

MATH 323 - EXERCISES 6

NOT FOR ASSESSMENT.

GENERATING FUNCTIONS

Recall the following definitions: for discrete random variable Y with pmf $p(y)$

- *moment-generating function* (mgf), $m(t)$: for $|t| \leq b$, some $b > 0$,

$$m(t) = \mathbb{E} [e^{tY}] = \sum_y e^{ty} p(y)$$

- *probability-generating function* (pgf), $G(t)$: For a non-negative discrete random variable Y , and for $|1 - t| \leq b$, some $b > 0$,

$$G(t) = \mathbb{E} [t^Y] = \sum_{y=0}^{\infty} t^y p(y) = p(0) + tp(1) + t^2 p(2) + \dots$$

These definitions hold under the condition that the sum is convergent.

- 1 Show how to compute $p(y)$ from $G(t)$. Hence show that if two pgfs $G_1(t)$ and $G_2(t)$ equal for all $|t| < 1$, then their corresponding pmfs, $p_1(y)$ and $p_2(y)$, are also equal for all y .

(Hint: differentiate. Also, recall that two real polynomials are equal everywhere on a bounded region if and only if their coefficients are equal).

- 2 Identify the distributions corresponding to the following mgfs and pgfs:

- (a) $m(t) = (1 + e^t)/2$.
- (b) $m(t) = e^t(1 + e^t + e^{2t})/3$.
- (c) $m(t) = (2 + e^t)^3/27$.
- (d) $G(t) = (2 + t)/3$.
- (e) $G(t) = e^{2t}e^{-2}$.
- (f) For some specific $c > 0$

$$G(t) = c(1 + t + t^2 + t^3 + \dots + t^N) = \frac{c(1 - t^{N+1})}{1 - t}$$

- 3 Show that the function

$$G(t) = 1 - (1 - t^2)^{1/2}$$

a valid pgf for t in a range to be identified.

- 4 Suppose that discrete random variable Y has mgf $m_Y(t)$ and pgf $G_Y(t)$. Define random variable X by $X = aY + b$ for constants a and b . Using the properties of expectations, derive the mgf and pgf of X in terms of $m_Y(t)$ and $G_Y(t)$.

- 5 Suppose that

$$p(y) = \frac{1}{2} p^{|y|-1} (1 - p) \quad y = \pm 1, \pm 2, \pm 3, \dots$$

for some p , $0 < p < 1$, and $p(y) = 0$ for all other values of y . Find the corresponding mgf $m(t)$.

- 6 Suppose Y is a non-negative discrete random variables with pmf $p(y) > 0$ for $y = 0, 1, 2, \dots$. Show that

$$\sum_{y=0}^{\infty} P(Y < y) t^y = \frac{t}{1 - t} G(t)$$

whenever the required sum is convergent.