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MATH 323 SECTION: 001
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STUDENT NAME:		McGILL ID:	
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INSTRUCTIONS

EXAM:	CLOSED BOOK ☒ OPEN BOOK ☐	
	SINGLE SIDED ☐ PRINTED ON BOTH SIDES ☒	
	MULTIPLE CHOICE ANSWER SHEETS ☐	
	NOTE: The Examination Security Monitor Program detects pairs of students with unusually similar answer patterns on multiple-choice exams. Data generated by this program can be used as admissible evidence, either to initiate or corroborate an investigation or a charge of cheating under Section 16 of the Code of Student Conduct and Disciplinary Procedures.	
	ANSWER IN BOOKLET ☒ EXTRA BOOKLETS PERMITTED ☒	
	ANSWER ON EXAM ☐	
	SHOULD THE EXAM BE: RETURNED ☒ KEPT BY STUDENT ☐	
CRIB SHEETS:	NOT PERMITTED ☒ PERMITTED ☐	
	Specifications: No crib sheets allowed.	
DICTIONARIES:	TRANSLATION ONLY ☒ REGULAR ☒ NONE ☐	
CALCULATORS:	NOT PERMITTED ☐ PERMITTED (Non-Programmable) ☒	
ANY SPECIAL INSTRUCTIONS:	Special Instructions on next page:	

MATH 323: Specific Instructions.

Please write your answers in the answer booklets provided. Please write clearly.

This paper contains five questions. Each question carries 20 marks. Credit will be given for all questions attempted. The total mark available is 100 but rescaling of the final mark may occur.

Calculators may be used. Dictionaries and Translation dictionaries are permitted.

A distribution sheet, a sheet of useful summation results, and tables of the standard Normal distribution, are provided.

The usual abbreviations are used: pmf indicates the probability mass function; pdf indicates the probability density function; cdf indicates the cumulative distribution function; mgf indicates the moment generating function; pgf indicates the probability generating function.

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1. (a) For events A, B and C in sample space S , suppose that

$$P(A) = 0.4 \quad P(B) = 0.3 \quad P(C) = 0.1.$$

Write down

- (i) an upper bound on $P(A \cup B \cup C)$, and 2 MARKS
- (ii) an upper bound on $P(A \cap B \cap C)$ 2 MARKS

which is, in each case, less than 1. Justify your answer. If it is not possible to find a bound lower than 1, explain why.

- (b) Suppose that two events E and F partition S . Show that

$$P(E \cap F) + P(E \cup F) = 1.$$

2 MARKS

- (c) A soccer squad comprises 6 defenders, 6 midfielders, and 4 attackers as its outfield players. The team manager decides to select the outfield team with 4 defenders, 3 midfielders, and 3 attackers. How many possible teams with this configuration could be selected ? 4 MARKS
- (d) In a multiple choice test, each question has three possible answers: A,B,C. How many possible answer patterns are there, if it is known that each answer letter must appear precisely five times each ? 5 MARKS
- (e) A bag contains 10 balls: three are Red, three are Blue, three are Green and one is White. Balls are to be selected from the bag, with all balls equally likely to be selected from those that are still in the bag at every selection. What is the probability that
 - (i) the White ball is not selected on any of the first ten selections, if selections are made *with replacement* at each stage ?; 2 MARKS
 - (ii) two Red balls are selected during the first five selections, if selections are made *without replacement* at each stage ? 3 MARKS

2. (a) Two eggs are needed for a cake recipe, and they are selected from a box of twelve eggs. Unfortunately, the box contains two bad eggs that will affect the taste of the cake:
- if no bad eggs are selected, then the cake will have an acceptable taste with probability 0.95;
 - if precisely one bad egg is selected, then the cake will have an acceptable taste with probability 0.60;
 - if two bad eggs are selected, then the cake will have an acceptable taste with probability 0.10.

What is the probability that

- (i) the cake does not have an acceptable taste ?; 5 MARKS
- (ii) given that the cake does not have an acceptable taste, what is the probability that precisely one bad egg was used ? 5 MARKS

Assume that all selections of two eggs from twelve are equally likely.

