

Chapter 1

Recap

Recap of hypothesis testing

- Consider the two hypotheses
 - ▶ H_0 is the **null hypothesis**
 - ▶ H_a is the **alternative hypothesis**
- Define **test statistic**, T , that permits assessment.
- Find the (null) distribution of T assuming that H_0 is **true**
- Compute the **observed** test statistic T_{obs} .
- Does T_{obs} lie in a **high probability region** of the null distribution ?

Statistical significance

Two approaches:

- Define the rejection region
- Compute the p-value (small p-value implies Reject H_0).

Confidence Intervals

A 95 % **confidence interval** for $\mu_A - \mu_B$

$$(\bar{x}_A - \bar{x}_B) \pm t_{\alpha/2, n_A + n_B - 2} SP \sqrt{\frac{1}{n_A} + \frac{1}{n_B}}$$

where

$$t_{\alpha/2, n_A + n_B - 2}$$

is the upper 0.025 point of the $t_{n_A + n_B - 2}$ distribution.

Designed Experiments: Terminology

Designed experiments: the experimenter **control** the levels of variables that may affect the variable of interest.

A **randomized experimental study** involves random allocation of treatments.

- **Response variable** (dependent variable):
- **Factors**
- **Factor Levels**
- **Treatments**
- **Experimental Units**

The Completely Randomized Design

A **completely randomized design** (CRD) is a design for which treatments are randomly assigned to experimental units, or in which random samples of experimental units are selected for each treatment.

The ANOVA F-test for CRDs

Suppose that there are k treatments:

TMT 1	Mean μ_1	n_1 experimental units
TMT 2	Mean μ_2	n_2 experimental units
$\vdots$	$\vdots$	
TMT k	Mean μ_k	n_k experimental units

x_{ij} is the response for unit j in treatment group i , for $j = 1, \dots, n_i$ and $i = 1, \dots, k$.

The ANOVA F-test for CRDs

We measure systematic variation using the statistic

$$SST = \sum_{i=1}^k n_i (\bar{x}_i - \bar{x})^2$$

SST - Sum of Squares for Treatments

We measure random variation using the statistic

$$SSE = \sum_{i=1}^k \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2 = (n - k)s_P^2$$

SSE - Sum of Squares for Error

The ANOVA F-test for CRDs

- MST - Mean Square for Treatments

$$MST = \frac{SST}{k - 1} = \frac{1}{k - 1} \sum_{i=1}^k n_i (\bar{x}_i - \bar{x})^2$$

- MSE - Mean Square for Error

$$MSE = \frac{SSE}{n - k} = \frac{1}{n - k} \sum_{i=1}^k \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2 = s_P^2$$

The ANOVA F-test for CRDs

Then the **test statistic** is

$$F = \frac{MST}{MSE} = \frac{\text{Average Variation due to Treatments}}{\text{Average Variation due to Errors}}$$

If H_0 is **true**, then

$$F \sim F_{k-1, n-k}$$

and the rejection region for a test at significance level α is the region to the right of the upper α point of this F distribution.

The ANOVA F-test for CRDs

If

$$SS = \sum_{i=1}^k \sum_{j=1}^{n_i} (x_{ij} - \bar{x})^2$$

is the **overall** or **total** sum of squares, then we have that

$$SS = SST + SSE$$

Assumptions behind the ANOVA F-test

1. **Independent** samples.
2. **Normal** distribution.
3. **Equal variances**

We may test the equal variances assumption formally using

Levene's test.

The ANOVA Table for the CRD

SOURCE	DF	SS	MS	F
TREATMENTS	$k - 1$	SST	$MST = \frac{SST}{(k - 1)}$	$F = \frac{MST}{MSE}$
ERROR	$n - k$	SSE	$MSE = \frac{SSE}{(n - k)}$	
TOTAL	$n - 1$	SS		

The ANOVA Table for the CRD

SOURCE	DF	SS	MS	F	p
TREATMENTS	$k - 1$	SST	MST	F	p
ERROR	$n - k$	SSE	MSE		
TOTAL	$n - 1$	SS			

Relaxing the assumptions

Permutation tests: We compute the test statistic for all possible relabellings consistent with H_0 , retaining the group sample sizes, and use these values to compute the rejection region and p-value.

