



McGill

**Fall 2018
Practice Final Examination**

Practice Final

PROBABILITY
MATH 323 SECTION: 001
SOLUTIONS

1. (a) (i) Bonferroni's inequality: $P(A \cup B \cup C) \leq P(A) + P(B) + P(C) = 0.8$. 2 MARKS
(ii) As $A \cap B \cap C$ is a subset of A , of B and of C

$$P(A \cap B \cap C) \leq \min\{P(A), P(B), P(C)\} = 0.1$$

2 MARKS

- (b) By definition of a partition, $E \cap F = \emptyset$ and $E \cup F = S$, so therefore by Axiom 1 and the rule for complements

$$P(E \cap F) + P(E \cup F) = P(\emptyset) + P(S) = 1.$$

2 MARKS

- (c) This is a sequence of separate combinatorial selections, and using the multiplication principle, the solution is

$$\binom{6}{4} \times \binom{6}{3} \times \binom{4}{3} = 15 \times 20 \times 4 = 1200.$$

where the first term is the number of ways of choosing the defenders (4 from 6), midfielders (3 from 6) and attackers (3 from 4) respectively. 4 MARKS

- (d) This is a multinomial/partition problem and the solution is

$$\frac{15!}{5! \times 5! \times 5!} = 756756.$$

5 MARKS

- (e) (i) The probability of getting the white ball is $1/10$ for each of the selections as selections are made with replacement, so the probability is

$$\left(\frac{9}{10}\right)^{10} = 0.3487.$$

2 MARKS

- (ii) This is given by the hypergeometric formula, with $N = 10$, $R = 3$, $n = 5$ and $y = 2$, that is

$$\frac{\binom{n}{y} \binom{N-n}{R-y}}{\binom{N}{R}} = \frac{\binom{5}{2} \binom{5}{1}}{\binom{10}{3}} = \frac{10 \times 5}{120} = \frac{50}{120} = 0.4167$$

or equivalently

$$\frac{\binom{R}{y} \binom{N-R}{n-y}}{\binom{N}{n}} = \frac{\binom{3}{2} \binom{7}{3}}{\binom{10}{5}} = \frac{3 \times 35}{252} = \frac{105}{252} = 0.4167.$$

3 MARKS

2. (a) Let E_i be the event that i bad eggs are selected for $i = 0, 1, 2$, and A be the event that the taste is acceptable. We have that

$$P(A|E_0) = 0.95 \quad P(A|E_1) = 0.60 \quad P(A|E_2) = 0.10$$

and can compute that

$$P(E_0) = \frac{10}{12} \times \frac{9}{11} = \frac{90}{132} \quad P(E_2) = \frac{2}{12} \times \frac{1}{11} = \frac{2}{132}$$

so that $P(E_1) = 40/132$.

- (i) Using the Theorem of Total Probability

$$P(A) = \sum_{i=0}^2 P(A|E_i)P(E_i) = 0.95 \times \frac{90}{132} + 0.60 \times \frac{40}{132} + 0.10 \times \frac{2}{132} = 0.8311.$$

so $P(A') = 0.1689$.

5 MARKS

- (ii) By Bayes Theorem

$$P(E_1|A') = \frac{P(A'|E_1)P(E_1)}{\sum_{i=0}^2 P(A'|E_i)P(E_i)} = \frac{0.40 \times \frac{40}{132}}{0.1689} = 0.7175.$$

5 MARKS

- (b) (i) If $R = 5$ is the number of Red balls in the bag of $N = 10$ balls, $n = 4$ is the sample size, and $y = 3$ is the number of Red balls in the sample, this is given by the hypergeometric formula as

$$\frac{\binom{R}{y} \binom{N-R}{n-y}}{\binom{N}{n}} = \frac{\binom{5}{3} \binom{5}{1}}{\binom{10}{4}} = \frac{10 \times 5}{210} = \frac{50}{210} = 0.2381.$$

4 MARKS

- (ii) We now need to use Bayes Theorem, with $P(R = r)$ constant across $r = 0, 1, 2, \dots, 10$ before the sample is taken

$$P(R = r|y = 3, n = 4, N = 10) = \frac{P(y = 3|n = 4, R = r, N = 10)P(R = r)}{\sum_{j=0}^N P(y = 3|n = 4, R = j, N = 10)P(R = j)}$$

