

MATH 323 - ASSIGNMENT 1

SOLUTIONS

1. We know that for any event A say, $0 \leq P(A) \leq 1$, and also for two events A and B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad (1)$$

by the general addition rule for probabilities. Therefore

$$P(A \cap B) + P(A \cup B) = P(A) + P(B). \quad (2)$$

In the question, we are told that the right hand side of the equation is a fixed value, $p_A + p_B = p$, say, and we want to study the two quantities on the left hand side. Write $x = P(A \cap B)$ and $y = P(A \cup B)$ for these quantities. Equation (2) then takes the form

$$x + y = p \quad (3)$$

and we need to consider possible values of x and y .

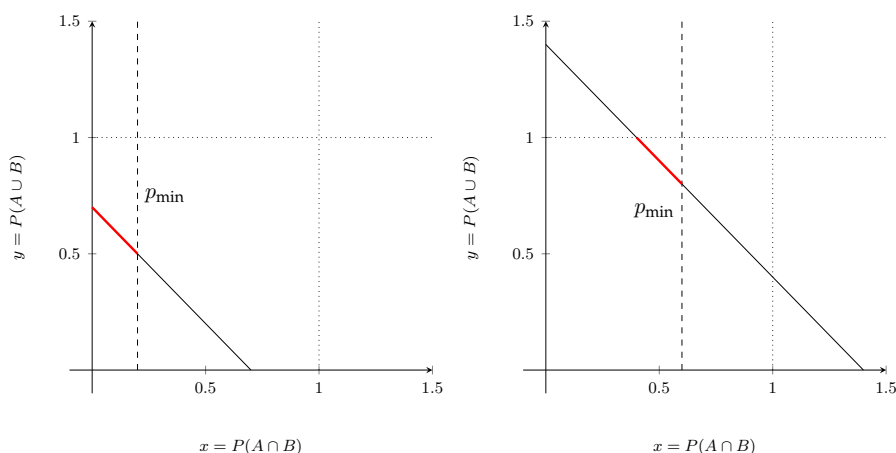
- For $P(A \cap B)$, we also know that $A \cap B \subseteq A$ and $A \cap B \subseteq B$, so therefore $P(A \cap B) \leq P(A) = p_A$ and $P(A \cap B) \leq P(B) = p_B$. Therefore

$$P(A \cap B) \leq \min\{p_A, p_B\} = p_{\min}.$$

- For $P(A \cup B)$, we also know that $A \subseteq A \cup B$ and $B \subseteq A \cup B$, so therefore $p_A = P(A) \leq P(A \cup B)$ and $p_B = P(B) \leq P(A \cup B)$. Therefore

$$p_{\max} = \max\{p_A, p_B\} \leq P(A \cup B).$$

We consider possible legitimate values of x and y by considering (3). The following figures depict possible scenarios: the left panel depicts $p \leq 1$, the right panel depicts $p \geq 1$.



In both panels, the solid line represents $x + y = p$; the dotted lines (and the axes) indicate the boundaries of one (and zero) that must hold for all probabilities; the dashed line is $p_{\min}$. The red coloured portion of the line depicts the legitimate values for both x (that is, $P(A \cap B)$) and y (that is, $P(A \cup B)$) – all legitimate configurations of probabilities must lie on this red portion. We can therefore read off from the pictures that for $p \leq 1$, we must have

$$0 \leq x \leq p_{\min} \quad p - p_{\min} \leq y \leq p$$

and for $p \geq 1$, we must have

$$p - 1 \leq x \leq p_{\min} \quad p - p_{\min} \leq y \leq 1.$$

Thus, to cover both possibilities, we need

$$\max\{0, p - 1\} \leq x \leq p_{\min} \quad p - p_{\min} \leq y \leq \min\{1, p\}.$$

(a) For $P(A \cup B)$, using the logic above

$$\max\{0, p_A + p_B - 1\} \leq P(A \cap B) \leq \min\{p_A, p_B\}$$

The left hand side follows from (1) by rearrangement

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

and the fact that $P(A \cup B)$ can be as large as 1.