Multiple Comparison of Means

If c tests are carried out each at level α , then the probability of **at least one** Type I error is

$$\alpha_F = 1 - (1 - \alpha)^c \simeq c\alpha$$

α_F is the **Familywise Error Rate**.

Setting the per-test $\alpha = \alpha_F/c$ is known as the

Bonferroni Correction.

Randomized Block Designs

A **randomized block design** used **matched** experimental units organized into sets known as **blocks** and assigns one member from the set to each treatment.

For k treatments

1. Compile b blocks of k experimental units, with each block comprising units that are **similar**.
2. Assign **one** unit per block to each treatment at random.

There are a total of $n = bk$ measured responses.

We wish to **compare treatments** whilst acknowledging that there may be **differences between the blocks**.

Testing for Equal Treatment Means

$$SS = SST + SSB + SSE$$

where

$$SST = \sum_{i=1}^k b(\bar{x}_i - \bar{x})^2 \quad SSB = \sum_{j=1}^b k(\bar{x}_j^{(B)} - \bar{x})^2 \quad SS = \sum_{i=1}^k \sum_{j=1}^b (x_{ij} - \bar{x})^2$$

SST: Sum of Squares for **Treatments**

SSB: Sum of Squares for **Blocks**

SS: **Total** Sum of Squares

SSE: Sum of Squares for **Errors**

Testing for Equal Treatment Means

Test statistic is

$$F = \frac{MST}{MSE}$$

where

$$MST = \frac{SST}{k - 1} \quad MSE = \frac{SSE}{n - b - k + 1}$$

Testing for Equal Treatment Means

To test

$$H_0 : \mu_1 = \cdots = \mu_k$$

H_a : At least one pair of μ_i s different.

use the test statistic

$$F = \frac{MST}{MSE}$$

If H_0 is true

$$F \sim F_{k-1, n-b-k+1}$$

Testing for Equal Treatment Means

ANOVA Table

SOURCE	DF	SS	MS	F
TMTS	$k - 1$	SST	MST	$F = MST/MSE$
BLOCKS	$b - 1$	SSB	MSB	
ERROR	$n - k - b + 1$	SSE	MSE	
TOTAL	$n - 1$	SS		

Testing for Equal Block Means

For

$$H_0 : \mu_1^{(B)} = \cdots = \mu_b^{(B)}$$

we use the F statistic

$$F = \frac{MSB}{MSE}$$

where

$$MSB = \frac{SSB}{b - 1}$$

If H_0 is true

$$F \sim F_{b-1, n-k-b+1}$$

Interactions

So far in the RBD, we must assume that the treatments behave in a similar way across blocks.

But perhaps the difference **does** depend on block; perhaps we have **interaction**:

- an interaction is present if the effect of one variable on the outcome is **modified** in the presence of a second variable
- for example, the effect of treatment may be different in different blocks.

RBD with balanced replication

Suppose we have r observations per block/treatment combination (termed **replicates**), so that we have

$$n = bkr$$

experimental units in total. This is **balanced replication**.

In this design, all the quantities

$$SST, SSB, SSE, SS$$

$$MST, MSB, MSE$$

can be defined, and an ANOVA F-test can be carried out - the only difference is that $n = bkr$.

RBD with balanced replication

ANOVA tests are now based on the decomposition

$$SS = SST + SSB + SSI + SSE$$

where SSI is the Sum of Squares for Interaction.

The three F statistics

$$F = \frac{MST}{MSE} \quad F = \frac{MSB}{MSE} \quad F = \frac{MSI}{MSE}$$

can be used to test for significant Treatment, Block and Interaction effects respectively.

RBD with balanced replication

Now

$$MSE = \frac{SSE}{\text{Error d.f.}} \quad \text{Error d.f.} = n - p$$

p is the total number of **parameters** fitted.

- No Interaction: $p = 1 + (b - 1) + (k - 1) = b + k - 1$
- Interaction: $p = bk$

Thus

- No Interaction: Error d.f. = $n - b - k + 1$
- Interaction: Error d.f. = $n - bk$

RBD with balanced replication

If

$$MSI = \frac{SSI}{(b-1)(k-1)}$$

is the Mean Square for Interaction, then

$$F = \frac{MSI}{MSE}$$

Under the null hypothesis of **no interaction**, then

$$F \sim F_{(b-1)(k-1), n-bk}$$

where $n = bkr$.

