events

$$(A_1 \cup \dots \cup A_k) \quad \text{and} \quad A_{k+1}.$$

Hence the result is true for $n = k + 1$.

(b) As for (a) except

$$P(A_1 \cup \dots \cup A_k \cup A_{k+1}) = P(A_1 \cup \dots \cup A_k) + P(A_{k+1}) - P((A_1 \cup \dots \cup A_k) \cap A_{k+1}). \quad (1)$$

Result follows by substituting required form for $P(A_1 \cup \dots \cup A_k)$, and re-writing

$$P((A_1 \cup \dots \cup A_k) \cap A_{k+1}) = P((A_1 \cap A_{k+1}) \cup \dots \cup (A_k \cap A_{k+1}))$$

(using distributivity) which is the union of k events, and hence can use the inductive hypothesis to re-write this final expression in the required form. Specifically, let $B_i = A_i \cap A_{k+1}$; then we have

$$\begin{aligned} P(B_1 \cup B_2 \cup \dots \cup B_k) &= \sum_i P(B_i) - \sum_i \sum_j P(B_i \cap B_j) + \dots (-1)^{k-1} P(B_1 \cap B_2 \cap \dots \cap B_k) \\ &= \sum_i P(A_i \cap A_{k+1}) - \sum_i \sum_j P(A_i \cap A_j \cap A_{k+1}) + \dots \\ &\quad (-1)^{k-1} P(A_1 \cap A_2 \cap \dots \cap A_k \cap A_{k+1}) \end{aligned}$$

as, for example,

$$B_i \cap B_j = (A_i \cap A_{k+1}) \cap (A_j \cap A_{k+1}) = A_i \cap A_j \cap A_{k+1}$$

Hence, using the inductive hypothesis to express

$$P(A_1 \cup \dots \cup A_k)$$

in (1) and adding in the final term, we complete the proof.