- (b) A bag contains 10 balls which are known to be Red or White. Four balls are selected from the bag, with all such selections deemed to be equally likely.
- (i) If the number of Red balls in the bag **known** to be five, compute the probability of obtaining three Red balls in the selection. 4 MARKS
 - (ii) If the number of Red balls in the bag is **unknown**, but it is believed *before* the selection that all possible numbers of Red balls in the bag are equally likely, derive an expression for the probability that number of Red balls is r , given that three of the balls in the sample are observed to be Red, for all suitable values of r . 6 MARKS

3. (a) Suppose that a discrete random variable Y has mgf $m(t)$ given by

$$m(t) = c(1 + 2e^t + 3e^{2t})$$

for $t \in \mathbb{R}$ for some c .

- (i) Find the value of c . 2 MARKS
- (ii) Find $\mathbb{E}[Y^3]$. 3 MARKS
- (iii) Find $\mathbb{E}[Y(Y - 1)]$. 3 MARKS
- (iv) Suppose that Y_1 and Y_2 are independent random variables having the same distribution as Y . Find the pmf of

$$X = Y_1 + Y_2.$$

6 MARKS

- (b) Suppose that Z_1 and Z_2 are independent Poisson random variables each with parameter λ . Let $S = Z_1 + Z_2$. Find the conditional distribution of Z_1 , given that $S = s$ for s such that $P(S = s) > 0$. 6 MARKS

Hint: if $P(S = s) > 0$ then

$$P(Z_1 = z | S = s) = \frac{P(Z_1 = z, S = s)}{P(S = s)}$$

for appropriate z .

4. (a) For each of the following transformation problems, compute the distribution (that is, compute the pmf, pdf or cdf, or identify by name) of the variable U for the stated initial variable Y .

(i) $U = 2Y$, if $Y \sim \text{Geometric}(p)$. 2 MARKS

(ii) $U = 2Y$, if $Y \sim \text{Exponential}(1)$. 2 MARKS

(iii) $U = \exp\{-Y/2\}$, if $Y \sim \text{Exponential}(2)$. 2 MARKS

(iv) $U = (Y - \mu)^2/4$, if $Y \sim \text{Normal}(\mu, 2^2)$. 2 MARKS

(v) $U = \Phi(Y)$, if $Y \sim \text{Normal}(0, 1)$, and $\Phi(\cdot)$ is the standard Normal cdf. 2 MARKS

- (b) Suppose that Y_1 and Y_2 are independent random variables with $Y_1 \sim \text{Exponential}(\beta_1)$ and $Y_2 \sim \text{Exponential}(\beta_2)$. Find expressions for

(i) $P(Y_1 > 2 | Y_1 > 1)$; 1 MARK

(ii) $P(Y_1 > 2 | Y_2 > 1)$; 1 MARK

(iii) $P(Y_1 + Y_2 > 3)$; 3 MARKS

(iv) $P(Y_1 + Y_2 > 3 | Y_1 > 2)$. 3 MARKS

If $\beta_1 = \beta_2 = 1$, derive the conditional distribution of Y_1 , given that $Y_1 + Y_2 = 1$.

2 MARKS

5. (a) Suppose that Y_1 and Y_2 are continuous random variables with joint pdf given by

$$f_{Y_1, Y_2}(y_1, y_2) = c \exp\{-2y_2\} \quad 0 \leq y_1 \leq y_2 < \infty$$

and zero otherwise for some constant c .

- (i) Find the value of c . 2 MARKS
- (ii) Find the marginal pdf of Y_1 and the marginal pdf of Y_2 . 4 MARKS
- (iii) Find the conditional pdf of Y_2 given that $Y_1 = y_1$, for $y_1 > 0$. 2 MARKS
- (iv) Are Y_1 and Y_2 independent? Justify your answer. 2 MARKS

- (b) Suppose that Y_1 and Y_2 are continuous random variables with joint pdf given by

$$f_{Y_1, Y_2}(y_1, y_2) = c(y_1^2 + 2y_1y_2 + y_2^2) \quad 0 < y_1 < l, 0 < y_2 < l$$

and zero otherwise for constants $l, c > 0$.

