

MATH 204: FORMULA SHEET

ANOVA

- **Completely Randomized Design:** One factor with k levels

Decomposition:

$$SS = SST + SSE$$

SOURCE	DF	SS	MS	F
TREATMENTS	$k - 1$	SST	MST	F
ERROR	$n - k$	SSE	MSE	
TOTAL	$n - 1$	SS		

F-test null distribution: $F \sim F_{k-1, n-k}$

- **Randomized Block Design:** Treatment factor with k levels, Blocking factor with b levels
 $r \geq 1$ replicates (complete and balanced), $n = bkr$.

► No interaction:

$$SS = SST + SSB + SSE$$

SOURCE	DF	SS	MS	F
TREATMENTS	$k - 1$	SST	MST	F_T
BLOCKS	$b - 1$	SSB	MSB	F_B
ERROR	$n - k - b + 1$	SSE	MSE	
TOTAL	$n - 1$	SS		

F-test null distributions: $F_T \sim F_{k-1, n-k-b+1}$, $F_B \sim F_{b-1, n-k-b+1}$

► Interaction

$$SS = SST + SSB + SSI + SSE$$

SOURCE	DF	SS	MS	F
TREATMENTS	$k - 1$	SST	MST	F_T
BLOCKS	$b - 1$	SSB	MSB	F_B
INTERACTION	$(b - 1)(k - 1)$	SSI	MSI	F_I
ERROR	$n - bk$	SSE	MSE	
TOTAL	$n - 1$	SS		

F-test null distributions: $F_T \sim F_{k-1, n-bk}$, $F_B \sim F_{b-1, n-bk}$, $F_I \sim F_{(b-1)(k-1), n-bk}$.

- **Factorial Design:** Factor A with a levels, Factor B with b levels
 $r \geq 1$ replicates (complete and balanced), $n = abr$

► No interaction

$$SS = SSA + SSB + SSE$$

SOURCE	DF	SS	MS	F
FACTOR A	$a - 1$	SSA	MSA	F_A
FACTOR B	$b - 1$	SSB	MSB	F_B
ERROR	$n - a - b + 1$	SSE	MSE	
TOTAL	$n - 1$	SS		

F-test null distributions: $F_A \sim F_{a-1, n-a-b+1}$, $F_B \sim F_{b-1, n-a-b+1}$

► Interaction

$$SS = SSA + SSB + SSI + SSE$$

SOURCE	DF	SS	MS	F
FACTOR A	$a - 1$	SSA	MSA	F_A
FACTOR B	$b - 1$	SSB	MSB	F_B
INTERACTION	$(a - 1)(b - 1)$	SSI	MSI	F_I
ERROR	$n - ab$	SSE	MSE	
TOTAL	$n - 1$	SS		

F-test null distributions: $F_A \sim F_{a-1, n-ab}$, $F_B \sim F_{b-1, n-ab}$, $F_I \sim F_{(a-1)(b-1), n-ab}$

- **General calculation rules:**

Total DF: $n - 1$

Total SS:

$$SS = \sum_{i=1}^n (x_i - \bar{x})^2$$

where the sum runs over all observed data points, and $\bar{x}$ is the overall mean.

If $*$ is T , A , B , or I ,

$$MS_* = \frac{SS_*}{DF} \quad F_* = \frac{MS_*}{MSE}$$

- **Multiplicity correction:** For c tests at level α , Familywise Error rate is

$$\alpha_F = 1 - (1 - \alpha)^c \simeq c\alpha$$

- **Post-hoc comparisons:** For the $k = 2$ comparison of means, a $100(1 - \alpha)\%$ confidence interval for $\mu_1 - \mu_2$ is

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2, n_1+n_2-2} s \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

where $t_{\alpha, \nu}$ is the upper α point of the t-distribution with ν degrees of freedom, and s is the estimate of residual error variance σ based on either the pooled two sample estimate s_P or the overall estimate given by the MSE from the ANOVA.