From the information given (that in a sample of 4 balls, three are Red and one is White), the only legitimate values of r to consider are $r = 3, 4, \dots, 9$ as we know that there are at least 3 and at most 9 Red balls. Then

$$P(y = 3 | n = 4, R = r, N = 10) = \frac{\binom{r}{y} \binom{N-r}{n-y}}{\binom{N}{n}} = \frac{\binom{r}{3} \binom{10-r}{1}}{\binom{10}{4}}$$

Cancelling $P(R = j)$ in numerator and denominator as it is constant, we have that

$$P(R = r | y = 3, n = 4, N = 10) = \frac{\binom{r}{3} \binom{10-r}{1}}{\sum_{j=3}^9 \binom{j}{3} \binom{10-j}{1}}$$

6 MARKS

3. (a) The mgf for a discrete and non-negative random variable is defined by

$$m(t) = \sum_{y=0}^{\infty} e^{ty} p(y) = p(0) + e^t p(1) + e^{2t} p(2) + \dots$$

and equating this to the given form demonstrates that we must have

$$p(0) = c \quad p(1) = 2c \quad p(2) = 3c.$$

with $p(y) = 0$ for all other values of y . Thus the corresponding random variable takes values on the set $\{0, 1, 2\}$ with positive probability.

- (i) We need $c + 2c + 3c = 1$ so $c = 1/6$. 2 MARKS

- (ii) We can compute using the derivative result:

$$\mathbb{E}[Y^3] = \frac{d^3}{dt^3} \{m(t)\}_{t=0} = m^{(3)}(0)$$

Here we obtain $\mathbb{E}[Y^3] = \frac{1}{6}(2 + 3 \times 2^3) = 26/6 = 4.333$. Alternatively, by direct calculation

$$\mathbb{E}[Y^3] = \sum_{y=0}^{\infty} y^3 p(y) = \frac{1}{6}(0^3 + 1^3 \times 2 + 2^3 \times 3) = 4.333.$$

3 MARKS

- (iii) Here we can use the pgf

$$G(t) = m(\ln t) = \frac{1}{6}(1 + 2t + 3t^2)$$

and the derivative result

$$\mathbb{E}[Y(Y-1)] = \frac{d^2}{dt^2} \{G(t)\}_{t=1} = G''(1)$$

and here we obtain $\mathbb{E}[Y(Y-1)] = 1$. Alternatively, by direct calculation

$$\mathbb{E}[Y(Y-1)] = \sum_{y=0}^{\infty} y(y-1)p(y) = \frac{1}{6}(3 \times 2) = 1.$$

3 MARKS

- (iv) Using the mgf result for the sum of independent random variables, we have

$$\begin{aligned} m_Y(t) &= m_{Y_1}(t)m_{Y_2}(t) = \frac{1}{36}(1 + 2e^t + 3e^{2t})^2 \\ &= \frac{1}{36}(1 + 4e^t + 10e^{2t} + 12e^{3t} + 9e^{4t}) \end{aligned}$$

and therefore, using the initial result we have that

$$p_Y(y) = \begin{cases} 1/36 & y = 0 \\ 4/36 & y = 1 \\ 10/36 & y = 2 \\ 12/36 & y = 3 \\ 9/36 & y = 4 \end{cases}$$

and $p(y) = 0$ for all other values of y .

6 MARKS

- (b) We first compute the distribution of S ; using the mgf result for the sum of independent random variables

$$m_S(t) = m_{Z_1}(t)m_{Z_2}(t) = \exp\{\lambda(e^t - 1)\} \exp\{\lambda(e^t - 1)\} = \exp\{2\lambda(e^t - 1)\}$$

so therefore $S \sim \text{Poisson}(2\lambda)$ by the uniqueness of mgfs. Now consider the joint probability $P(Z_1 = z, S = s)$. We have

$$P(Z_1 = z, S = s) \equiv P(Z_1 = z, Z_2 = s - z) = P(Z_1 = z)P(Z_2 = s - z)$$

and hence

$$P(Z_1 = z, S = s) = e^{-\lambda} \frac{\lambda^z}{z!} e^{-\lambda} \frac{\lambda^{s-z}}{(s-z)!}$$

Therefore for $s > 0$ and $0 \leq z \leq s$

$$\begin{aligned} P(Z_1 = z | S = s) &= \frac{P(Z_1 = z, S = s)}{P(S = s)} = \frac{\frac{e^{-2\lambda} \lambda^s}{z!(s-z)!}}{\frac{e^{-2\lambda} (2\lambda)^s}{s!}} \\ &= \frac{s!}{z!(s-z)!} \left(\frac{1}{2}\right)^s \end{aligned}$$

and therefore given that $S = s$, Z_1 has a conditional $\text{Binomial}(s, 1/2)$ distribution.