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(b) For $P(A \cup B)$, direct from (1) and as $P(A \cap B) \leq \min\{p_A, p_B\}$ from (a), we have that

$$p_A + p_B - \min\{p_A, p_B\} \leq P(A \cup B) \leq \min\{1, p_A + p_B\}.$$

But

$$p_A + p_B - \min\{p_A, p_B\} = \max\{p_A, p_B\}$$

so therefore

$$\max\{p_A, p_B\} \leq P(A \cup B) \leq \min\{1, p_A + p_B\}.$$

agreeing with the previous deduced lower bound.

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2. We are given the following probabilities

$$\begin{aligned} P(A) &= 0.20 & P(A \cap B) &= 0.08 & P(A \cap B \cap C) &= 0.02 \\ P(B) &= 0.16 & P(A \cap C) &= 0.05 \\ P(C) &= 0.14 & P(B \cap C) &= 0.04 \end{aligned}$$

From this we have

$$P(A \cap B) = P(A \cap B \cap C) + P(A \cap B \cap C')$$

$$\implies P(A \cap B \cap C') = P(A \cap B) - P(A \cap B \cap C) = 0.08 - 0.02 = 0.06.$$

$$P(A \cap C) = P(A \cap B \cap C) + P(A \cap B' \cap C)$$

$$\implies P(A \cap B' \cap C) = P(A \cap C) - P(A \cap B \cap C) = 0.05 - 0.02 = 0.03.$$

$$P(B \cap C) = P(A \cap B \cap C) + P(A' \cap B \cap C)$$

$$\implies P(A' \cap B \cap C) = P(B \cap C) - P(A \cap B \cap C) = 0.04 - 0.02 = 0.02.$$

Then

$$\begin{aligned} P(A) &= P(A \cap B \cap C) + P(A \cap B \cap C') + P(A \cap B' \cap C) + P(A \cap B' \cap C') \\ 0.20 &\quad 0.02 \quad 0.06 \quad 0.03 \quad P(A \cap B' \cap C') \end{aligned}$$

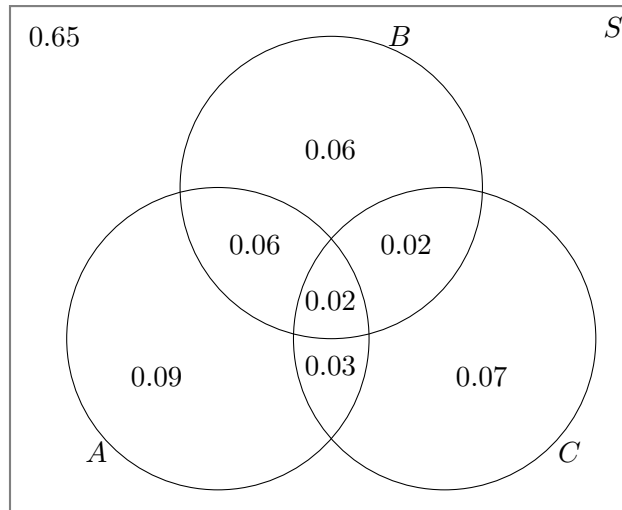
so therefore $P(A \cap B' \cap C') = 0.09$. Using the same method

$$P(A' \cap B \cap C') = 0.06 \quad P(A' \cap B' \cap C) = 0.07.$$

Therefore, we have for the fundamental partition of S that

$$\begin{array}{ll} P(A \cap B \cap C) = 0.02 & P(A' \cap B \cap C) = 0.02 \\ P(A \cap B \cap C') = 0.06 & P(A' \cap B \cap C') = 0.06 \\ P(A \cap B' \cap C) = 0.03 & P(A' \cap B' \cap C) = 0.07 \\ P(A \cap B' \cap C') = 0.09 & P(A' \cap B' \cap C') = 0.65 \end{array}$$

which we can display in the following Venn diagram:



(a) 'the selected person reads none of the newspapers' is represented $A' \cap B' \cap C'$, so

$$P(A' \cap B' \cap C') = 0.65.$$

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(b) 'the selected person reads precisely one of the newspapers' is represented

$$(A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C)$$

which occurs with probability $0.09 + 0.06 + 0.07 = 0.22$.