REGRESSION

- **Simple Linear Regression:** We consider the model

$$Y = \beta_0 + \beta_1 x + \epsilon \quad \text{or} \quad E[Y|x] = \beta_0 + \beta_1 x$$

where β_0 and β_1 are the **intercept** and **slope** parameters, ϵ is a random variable with expectation zero and variance σ^2 . Under assumptions:

- **Least-squares criterion:** Choose $\hat{\beta}_0$ and $\hat{\beta}_1$ to minimize the **sum of squared errors**

$$SSE(\beta_0, \beta_1) = \sum_{i=1}^n e_i^2 = \sum_{i=1}^n (y_i - (\beta_0 + \beta_1 x_i))^2$$

- **Sample means:**

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i \quad \bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$$

- **Sums of Squares:**

$$SS_{xx} = \sum_{i=1}^n (x_i - \bar{x})^2 \quad SS_{xy} = \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) \quad SS_{yy} = \sum_{i=1}^n (y_i - \bar{y})^2$$

- **Least-squares estimates are:**

$$\hat{\beta}_1 = \frac{SS_{xy}}{SS_{xx}} \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

- **Fitted-values:** $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$

- **Residuals:** $\hat{e}_i = y_i - \hat{y}_i$

- **Residual error variance estimate:**

$$\hat{\sigma}^2 = \frac{SSE(\hat{\beta}_0, \hat{\beta}_1)}{n - 2} = s^2$$

- **Hypothesis tests:** to test

$$H_0 : \beta_j = 0$$

$$H_a : \beta_j \neq 0$$

for $j = 0, 1$ we use the test statistic

$$T = \frac{\hat{\beta}_j}{\text{e.s.e}(\hat{\beta}_j)}$$

where $\text{e.s.e}(\hat{\beta}_1)$ is the estimated standard error of $\hat{\beta}_j$, computed as

$$\text{e.s.e}(\hat{\beta}_0) = \sqrt{\frac{1}{n} \left(1 + \frac{n\bar{x}^2}{SS_{xx}} \right)} \quad \text{e.s.e}(\hat{\beta}_1) = \frac{s}{\sqrt{SS_{xx}}}$$

If H_0 is true

$$T = \frac{\hat{\beta}_j}{\text{e.s.e}(\hat{\beta}_j)} \sim t_{n-2}$$

► **Correlation coefficient:**

$$r = \frac{SS_{xy}}{\sqrt{SS_{xx}SS_{yy}}}$$

For testing

$$H_0 : \rho = 0$$

$$H_a : \rho \neq 0$$

use the test statistic

$$T = \frac{r}{\sqrt{(1-r^2)/(n-2)}}$$

If H_0 is true, then approximately $T \sim t_{n-2}$.

► **R^2 statistic** (R-squared statistic)

$$R^2 = r^2 = \frac{SSR}{SS} = \frac{SS_{yy} - SSE}{SS_{yy}} = 1 - \frac{SSE}{SS_{yy}}$$

Adjusted R-squared statistic

$$R_{adj}^2 = 1 - \frac{SSE/(n-k-1)}{SS/(n-1)}$$

► **Prediction:** A $100(1-\alpha)\%$ **confidence interval** for the *mean* at $x = x_p$ is

$$(\hat{\beta}_0 + \hat{\beta}_1 x_p) \pm t_{\alpha/2, n-2} s \sqrt{\frac{1}{n} + \frac{(x_p - \bar{x})^2}{SS_{xx}}}$$

whereas for an *individual* new value the **prediction interval** is

$$(\hat{\beta}_0 + \hat{\beta}_1 x_p) \pm t_{\alpha/2, n-2} s \sqrt{1 + \frac{1}{n} + \frac{(x_p - \bar{x})^2}{SS_{xx}}}$$

► **ANOVA table**

SOURCE	DF	SS	MS	F
REGRESSION	1	SSR	$MSR = \frac{SSR}{1}$	$F = \frac{MSR}{MSE}$
ERROR	$n-2$	SSE	$MSE = \frac{SSE}{(n-2)}$	
TOTAL	$n-1$	SS		

where

$$SS = SSR + SSE$$

and if $H_0 : \beta_1 = 0$ is true, $F \sim F_{1, n-2}$.

- **Multiple linear regression and the Linear Model:** For the model

$$Y_i = \beta_0 + \sum_{j=1}^k \beta_j x_{ij} + \epsilon_i$$

where the predictors x_{ij} may be continuous covariates, polynomial terms, dummy variables resulting from factor predictors or interactions we may fit using least-squares.

► **Adjusted R-squared statistic**

$$R_{adj}^2 = 1 - \frac{SSE/(n - k - 1)}{SS/(n - 1)}$$

► **Hypothesis tests:**

$$H_0 : \beta_j = 0$$

$$H_a : \beta_j \neq 0$$

$$T_j = \frac{\hat{\beta}_j}{\text{e.s.e.}(\hat{\beta}_j)}$$

If H_0 is true, $T_j \sim t_{n-k-1}$.