6 MARKS

4. (a) (i) By discrete transformation theory U takes values on $\{2, 4, 6, \dots\}$, and for u in this set

$$p_U(u) = P(U = u) = P(2Y = u) = P(Y = u/2) = (1 - p)^{u/2-1}p$$

with $p_U(u)$ zero for all other values of u .

2 MARKS

- (ii) From first principles, for $u > 0$

$$F_U(u) = P(U \leq u) = P(2Y \leq u) = P(Y \leq u/2) = F_Y(u/2) = 1 - e^{-u/2}$$

and so $U \sim \text{Exponential}(2)$.

2 MARKS

- (iii) From first principles, for $0 < u < 1$

$$\begin{aligned} F_U(u) &= P(U \leq u) = P(\exp\{-Y/2\} \leq u) \\ &= P(Y \geq -2 \ln u) \\ &= 1 - F_Y(-2 \ln u) \\ &= 1 - (1 - \exp\{-(-2 \ln u)/2\}) \\ &= u \end{aligned}$$

and so $U \sim \text{Uniform}(0, 1)$.

2 MARKS

- (iv) We have that if $Z = (Y - \mu)/2$, then by the result for linear transformations of Normal random variables, $Z \sim \text{Normal}(0, 1)$, and $U = Z^2$. Thus

$$F_U(u) = P(U \leq u) = P(Z^2 \leq u) = P(-\sqrt{u} \leq Z \leq \sqrt{u}) = F_Z(\sqrt{u}) - F_Z(-\sqrt{u}).$$

Differentiating we have

$$\begin{aligned} f_U(u) &= \frac{1}{2\sqrt{u}}f_Z(\sqrt{u}) + \frac{1}{2\sqrt{u}}f_Z(-\sqrt{u}) \\ &= \frac{1}{\sqrt{u}}f_Z(\sqrt{u}) && \text{as } f_Z(z) \text{ is symmetric} \\ &= \frac{1}{\sqrt{2\pi}}u^{1/2-1} \exp\left\{-\frac{u}{2}\right\} \end{aligned}$$

so therefore $U \sim \text{Gamma}(1/2, 2) \equiv \text{Chisquared}(1)$.

2 MARKS

- (v) Note that $\Phi(\cdot)$ is a monotonic increasing transform mapping $\mathbb{R}$ to $(0, 1)$. Thus, from first principles, for $0 < u < 1$,

$$F_U(u) = P(U \leq u) = P(\Phi(Y) \leq u) = P(Y \leq \Phi^{-1}(u))$$

Thus

$$F_U(u) = \Phi(\Phi^{-1}(u)) = u$$

so $U \sim \text{Uniform}(0, 1)$.

2 MARKS

(b) (i) By conditional probability

$$P(Y_1 > 2 | Y_1 > 1) = \frac{P(Y_1 > 2, Y_2 > 1)}{P(Y_2 > 1)} = \frac{P(Y_1 > 2)}{P(Y_2 > 1)} = \frac{e^{-2/\beta_1}}{e^{-1/\beta_1}} = e^{-1/\beta_1}.$$

1 MARK

(ii) Y_1 and Y_2 are independent, so

$$P(Y_1 > 2 | Y_2 > 1) \equiv P(Y_1 > 2) = e^{-2/\beta_1}.$$

1 MARK

(iii) If $\beta_1 \neq \beta_2$, let $\lambda_1 = 1/\beta_1$ and $\lambda_2 = 1/\beta_2$. Then

$$\begin{aligned} P(Y_1 + Y_2 > 3) &= 1 - P(Y_1 + Y_2 \leq 3) \\ &= 1 - \int_0^3 \left\{ \int_0^{3-y_1} f_{Y_1}(y_1) f_{Y_2}(y_2) dy_2 \right\} dy_1 \\ &= 1 - \int_0^3 \left\{ \int_0^{3-y_1} \lambda_1 \exp\{-\lambda_2 y_2\} dy_2 \right\} \lambda_1 \exp\{-\lambda_1 y_1\} dy_1 \\ &= 1 - \int_0^3 \left[-\exp\{-\lambda_2 y_2\} \right]_0^{3-y_1} \lambda_1 \exp\{-\lambda_1 y_1\} dy_1 \\ &= 1 - \int_0^3 (1 - \exp\{-\lambda_2(3 - y_1)\}) \lambda_1 \exp\{-\lambda_1 y_1\} dy_1 \\ &= 1 - \left[-\exp\{-\lambda_1 y_1\} \right]_0^3 + \lambda_1 e^{-3\lambda_2} \int_0^3 \exp\{y_1(\lambda_2 - \lambda_1)\} dy_1 \\ &= \exp\{-3\lambda_1\} + \lambda_1 e^{-3\lambda_2} \left[\left(\frac{1}{\lambda_2 - \lambda_1} \right) \exp\{y_1(\lambda_2 - \lambda_1)\} \right]_0^3 \\ &= \exp\{-3\lambda_1\} + \frac{\lambda_1 e^{-3\lambda_2}}{\lambda_2 - \lambda_1} (\exp\{3(\lambda_2 - \lambda_1)\} - 1) \end{aligned}$$

