

MATH 323 - ASSIGNMENT 2

SOLUTIONS

1. (a) We must have that $\sum_{y=1}^{10} p(y) = 1$ so $c = 1/55$ by direct computation. Similarly, by direct calculation

$$\mathbb{E}[Y] = \sum_{y=1}^{10} \frac{1}{55} y^2 = \frac{385}{55} = 7$$

and

$$\mathbb{E}[Y^2] = \sum_{y=1}^{10} \frac{1}{55} y^3 = \frac{3025}{55} = 55 \quad \therefore \quad \mathbb{V}[Y] = \mathbb{E}[Y^2] - \{\mathbb{E}[Y]\}^2 = 55 - 49 = 6.$$

3 MARKS

- (b) We notice that the series

$$\frac{1}{y} \left(\frac{1}{2}\right)^y = \frac{1}{2} + \frac{1}{2} \left(\frac{1}{2}\right)^2 + \frac{1}{3} \left(\frac{1}{2}\right)^3 + \dots$$

is precisely the logarithmic series (see the summations handout) with common term $1/2$, so

$$\sum_{y=1}^{\infty} \frac{1}{y} \left(\frac{1}{2}\right)^y = -\ln(1 - 1/2) = \ln 2.$$

so $c = 1/\ln 2 = 1.442695$.

2 MARKS

Also, using the geometric series

$$\mathbb{E}[Y] = \sum_{y=1}^{\infty} y \frac{1}{\ln 2} \frac{1}{y} \left(\frac{1}{2}\right)^y = \frac{1}{2 \ln 2} \sum_{y=0}^{\infty} \left(\frac{1}{2}\right)^y = \frac{1}{2 \ln 2} \frac{1}{1 - \frac{1}{2}} = \frac{1}{\ln 2} = 1.442695.$$

2 MARKS

- (c) In class we computed

$$\mathbb{E}[Y^2] = \mathbb{E}[Y(Y-1)] + \mathbb{E}[Y] = n(n-1)p^2 + np$$

so here $\mathbb{E}[Y^2] = 20 \times 19 \times (3/4)^2 + 20 \times (3/4) = 228.75$.

2 MARKS

- (d) Using the method from lectures:

$$\mathbb{E}[Y^3] = \mathbb{E}[Y(Y-1)(Y-2)] + \mathbb{E}[3Y^2 - 2Y] = \mathbb{E}[Y(Y-1)(Y-2)] + 3\mathbb{E}[Y(Y-1)] + \mathbb{E}[Y].$$

We computed $\mathbb{E}[Y(Y-1)] = 2(1-p)/p^2$ and $\mathbb{E}[Y] = 1/p$ in class (here we have $p = 1/2$), and using the same method – as given on the summations handout, differentiating the geometric series – we have that

$$\mathbb{E}[Y(Y-1)(Y-2)] = \sum_{y=1}^{\infty} y(y-1)(y-2)q^{y-1}p = \sum_{y=3}^{\infty} y(y-1)(y-2)q^{y-1}p = pq^2 \frac{6}{(1-q)^4} = \frac{6q^2}{p^3}.$$

Thus

$$\mathbb{E}[Y^3] = \frac{6(1-1/2)^2}{(1/2)^3} + 3 \frac{2(1-1/2)}{(1/2)^2} + \frac{1}{(1/2)} = 26.$$

3 MARKS

2. (a) This is the Geometric distribution with parameter $p = 3/36 = 1/12$, as there are precisely three equally likely combinations of scores that can give a total score of 10. 2 MARKS

(b) This is $Binomial(200, 0.75)$, from first principles. 1 MARKS

(c) This is the Geometric distribution with parameter p given by

$$p = \frac{\binom{80}{5} \binom{120}{5}}{\binom{200}{10}} \simeq 0.20407$$

that is, by the hypergeometric probability.

2 MARKS

3. This is the Geometric distribution with parameter p given by the Theorem of Probability as

$$p = P(\text{Passes a test}|A)P(A) + P(\text{Passes a test}|A')P(A')$$

where A is the event that the driver takes a test at centre A. That is

$$p = \frac{1}{2}p_A + \frac{1}{2}p_B$$

as it is this probability that determines the chance of passing on any individual test. Thus

$$p(y) = (1 - p)^{y-1}p \quad y = 1, 2, 3, \dots$$

3 MARKS

Note that the question refers to multiple coin tosses, and that the coin-tossing procedure is used to determine the centre at each test. If a single toss were to determine the centre for **all** tests, then the pmf would also be determined by the Theorem of Total Probability, but would be

$$p(y) = P(Y = y) = P(Y = y|A)P(A) + P(Y = y|A')P(A')$$

where A is the event that the driver takes all the tests at centre A. That is

$$p(y) = \frac{1}{2}(1 - p_A)^{y-1}p_A + \frac{1}{2}(1 - p_B)^{y-1}p_B \quad y = 1, 2, 3, \dots$$