► **ANOVA:** For the overall test

SOURCE	DF	SS	MS	F
REGRESSION	k	SSR	MSR	F
ERROR	$n - k - 1$	SSE	MSE	
TOTAL	$n - 1$	SS		

where

$$MSR = \frac{SSR}{k} \quad MSE = \frac{SSE}{n - k - 1} \quad F = \frac{MSR}{MSE}$$

Here

$$H_0 : \beta_1 = \beta_2 = \dots = \beta_k = 0$$

$$H_a : \text{At least one } \beta_j \neq 0$$

If H_0 is true, $F \sim F_{k, n-k-1}$.

- **ANOVA decomposition:** If the model is formed from K separate predictors or interactions that comprise the k non-intercept terms. then we have a decomposition

$$SSR = SSR_1 + SSR_2 + \dots + SSR_K$$

eg for two predictors X_1, X_2 , the model written

$$X_1 + X_2 + X_1 : X_2$$

has $K = 3$ terms, and the model

$$X_1 + X_2$$

has $K = 2$ terms.

The number of parameters (degrees of freedom) associated with each kind of term is

- continuous predictor: 1
- factor predictor with a levels: $(a - 1)$
- interaction: the product of the number of parameters for each component.

If

$$k = k_1 + k_2 + \cdots + k_K$$

then we may form the ANOVA table

SOURCE	DF	SS	MS	F
TERM 1	k_1	SSR_1	MSR_1	F_1
TERM 2	k_2	SSR_2	MSR_2	F_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
TERM K	k_K	SSR_K	MSR_K	F_K
ERROR	$n - k - 1$	SSE	MSE	
TOTAL	$n - 1$	SS		

but we may only interpret the F values as representing the impact of the term concerned if we have balance across all combinations of levels of the factor predictors.

► **F-tests for Comparing Nested Models:** To compare two nested models

$$\text{Complete Model } E[Y|x] = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k$$

$$\text{Reduced Model } E[Y|x] = \beta_0 + \beta_1 x_1 + \cdots + \beta_g x_g$$

where $g < k$, we test the hypothesis

$$H_0 : \beta_{g+1} = \beta_{g+2} = \cdots = \beta_k = 0$$

$$H_a : \text{At least one of these } \beta_j \neq 0$$

We form the test statistic

$$F = \frac{(SSE_R - SSE_C)/(k - g)}{SSE_C/(n - k - 1)}$$

where SSE_C is the SSE under the complete model, and SSE_R is the SSE under the reduced model. If H_0 is true, then $F \sim F_{k-g, n-k-1}$.

We aim to find the simplest adequate model by searching the model space.

NON-PARAMETRIC METHODS

- **Chi-squared tests:** For a **multinomial** experiment with k categories, let $n_1, n_2, \dots, n_k$ be the numbers of experimental units in the k categories, where $n_1 + n_2 + \dots + n_k = n$.

► **One-way classification.** To test a hypothesis H_0 that fully specifies $p_1, \dots, p_k$,

$$H_0 : p_1 = p_1^{(0)}, p_2 = p_2^{(0)}, \dots, p_k = p_k^{(0)}$$

we use the test statistic

$$X^2 = \sum_{i=1}^k \frac{(n_i - np_i^{(0)})^2}{np_i^{(0)}}$$

We write $\hat{n}_i = np_i^{(0)}$. If H_0 is true, $X^2 \sim \chi_{k-1}^2$ approximately.

- **Two-way classification:** For cross-classification for two discrete factors A and B with r and c , to test the hypothesis

H_0 : Factor A and Factor B levels are assigned independently

we use the same test statistic that can be rewritten

$$X^2 = \sum_{i=1}^r \sum_{j=1}^c \frac{(n_{ij} - \hat{n}_{ij})^2}{\hat{n}_{ij}}$$

where

$$\hat{n}_{ij} = \frac{n_{i.} n_{.j}}{n} \quad n_{i.} = \sum_{j=1}^c n_{ij} \quad n_{.j} = \sum_{i=1}^r n_{ij}.$$

where $n_{i.}$ and $n_{.j}$ are the row and column totals for row i and column j respectively. If H_0 is true, $X^2 \sim \chi_{(r-1)(c-1)}^2$ approximately.

