

MATH 323 - EXERCISES 2

NOT FOR ASSESSMENT.

ELEMENTARY PROBABILITY

1. A person selected from a population and subjected to screening for a particular disease is either a sufferer (D) or not (D'). The screening will either result in a positive test (T), or a negative (T'). If the result of the test is used to classify the person, write down the event "person misclassified".

Suppose now that an X-ray is also taken and either gives a positive (X), or a negative (X') indication, and also a doctor carries out an examination resulting in a positive (A), or negative (A') assessment. If a person is classified as a sufferer if and only if the doctor and at least one of the tests points in this direction, represent the event "correct classification made".

2. Given two events $A, B \subseteq S$, prove that the probability of *one and only one* of them occurring is

$$P(A) + P(B) - 2P(A \cap B).$$

3. Consider the following statements, which are claimed to be true for events A_1, A_2 in a sample space S :

- (a) $P(A_1) = 0 \implies P(A_1 \cup A_2) = 0$
- (b) $P(A_1) = P(A'_2) \implies A'_1 = A_2$
- (c) $A_1 \subseteq A_2$ and $P(A_1) = P(A'_2) \implies P(A_1) \leq 1/2$
- (d) $P(A'_1) = x_1, P(A'_2) = x_2 \implies P(A_1 \cup A_2) \geq 1 - x_1 - x_2$

In each case, either prove that the statement is true for all S, A_1, A_2 , or provide a specific counter-example to show that there exists S, A_1, A_2 for which it is false

4. Given two events $A, B \subseteq S$, prove that

- (a) $P(A' \cap B) = P(B) - P(A \cap B)$
- (b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- (c) $A \subseteq B \implies P(A) \leq P(B)$
- (d) $P(A \cap B) \geq P(A) + P(B) - 1$

(d) is known as *Bonferroni's Inequality*.

5. Suppose that A and B are events such that $P(A) = x, P(B) = y$ and $P(A \cap B) = z$. Express the following terms of x, y and z :

- (a) $P(A' \cup B')$
- (b) $P(A' \cap B)$
- (c) $P(A' \cup B)$
- (d) $P(A' \cap B')$.

6. Probability Axiom I can be equivalently stated as:

$$(I)^* : 0 \leq P(A) \leq 1 \text{ for all } A \subseteq S,$$

Assuming that Axioms $(I)^*, (II)$ and (III) hold, prove that $P(A) \geq 0$ for all $A \subseteq S$.

7. Use mathematical induction to prove that, for any events $A_1, A_2, \dots, A_n$ in S :

(a) *Boole's Inequality*

$$P(A_1 \cup A_2 \cup \dots \cup A_n) \leq P(A_1) + \dots + P(A_n)$$

(b) *General Addition Rule*

$$\begin{aligned} P(A_1 \cup A_2 \cup \dots \cup A_n) = & \sum_i P(A_i) - \sum_i \sum_j P(A_i \cap A_j) \\ & + \sum_i \sum_j \sum_k P(A_i \cap A_j \cap A_k) - \\ & \dots + (-1)^{n-1} P(A_1 \cap A_2 \cap \dots \cap A_n) \end{aligned}$$

In (b), summations are over *distinct* sets of subscripts, each counted once only. For example, if $n = 3$, the result would correspond to

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3).$$