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(c) 'the selected person reads at least A and B, **given** that they read at least one newspaper':

- 'reads at least one newspaper': this is $A \cup B \cup C$, and from (a)

$$P(A \cup B \cup C) = 1 - P((A \cup B \cup C)') = 1 - P(A' \cap B' \cap C') = 1 - 0.65 = 0.35.$$

- 'reads at least A and B': this is $A \cap B$, as

$$A \cap B = (A \cap B \cap C) \cup (A \cap B \cap C'),$$

that is reads A and B and C, or reads A and B but not C. We are given $P(A \cap B) = 0.08$.

Therefore

$$P(A \cap B | A \cup B \cup C) = \frac{P((A \cap B) \cap (A \cup B \cup C))}{P(A \cup B \cup C)} = \frac{P(A \cap B)}{P(A \cup B \cup C)} = \frac{0.08}{0.35} = 0.229.$$

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3. Consider the case of there being k black balls, and $n - k$ red balls. There are

$$\binom{n}{k}$$

(equally likely) ways of arranging the balls to be distributed to the n students. Of these, there are

$$\binom{n-1}{k-1}$$

arrangements that assign a black ball to position r , for $r = 1, 2, \dots, 18$; we extract one black ball from the k available, insert that in position r , and then arrange the remaining $k - 1$ black balls and the $n - k$ red balls in any order. Thus the probability required is

$$\frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{(n-1)!}{(k-1)!(n-k)!} \times \frac{k!(n-k)!}{n!} = \frac{k}{n}.$$

Thus for (a) the probability is $1/18$, and for (b) the probability is $2/18$ (or $1/9$), with both numbers being the same for each r . **Therefore, it does not matter where in the sequence the student chooses, their probability of getting an A is unchanged.** Alternative constructions can also be used:

(a) To get an A, that is, the black ball, on the r th selection, there must be $r - 1$ previous selections of red balls with the selections being made without replacement each time. If $r = 1$, then the probability of getting an A is $1/18$. For $r = 2, 3, \dots, 18$, the probability can be computed as

$$\frac{17}{18} \times \frac{16}{17} \times \dots \times \frac{18-r+1}{18-r+2} \times \frac{1}{18-r+1} = \frac{1}{18}$$

that is, the probability is **identical** for each r .

A combinatorial argument yields the same result: let A_r denote the event that the black ball is selected on the r th selection. Then A_r corresponds to choosing $r - 1$ of the red balls from the 17 available, and then choosing the one black ball; in this version of the experiment, the number of possible sequences of r choices is P_r^{18} . Thus for $r \geq 2$,

$$P(A_r) = \frac{n_{A_r}}{n_{S_r}} = \frac{P_{r-1}^{17} \times P_1^1}{P_r^{18}} = \frac{17 \times 16 \times \dots \times (17 - (r-1) + 1) \times 1}{18 \times 17 \times \dots \times (18 - r + 1)} = \frac{1}{18}.$$

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(b) To receive an A under the new setting, again let A_r denote the event that a student receives an A when selecting the r th ball. We then have the decomposition $A_r = A_{1r} \cup A_{2r}$, where A_{1r} indicates that the A corresponds to the **first** black ball selected, and A_{2r} indicates that the A corresponds to the **second** black ball selected; A_{1r} and A_{2r} are mutually exclusive, so

$$P(A_r) = P(A_{1r}) + P(A_{2r}).$$

For $P(A_{1r})$, we have to select $r - 1$ red balls from the 16 available, and then one of the two black balls on the r th selection, when there are $18 - r + 1$ balls remaining. Hence

$$P(A_{1r}) = \frac{\binom{16}{r-1}}{\binom{18}{r-1}} \times \frac{2}{18-r+1} = \frac{2(18-r)}{18 \times 17}$$

For $P(A_{2r})$, we can use a combinatorial argument as above: for $r \geq 2$, the event A_{2r} corresponds to choosing precisely one of the two black balls, and $r - 2$ of the 16 red balls in the first $r - 1$ selections, and then choosing the remaining black ball on the r th selection. Then

$$P(A_{2r}) = \frac{\binom{16}{r-2} \times \binom{2}{1}}{\binom{18}{r-1}} \times \frac{1}{18-r+1} = \frac{2(r-1)}{18 \times 17}.$$

Therefore

$$P(A_r) = \frac{2(18-r)}{18 \times 17} + \frac{2(r-1)}{18 \times 17} = \frac{2}{18} = \frac{1}{9}.$$

Again, this is **identical** for each r .

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