- **Odds ratio for 2×2 tables:** For a 2×2 table formed by cross-categorization by the two levels of factor A and factor B, we wish to assess the evidence for association between the factors. The true **population** odds ratio is

$$\psi = \frac{p_{11}/(1 - p_{11})}{p_{12}/(1 - p_{12})}$$

where p_{1j} is the probability of having level 1 for factor A and level j for factor B, for $j = 1, 2$. The **sample** odds ratio is

$$\text{O.R.} = \frac{n_{11}n_{22}}{n_{12}n_{21}} = \hat{\psi}$$

The test statistic to test

H_0 : No association between factor A and factor B

which is equivalent to $H_0 : \psi = 1$ or $H_0 : p_{11} = p_{12}$ is

$$Z = \frac{\log n_{11} + \log n_{22} - \log n_{12} - \log n_{21}}{\sqrt{\frac{1}{n_{11}} + \frac{1}{n_{12}} + \frac{1}{n_{21}} + \frac{1}{n_{22}}}}$$

If H_0 is true, then $Z \sim \text{Normal}(0, 1)$ approximately, where log means $\ln$ or *natural log*.

- **One sample non-parametric test:** To test the null hypothesis about population median, η

$$H_0 : \eta = \eta_0$$

we use the **sign test**: the test statistic S depends on the alternative hypothesis, H_a .

- For **one-sided** tests, to test

$$H_a : \eta > \eta_0 \quad S = \text{Number of observations greater than } \eta_0$$

$$H_a : \eta < \eta_0 \quad S = \text{Number of observations less than } \eta_0$$

If H_0 is **true**, it follows that

$$S \sim \text{Binomial} \left(n, \frac{1}{2} \right)$$

The p-value is $p = \Pr[X \geq S_{obs}]$ where $X \sim \text{Binomial}(n, 1/2)$.

- For a **two-sided** test,

$$H_a : \eta \neq \eta_0 \quad S = \max\{S_1, S_2\}$$

where S_1 and S_2 are the counts of the number of observations less than, and greater than, η_0 respectively. The p-value is $p = 2 \Pr[X \geq S_{obs}]$ where $X \sim \text{Binomial}(n, 1/2)$.

- **Two independent samples non-parametric test:** To test equality of population medians

$$H_0 : \eta_1 = \eta_2$$

we label the samples such that $n_1 > n_2$. Denote the **sum of the ranks** for the data from sample 2 by R_2 .

Wilcoxon Rank Sum Statistic: $W = R_2$

$$\text{Mann-Whitney U Statistic: } U = R_2 - \frac{n_2(n_2 + 1)}{2}$$

- **Large Sample Test** If $n_2 \geq 10$, a large sample test based on the Z statistic

$$Z = \frac{U - \frac{n_1 n_2}{2}}{\sqrt{\frac{n_1 n_2 (n_1 + n_2 + 1)}{12}}}$$

can be used. If $H_0 : \eta_1 = \eta_2$, is true $Z \sim \text{Normal}(0, 1)$, approximately. The rejection regions are

$$H_a : \eta_1 > \eta_2 \quad \text{then} \quad C_R = -1.645$$

$$H_a : \eta_1 < \eta_2 \quad \text{then} \quad C_R = 1.645$$

$$H_a : \eta_1 \neq \eta_2 \quad \text{then} \quad C_R = \pm 1.960$$

- **Small Sample Test** If $n_1 < 10$, we use an exact test based on R_2

$$H_a : \eta_1 > \eta_2 \quad \text{Rejection Region is} \quad R_2 \leq T_L \quad (\alpha = 0.025)$$

$$H_a : \eta_1 < \eta_2 \quad \text{Rejection Region is} \quad R_2 \geq T_U \quad (\alpha = 0.025)$$

$$H_a : \eta_1 \neq \eta_2 \quad \text{Rejection Region is} \quad R_2 \leq T_L \text{ or } R_2 \geq T_U \quad (\alpha = 0.05)$$

The values T_L and T_U are tabulated for different sizes of n_1 and n_2 .