ANOVA Table

SOURCE	DF	SS	MS	F
TMTS	$k - 1$	SST	MST	F_T
BLOCKS	$b - 1$	SSB	MSB	F_B
INTERACTION	$(b - 1)(k - 1)$	SSI	MSI	F_I
ERROR	$(n - bk)$	SSE	MSE	
TOTAL	$n - 1$	SS		

where

$$MST = \frac{SST}{k - 1} \quad MSB = \frac{SSB}{b - 1}$$

$$MSI = \frac{SSI}{(b - 1)(k - 1)} \quad MSE = \frac{SSE}{n - bk}$$

and

$$F_T = \frac{MST}{MSE} \quad F_B = \frac{MSB}{MSE} \quad F_I = \frac{MSI}{MSE}$$

Factorial designs

FACTOR A with a levels

FACTOR B with b levels

FACTOR C with c levels

$\vdots$

A **complete factorial experiment** is one in which every combination of the factors is utilized.

- the number of treatments is equal to the total number of factor-level combinations.

Factorial designs

Consider two factor experiments:

FACTOR A with a levels

FACTOR B with b levels

so there are ab treatments in total. We can study the effect of

- Factor A and Factor B (the **main effects**)
- the **interaction**

provided we have **balanced replication**.

Factorial designs

We construct ANOVA F-tests based on the decomposition

$$SS = SSA + SSB + SSI + SSE$$

- Sum of Squares for Treatments due to Factor A (*SSA*)

$$SSA = \sum_{i=1}^a br(\bar{x}_{i.} - \bar{x}_{..})^2$$

- Sum of Squares for Treatments due to Factor B (*SSB*)

$$SSB = \sum_{j=1}^b ar(\bar{x}_{.j} - \bar{x}_{..})^2$$

- Sum of Squares for Interaction (*SSI*)

$$SSI = \sum_{i=1}^a \sum_{j=1}^b r(\bar{x}_{ij} - \bar{x}_{i.} - \bar{x}_{.j} + \bar{x}_{..})^2$$

Factorial designs

These allow computation of

SSA, SSB, SSI, SS, SSE

MSA, MSB, MSI, MSE

using the degrees of freedom identical to those in the RBD with replication.

ANOVA Table

SOURCE	DF	SS	MS	F
FACTOR A	$a - 1$	SSA	MSA	F_A
FACTOR B	$b - 1$	SSB	MSB	F_B
INTERACTION	$(a - 1)(b - 1)$	SSI	MSI	F_{AB}
ERROR	$(n - ab)$	SSE	MSE	
TOTAL	$n - 1$	SS		

Test 'No effect of Factor A' : $F_A \sim F_{a-1, n-ab}$

Test 'No effect of Factor B' : $F_B \sim F_{b-1, n-ab}$

Test 'No Interaction' : $F_{AB} \sim F_{(a-1)(b-1), n-ab}$

Legal Models

For a two factor design:

MODEL	FACTOR	INTERACTION
NULL	NONE	NO
A	A	NO
B	B	NO
A+B	A,B	NO
A+B+A:B	A,B	YES

that is, if the interaction is significant, the only model you should report is

$$A + B + A:B$$

Parameterizations for two factor designs

The cell means m_{ij} can be written

$$\begin{aligned}m_{ij} &= \beta_1 & i = 1, j = 1 \\&= \beta_1 + \beta_i^{(A)} & i = 2, \dots, a, j = 1 \\&= \beta_1 + \beta_j^{(B)} & i = 1, j = 2, \dots, b \\&= \beta_1 + \beta_i^{(A)} + \beta_j^{(B)} + \gamma_{ij}^{(AB)} & i = 2, \dots, a \\& & j = 2, \dots, b\end{aligned}$$

where

$$\begin{aligned}\beta_i^{(A)} &: \text{contrasts for factor A} \\ \beta_j^{(B)} &: \text{contrasts for factor B} \\ \gamma_{ij}^{(AB)} &: \text{interaction}\end{aligned}$$

Parameterizations for two factor designs

The ab parameters are

β_1	1
$\beta_2^{(A)}, \dots, \beta_a^{(A)}$	$(a - 1)$
$\beta_2^{(B)}, \dots, \beta_b^{(B)}$	$(b - 1)$
$\gamma_{ij}^{(AB)}, i = 2, \dots, a, j = 2, \dots, b$	$(a - 1)(b - 1)$
Total	ab