If $\beta_1 = \beta_2 = \beta$ say, then using mgfs we have that

$$Y = Y_1 + Y_2 \sim \text{Gamma}(2, \beta)$$

so if $\lambda = 1/\beta$

$$P(Y > 3) = 1 - P(Y \leq 3) = 1 - \int_0^3 \lambda^2 y \exp\{-\lambda y\} dy$$

and integrating by parts we have that

$$\begin{aligned} P(Y > 3) &= 1 - [-\lambda y \exp\{-\lambda y\}]_0^3 - \lambda \int_0^3 \exp\{-\lambda y\} dy \\ &= 1 + 3\lambda \exp\{-3\lambda\} - (1 - \exp\{-3\lambda\}) \\ &= \exp\{-3\lambda\}(1 + 3\lambda) \end{aligned}$$

3 MARKS

(iv) We have by definition of conditional probability

$$P(Y_1 + Y_2 > 3 | Y_1 > 2) = \frac{P(Y_1 + Y_2 > 3, Y_1 > 2)}{P(Y_1 > 2)}$$

We can express the numerator

$$\begin{aligned} P(Y_1 + Y_2 > 3, Y_1 > 2) &= P(Y_1 + Y_2 > 3, 2 < Y_1 < 3) + P(Y_1 + Y_2 > 3, Y_1 > 3) \\ &= \int_2^3 \left\{ \int_{3-y_1}^{\infty} f_{Y_2}(y_2) dy_2 \right\} f_{Y_1}(y_1) dy_1 + P(Y_1 > 3) \end{aligned}$$

where the first term can be computed using the method from (iii).

3 MARKS

If $\beta_1 = \beta_2 = 1$, by definition of the conditional density

$$f_{Y_1|Y}(y_1|1) = \frac{f_{Y_1,Y}(y_1, 1)}{f_Y(1)} = \frac{f_{Y_1,Y_2}(y_1, 1 - y_1)}{f_Y(1)} = \frac{f_{Y_1}(y_1)f_{Y_2}(1 - y_1)}{f_Y(1)}$$

Thus, up to proportionality,

$$f_{Y_1|Y}(y_1|1) \propto \exp\{-y_1\} \exp\{-(1 - y_1)\} \propto 1 \quad 0 < y_1 < 1$$

so given $Y = 1$, Y_1 is conditionally distributed as $Uniform(0, 1)$.

2 MARKS

5. (a) (i) We have that

$$\frac{1}{c} = \int_0^\infty \int_0^{y_2} f_{Y_1, Y_2}(y_1, y_2) dy_1 dy_2$$

For the integral we have

$$\begin{aligned} \int_0^\infty \int_0^{y_2} f_{Y_1, Y_2}(y_1, y_2) dy_1 dy_2 &= \int_0^\infty \int_0^{y_2} \exp\{-2y_2\} dy_1 dy_2 \\ &= \int_0^\infty y_2 \exp\{-2y_2\} dy_2 \\ &= \frac{\Gamma(2)}{2^2} = \frac{1}{4} \end{aligned}$$

as the integrand is a $Gamma(2, 1/2)$ pdf. Thus $c = 4$.

2 MARKS

(ii) We have for $y_1 > 0$

$$f_{Y_1}(y_1) = \int_{y_1}^\infty 4 \exp\{-2y_2\} dy_2 = [-2 \exp\{-2y_2\}]_{y_1}^\infty = 2 \exp\{-2y_1\}$$

with the pdf zero otherwise. Thus $Y_1 \sim Exponential(1/2)$.