- **Two dependent samples non-parametric test:** For n paired data define the within-pair differences

$$x_i = y_{i1} - y_{i2} \quad i = 1, \dots, n$$

To test the null hypothesis

$$H_0 : \text{No change between first and second measurements}$$

we can use the **Wilcoxon Signed Rank** test:

- If any $x_i = 0$, discard this value and adjust the sample size.
- Sort the **absolute values** $s_1, \dots, s_n$ into **ascending** order,
- Assign ranks 1 up to n .
- Compute T_+ and T_- , where

$$T_+ = \text{Sum of ranks for those } x_i > 0 \quad T_- = \text{Sum of ranks for those } x_i < 0$$

To test against

- (1) H_a : Significant **decrease** between first and second measurements
- (2) H_a : Significant **increase** between first and second measurements
- (3) H_a : Significant **change** between first and second measurements

- **Small Sample Test:** The critical value in the test is denoted T_0 .
 - For (1), we use T_- ; if $T_- \leq T_0$, we **reject** H_0 in favour of H_a
 - For (2), we use T_+ ; if $T_+ \leq T_0$, we **reject** H_0 in favour of H_a .
 - For (3), we use $T = \min\{T_-, T_+\}$; if $T \leq T_0$, we **reject** H_0
- **Large Sample Test:** For $n \geq 25$, we can use a large sample version of the test based on T_+ , and the Z statistic

$$Z = \frac{T_+ - \frac{n(n+1)}{4}}{\sqrt{\frac{n(n+1)(2n+1)}{24}}}$$

If H_0 is **true**, then $Z \sim \text{Normal}(0, 1)$ approximately, so that the test at $\alpha = 0.05$ uses the following critical values

- For H_a (1) use $C_R = 1.645$
- For H_a (2) use $C_R = -1.645$
- For H_a (3) use $C_R = \pm 1.960$

Table 1: Upper $\alpha = 0.05$ points of F_{ν_1, ν_2} distribution

ν_2	ν_1											
	1	2	3	4	5	6	7	8	9	10	11	12
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	242.98	243.91
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.40	19.41
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.76	8.74
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.94	5.91
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.70	4.68
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.03	4.00
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.60	3.57
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.31	3.28
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10	3.07
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.94	2.91
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.82	2.79
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.72	2.69
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.63	2.60
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.57	2.53
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.51	2.48
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.46	2.42
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.41	2.38
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.37	2.34
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.34	2.31
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.31	2.28
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.28	2.25
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.26	2.23
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.24	2.20
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.22	2.18
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.20	2.16
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.18	2.15
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.17	2.13
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.15	2.12
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.14	2.10
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.13	2.09
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.04	2.00
50	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07	2.03	1.99	1.95
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.95	1.92
70	3.98	3.13	2.74	2.50	2.35	2.23	2.14	2.07	2.02	1.97	1.93	1.89
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	2.00	1.95	1.91	1.88
90	3.95	3.10	2.71	2.47	2.32	2.20	2.11	2.04	1.99	1.94	1.90	1.86
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97	1.93	1.89	1.85
110	3.93	3.08	2.69	2.45	2.30	2.18	2.09	2.02	1.97	1.92	1.88	1.84
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.87	1.83
Inf	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.79	1.75

If required d.f. is not tabulated, approximate by taking the nearest tabulated value.

Table 2: Upper α points of the χ^2_ν distribution

ν	α				ν	α			
	0.100	0.050	0.010	0.001		0.100	0.050	0.010	0.001
1	2.706	3.841	6.635	10.828	11	17.275	19.675	24.725	31.264
2	4.605	5.991	9.210	13.816	12	18.549	21.026	26.217	32.909
3	6.251	7.815	11.345	16.266	13	19.812	22.362	27.688	34.528
4	7.779	9.488	13.277	18.467	14	21.064	23.685	29.141	36.123
5	9.236	11.070	15.086	20.515	15	22.307	24.996	30.578	37.697
6	10.645	12.592	16.812	22.458	16	23.542	26.296	32.000	39.252
7	12.017	14.067	18.475	24.322	17	24.769	27.587	33.409	40.790
8	13.362	15.507	20.090	26.124	18	25.989	28.869	34.805	42.312
9	14.684	16.919	21.666	27.877	19	27.204	30.144	36.191	43.820
10	15.987	18.307	23.209	29.588	20	28.412	31.410	37.566	45.315