- (i) Find the joint cdf for Y_1 and Y_2 , $F_{Y_1, Y_2}(y_1, y_2)$, for $(y_1, y_2) \in \mathbb{R}^2$. 4 MARKS
- (ii) Find the value of c in terms of l . 2 MARKS
- (iii) Are Y_1 and Y_2 independent? Justify your answer. 2 MARKS
- (iv) Are random variables

$$U_1 = (Y_1 + Y_2) \quad U_2 = (Y_1 - Y_2)$$

uncorrelated? Justify your answer. 2 MARKS

DISCRETE DISTRIBUTIONS

	$\mathcal{Y}$	PARAMETERS	$p(y)$	$F(y)$	$\mathbb{E}[Y]$	$\text{Var}[Y]$	$m(t)$
<i>Bernoulli</i> (p)	$\{0, 1\}$	$p \in (0, 1)$	$p^y(1-p)^{1-y}$		p	$p(1-p)$	$1-p+pe^t$
<i>Binomial</i> (n, p)	$\{0, 1, \dots, n\}$	$n \in \mathbb{Z}^+, p \in (0, 1)$	$\binom{n}{y} p^y(1-p)^{n-y}$		np	$np(1-p)$	$(1-p+pe^t)^n$
<i>Poisson</i> (λ)	$\{0, 1, 2, \dots\}$	$\lambda \in \mathbb{R}^+$	$\frac{e^{-\lambda} \lambda^y}{y!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
<i>Geometric</i> (p)	$\{1, 2, \dots\}$	$p \in (0, 1)$	$(1-p)^{y-1}p$	$1-(1-p)^y$	$\frac{1}{p}$	$\frac{(1-p)}{p^2}$	$\frac{pe^t}{1-e^t(1-p)}$
<i>NegBinomial</i> (r, p)	$\{r, r+1, \dots\}$	$r \in \mathbb{Z}^+, p \in (0, 1)$	$\binom{y-1}{r-1} p^r(1-p)^{y-r}$		$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left(\frac{pe^t}{1-e^t(1-p)}\right)^r$
or	$\{0, 1, 2, \dots\}$	$r \in \mathbb{Z}^+, p \in (0, 1)$	$\binom{r+y-1}{r-1} p^r(1-p)^y$		$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$	$\left(\frac{p}{1-e^t(1-p)}\right)^r$
<i>Hypergeom</i> (N, r, n)	$\{\max\{0, n-N+r\}, \dots, \min\{n, r\}\}$	$N \geq r, n$	$\frac{\binom{r}{y} \binom{N-r}{n-y}}{\binom{N}{n}}$		$n \binom{\frac{r}{N}}{\frac{r}{N}} \left(1 - \frac{r}{N}\right) \left(\frac{N-n}{N-1}\right)$		
or			$\frac{\binom{n}{y} \binom{N-n}{r-y}}{\binom{N}{r}}$				

