

MATH 204: PRINCIPLES OF STATISTICS 2

WINTER 2026

LINEAR REPARAMETERIZATION OF ANOVA MODELS

For the two-way layout for factors A and B, we may use the data generating model

$$X_{ijt} \sim \text{Normal}(m_{ij}, \sigma^2)$$

for

- $i = 1, \dots, a$ levels of Factor A
- $j = 1, \dots, b$ levels of Factor B
- $t = 1, \dots, r$ for the balanced replication.

with the mean model for m_{ij} broken down into components as follows

$$\begin{aligned} m_{ij} &= \beta_1 & i = 1, j = 1 \\ &= \beta_1 + \beta_i^{(A)} & i = 2, \dots, a, j = 1 \\ &= \beta_1 + \beta_j^{(B)} & i = 1, j = 2, \dots, b \\ &= \beta_1 + \beta_i^{(A)} + \beta_j^{(B)} + \gamma_{ij}^{(AB)} & i = 2, \dots, a, j = 2, \dots, b \end{aligned}$$

The data fall into cells of the $a \times b$ table, where the cell means are m_{ij} . For example, for $a = 3, b = 4$.

		Factor B			
		1	2	3	4
Factor A	1	m_{11}	m_{12}	m_{13}	m_{14}
	2	m_{21}	m_{22}	m_{23}	m_{24}
	3	m_{31}	m_{32}	m_{33}	m_{34}

that is

		Factor B			
		1	2	3	4
Factor A	1	β_1	$\beta_1 + \beta_2^{(B)}$	$\beta_1 + \beta_3^{(B)}$	$\beta_1 + \beta_4^{(B)}$
	2	$\beta_1 + \beta_2^{(A)}$	$\beta_1 + \beta_2^{(A)} + \beta_2^{(B)} + \gamma_{22}^{(AB)}$	$\beta_1 + \beta_2^{(A)} + \beta_3^{(B)} + \gamma_{23}^{(AB)}$	$\beta_1 + \beta_2^{(A)} + \beta_4^{(B)} + \gamma_{24}^{(AB)}$
	3	$\beta_1 + \beta_3^{(A)}$	$\beta_1 + \beta_3^{(A)} + \beta_2^{(B)} + \gamma_{32}^{(AB)}$	$\beta_1 + \beta_3^{(A)} + \beta_3^{(B)} + \gamma_{33}^{(AB)}$	$\beta_1 + \beta_3^{(A)} + \beta_4^{(B)} + \gamma_{34}^{(AB)}$

This model has ab parameters.

We may see the connection between the two different representations by examining the linear system of equations obtained by vectorizing the quantities into 12×1 column vectors:

$$\begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \\ m_{12} \\ m_{22} \\ m_{32} \\ m_{13} \\ m_{23} \\ m_{33} \\ m_{14} \\ m_{24} \\ m_{34} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2^{(A)} \\ \beta_3^{(A)} \\ \beta_2^{(B)} \\ \beta_3^{(B)} \\ \beta_4^{(B)} \\ \gamma_{22}^{(AB)} \\ \gamma_{32}^{(AB)} \\ \gamma_{23}^{(AB)} \\ \gamma_{33}^{(AB)} \\ \gamma_{24}^{(AB)} \\ \gamma_{34}^{(AB)} \end{bmatrix}$$

or in vector/matrix form $\mathbf{m} = \mathbf{L}\beta$ where $\mathbf{L}$ is the 12×12 matrix. In fact, the matrix $\mathbf{L}$ is non-singular, and its inverse is

$$\mathbf{L}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

so we may write $\beta = \mathbf{L}^{-1}\mathbf{m}$. This gives the reverse relationships

$$\begin{aligned} \beta_1 &= m_{11} \\ \beta_2^{(A)} &= m_{21} - m_{11} \\ \beta_3^{(A)} &= m_{31} - m_{11} \\ \beta_2^{(B)} &= m_{13} - m_{11} \\ \beta_3^{(B)} &= m_{23} - m_{11} \\ \beta_4^{(B)} &= m_{33} - m_{11} \\ \gamma_{22}^{(AB)} &= m_{22} - m_{12} - m_{21} + m_{11} \\ \gamma_{32}^{(AB)} &= m_{32} - m_{12} - m_{31} + m_{11} \\ \gamma_{23}^{(AB)} &= m_{23} - m_{13} - m_{21} + m_{11} \\ \gamma_{33}^{(AB)} &= m_{33} - m_{13} - m_{31} + m_{11} \\ \gamma_{24}^{(AB)} &= m_{24} - m_{14} - m_{21} + m_{11} \\ \gamma_{34}^{(AB)} &= m_{34} - m_{14} - m_{31} + m_{11} \end{aligned}$$