Table 3: Upper α points of the t_ν distribution

ν	α					ν	α				
	0.100	0.050	0.025	0.010	0.001		0.100	0.050	0.025	0.010	0.001
1	3.078	6.314	12.706	31.821	318.309	16	1.337	1.746	2.120	2.583	3.686
2	1.886	2.920	4.303	6.965	22.327	17	1.333	1.740	2.110	2.567	3.646
3	1.638	2.353	3.182	4.541	10.215	18	1.330	1.734	2.101	2.552	3.610
4	1.533	2.132	2.776	3.747	7.173	19	1.328	1.729	2.093	2.539	3.579
5	1.476	2.015	2.571	3.365	5.893	20	1.325	1.725	2.086	2.528	3.552
6	1.440	1.943	2.447	3.143	5.208	21	1.323	1.721	2.080	2.518	3.527
7	1.415	1.895	2.365	2.998	4.785	22	1.321	1.717	2.074	2.508	3.505
8	1.397	1.860	2.306	2.896	4.501	23	1.319	1.714	2.069	2.500	3.485
9	1.383	1.833	2.262	2.821	4.297	24	1.318	1.711	2.064	2.492	3.467
10	1.372	1.812	2.228	2.764	4.144	25	1.316	1.708	2.060	2.485	3.450
11	1.363	1.796	2.201	2.718	4.025	26	1.315	1.706	2.056	2.479	3.435
12	1.356	1.782	2.179	2.681	3.930	27	1.314	1.703	2.052	2.473	3.421
13	1.350	1.771	2.160	2.650	3.852	28	1.313	1.701	2.048	2.467	3.408
14	1.345	1.761	2.145	2.624	3.787	29	1.311	1.699	2.045	2.462	3.396
15	1.341	1.753	2.131	2.602	3.733	30	1.310	1.697	2.042	2.457	3.385
Inf							1.282	1.645	1.960	2.326	3.090

d.f. equal to infinity corresponds to the standard Normal distribution.

Table 4: MWW: Critical Values T_L and T_U (α one-tailed, 2α two-tailed)

n_2	α	n_1															
		3		4		5		6		7		8		9		10	
		T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U
3	0.025	5	16	6	18	6	21	7	23	7	26	8	28	8	31	9	33
	0.05	6	15	7	17	7	20	8	22	9	24	9	27	10	29	11	31
4	0.025	6	18	11	25	12	28	12	32	13	35	14	38	15	41	16	44
	0.05	7	17	12	24	13	27	14	30	15	33	16	36	17	39	18	42
5	0.025	6	21	12	28	18	37	19	41	20	45	21	49	22	53	24	56
	0.05	7	20	13	27	19	36	20	40	22	43	24	46	25	50	26	54
6	0.025	7	23	12	32	19	41	26	52	28	56	29	61	31	65	32	70
	0.05	8	22	14	30	20	40	28	50	30	54	32	58	33	63	35	67
7	0.025	7	26	13	35	20	45	28	56	37	68	39	73	41	78	43	83
	0.05	9	24	15	33	22	43	30	54	39	66	41	71	43	76	46	80
8	0.025	8	28	14	38	21	49	29	61	39	73	49	87	51	93	54	98
	0.05	9	27	16	36	24	46	32	58	41	71	52	84	54	90	57	95
9	0.025	8	31	15	41	22	53	31	65	41	78	51	93	63	108	66	114
	0.05	10	29	17	39	25	50	33	63	43	76	54	90	66	105	69	111
10	0.025	9	33	16	44	24	56	32	70	43	83	54	98	66	114	79	131
	0.05	11	31	18	42	26	54	35	67	46	80	57	95	69	111	83	127

Table 5: Wilcoxon Signed Rank Test: Critical Values T_0 (α one-tailed, 2α two-tailed)

α	n											
	5	6	7	8	9	10	11	12	13	14	15	16
0.05	1	2	4	6	8	11	14	17	21	26	30	36
0.025		1	2	4	6	8	11	14	17	21	25	30
0.010			0	2	3	5	7	10	13	16	20	24
0.005				0	2	3	5	7	10	13	16	19
	17	18	19	20	21	22	23	24	25	26	27	28
0.05	41	47	54	60	68	75	83	92	101	110	120	130
0.025	35	40	46	52	59	66	73	81	90	98	107	117
0.010	28	33	38	43	49	56	62	69	77	85	93	102
0.005	23	28	32	37	43	49	55	61	68	76	84	92
	29	30	31	32	33	34	35	36	37	38	39	40
0.05	141	152	163	175	188	201	214	228	242	256	271	287
0.025	127	137	148	159	171	183	195	208	222	235	250	264
0.010	111	120	130	141	151	162	174	186	198	211	224	238
0.005	100	109	118	128	138	149	160	171	183	195	208	221
	41	42	43	44	45	46	47	48	49	50		
0.05	303	319	336	353	371	389	408	427	446	466		
0.025	279	295	311	327	344	361	379	397	415	434		
0.010	252	267	281	297	313	329	345	362	380	398		
0.005	234	248	262	277	292	307	323	339	356	373		