CONTINUOUS DISTRIBUTIONS							
	$\mathcal{Y}$	PARAMETERS	$p(y)$	$F(y)$	$\mathbb{E}[Y]$	$\text{Var}[Y]$	$m(t)$
<i>Uniform</i> (θ_1, θ_2) (standard model $\theta_1 = 0, \theta_2 = 1$)	(θ_1, θ_2)	$\theta_1 < \theta_2 \in \mathbb{R}$	$\frac{1}{\theta_2 - \theta_1}$	$\frac{y - \theta_1}{\theta_2 - \theta_1}$	$\frac{(\theta_1 + \theta_2)}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{\theta_2 t} - e^{\theta_1 t}}{t(\theta_2 - \theta_1)}$
<i>Exponential</i> (β) (standard model $\beta = 1$)	$\mathbb{R}^+$	$\beta \in \mathbb{R}^+$	$\frac{1}{\beta} e^{-y/\beta}$	$1 - e^{-y/\beta}$	β	β^2	$\left(\frac{1}{1 - \beta t}\right)$
<i>Gamma</i> (α, β) (standard model $\beta = 1$)	$\mathbb{R}^+$	$\alpha, \beta \in \mathbb{R}^+$	$\frac{1}{\beta^\alpha \Gamma(\alpha)} y^{\alpha-1} e^{-y/\beta}$ where $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$		$\alpha\beta$	$\alpha\beta^2$	$\left(\frac{1}{1 - \beta t}\right)^\alpha$
<i>Normal</i> (μ, σ^2) (standard model $\mu = 0, \sigma = 1$)	$\mathbb{R}$	$\mu \in \mathbb{R}, \sigma \in \mathbb{R}^+$	$\frac{1}{\sqrt{2\pi}\sigma^2} \exp\left\{-\frac{(y - \mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$\exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$
<i>Beta</i> (α, β)	$(0, 1)$	$\alpha, \beta \in \mathbb{R}^+$	$\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1} (1 - y)^{\beta-1}$		$\frac{\alpha}{\alpha + \beta}$	$\frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$	
<i>Weibull</i> (α, β) (standard model $\beta = 1$)	$\mathbb{R}^+$	$\alpha, \beta \in \mathbb{R}^+$	$\alpha\beta y^{\alpha-1} e^{-\beta y^\alpha}$	$1 - e^{-\beta y^\alpha}$	$\frac{\Gamma(1 + 1/\alpha)}{\beta^{1/\alpha}}$	$\frac{\Gamma(1 + 2/\alpha) - \Gamma(1 + 1/\alpha)^2}{\beta^{2/\alpha}}$	
<i>Student</i> (ν)	$\mathbb{R}$	$\nu \in \mathbb{R}^+$	$\frac{\Gamma((\nu + 1)/2)}{\Gamma(\nu/2)\sqrt{\pi\nu}} \{1 + y^2/\nu\}^{-(\nu+1)/2}$		0 (if $\nu > 1$)	$\frac{\nu}{\nu - 2}$ (if $\nu > 2$)	
<i>Pareto</i> (θ, α)	$\mathbb{R}^+$	$\theta, \alpha \in \mathbb{R}^+$	$\frac{\alpha\theta^\alpha}{(\theta + y)^{\alpha+1}}$	$1 - \left(\frac{\theta}{\theta + y}\right)^\alpha$	$\frac{\theta}{\alpha - 1}$ (if $\alpha > 1$)	$\frac{\alpha\theta^2}{(\alpha - 1)(\alpha - 2)}$ (if $\alpha > 2$)	

For linear transformation $X = \mu + \sigma Y$

$$f_X(x) = f_Y\left(\frac{x - \mu}{\sigma}\right) \frac{1}{\sigma} \quad F_X(x) = F_Y\left(\frac{x - \mu}{\sigma}\right) \quad m_X(t) = e^{it} m_Y(\sigma t) \quad \mathbb{E}[X] = \mu + \sigma \mathbb{E}[Y] \quad \mathbb{V}[X] = \sigma^2 \mathbb{V}[Y]$$

Useful Series Summations

GEOMETRIC: For $|t| < 1$,

$$\frac{1}{1-t} = 1 + t + t^2 + \dots = \sum_{k=0}^{\infty} t^k$$

For the finite series, for $n \geq 1$

$$\frac{1-t^{n+1}}{1-t} = 1 + t + \dots + t^n$$

EXPONENTIAL:

$$e^t = 1 + t + \frac{t^2}{2!} + \dots = \sum_{k=0}^{\infty} \frac{t^k}{k!}$$

$$e^t = \lim_{n \rightarrow \infty} \left(1 + \frac{t}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 - \frac{t}{n}\right)^{-n}$$

$$e^{-t} = \lim_{n \rightarrow \infty} \left(1 - \frac{t}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{t}{n}\right)^{-n}$$

BINOMIAL: For $n > 0$,

$$(1+t)^n = 1 + nt + \frac{n(n-1)}{2!}t^2 + \dots + nt^{n-1} + t^n = \sum_{k=0}^n \binom{n}{k} t^k$$

NEGATIVE BINOMIAL: for $n > 0$ and $|t| < 1$

$$\frac{1}{(1-t)^{n+1}} = 1 + (n+1)t + \frac{(n+1)(n+2)}{2!}t^2 + \dots = \sum_{k=0}^{\infty} \binom{n+k}{k} t^k$$

LOGARITHMIC: for $|t| < 1$,

$$-\log(1-t) = t + \frac{t^2}{2} + \frac{t^3}{3} + \dots = \sum_{k=1}^{\infty} \frac{t^k}{k}$$

$$\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \dots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{t^k}{k}$$

Table of the standard Normal distribution

Entries in table are $P(Y \leq y)$

y	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990