2 MARKS

In the same way, for $y_2 > 0$,

$$f_{Y_2}(y_2) = \int_0^{y_2} 4 \exp\{-2y_2\} dy_1 = 4y_2 \exp\{-2y_2\}$$

with the pdf zero otherwise. Thus $Y_2 \sim Gamma(2, 1/2)$.

2 MARKS

(iii) The conditional pdf of Y_2 given that $Y_1 = y_1$, for $y_1 > 0$ is given by

$$f_{Y_2|Y_1}(y_2|y_1) = \frac{f_{Y_1, Y_2}(y_1, y_2)}{f_{Y_1}(y_1)}$$

and for $y_2 \geq y_1$

$$f_{Y_2|Y_1}(y_2|y_1) = \frac{4 \exp\{-2y_2\}}{2 \exp\{-2y_1\}} = 2 \exp\{-2(y_2 - y_1)\}.$$

This is the so-called *shifted Exponential* distribution; given $Y_1 = y_1$, the random variable $Y_2 - y_1$ is *Exponential(2)* distributed.

2 MARKS

(iv) Y_1 and Y_2 are not independent, as the support of the joint pdf is not a Cartesian product of the two marginal supports.

2 MARKS

- (b) (i) For $0 < y_1 < l$ and $0 < y_2 < l$

$$\begin{aligned}
 F_{Y_1, Y_2}(y_1, y_2) &= \int_0^{y_1} \int_0^{y_2} c(t_1^2 + 2t_1 t_2 + t_2^2) dt_2 dt_1 \\
 &= c \int_0^{y_1} \left[t_1^2 t_2 + t_1 t_2^2 + \frac{t_2^3}{3} \right]_0^{y_2} dt_1 \\
 &= c \int_0^{y_1} \left(t_1^2 y_2 + t_1 y_2^2 + \frac{y_2^3}{3} \right) dt_1 \\
 &= c \left[\frac{t_1^3}{3} y_2 + \frac{t_1^2}{2} y_2^2 + \frac{y_2^3}{3} t_1 \right]_0^{y_1} \\
 &= c \left(\frac{y_1^3 y_2}{3} + \frac{y_1^2 y_2^2}{2} + \frac{y_1 y_2^3}{3} \right)
 \end{aligned}$$

and as we need $F_{Y_1, Y_2}(l, l) = 1$, we must have

$$\frac{1}{c} = \frac{7l^4}{6}.$$

so therefore $c = 6/(7l^4)$. Thus, we have

$$F_{Y_1, Y_2}(y_1, y_2) = \begin{cases} 0 & y_1 \leq 0 \text{ or } y_2 \leq 0 \\ c \left(\frac{y_1^3 y_2}{3} + \frac{y_1^2 y_2^2}{2} + \frac{y_1 y_2^3}{3} \right) & 0 < y_1 < l, 0 < y_2 < l \\ c \left(\frac{y_1^3 l}{3} + \frac{y_1^2 l^2}{2} + \frac{y_1 l^3}{3} \right) & 0 < y_1 < l, y_2 \geq l \\ c \left(\frac{l^3 y_2}{3} + \frac{l^2 y_2^2}{2} + \frac{l y_2^3}{3} \right) & 0 < y_2 < l, y_1 \geq l \\ 1 & y_1 \geq l \text{ and } y_2 \geq l \end{cases}.$$

4 MARKS

- (ii) As above $c = 6/(7l^4)$.

2 MARKS

- (iii) Y_1 and Y_2 are not independent as the joint pdf is not the product of a function of y_1 and a function of y_2 .

2 MARKS

- (iv) We have that

$$\begin{aligned}
 \text{Cov}(U_1, U_2) &= \mathbb{E}[U_1 U_2] - \mathbb{E}[U_1] \mathbb{E}[U_2] \\
 &= \mathbb{E}[Y_1 + Y_2](Y_1 - Y_2) - \mathbb{E}[Y_1 + Y_2] \mathbb{E}[(Y_1 - Y_2)] \\
 &= \mathbb{E}[Y_1^2 - Y_2^2] - (\mu_1 + \mu_2)(\mu_1 - \mu_2)
 \end{aligned}$$

say. However, the joint pdf is symmetric in y_1 and y_2 , so Y_1 and Y_2 have the same marginal distributions. Hence $\mu_1 = \mu_2$ and

$$\mathbb{E}[Y_1^2 - Y_2^2] = \mathbb{E}[Y_1^2] - \mathbb{E}[Y_2^2] = 0$$

so $\text{Cov}(U_1, U_2) = 0$ and U_1 and U_2 are uncorrelated.

2 MARKS