MATH 204: Principles of Statistics 2

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Notes

Textbook: McClave and Sincich (2012), *Statistics* (12th Edition), Chapters 10-14.

Note that the 10th and 11th Editions of McClave and Sincich contains essentially the same material.

Prerequisites: MATH 203 (or equivalent)

Some statistical computing knowledge useful.

Method of Assessment:

- Assignments
- Mid-Term
- Final

See syllabus handout for precise details.

Notes

Course Objectives

- Extensions of MATH 203 topics to other practical experimental contexts
- Introduction to statistical computation using standard software (R)
- Practice in the use of statistical methods, in particular, hypothesis testing and linear modelling.

Notes

Three main sections

I. THE ANALYSIS OF VARIANCE AND DESIGNED EXPERIMENTS

II. LINEAR REGRESSION MODELLING

III. NON-PARAMETRIC TESTING

Notes

Typical experimental scenario

- two different groups of subjects
- single observation/measurement made on each subject
- scientific question of interest

  ARE THE TWO GROUPS OF SUBJECTS SIGNIFICANTLY DIFFERENT IN TERMS OF THEIR MEASURED VALUES ?

Notes

Example: Pre-Natal Care

Objective: To compare the birthweights of babies in two groups of mothers.

- GROUP A: Received five or fewer pre-natal visits
- GROUP B: Received more than five pre-natal visits

Do the GROUP A babies have significantly different birthweights from those from GROUP B ?

Notes

Data: Birthweights (grammes)

- GROUP A: 10 subjects

2164	2600	2184	2080	1820
2496	2184	2080	2184	2576

- GROUP B: 7 subjects

3224	2704	2912	2444	3120
		2912	3848	

Notes

First step in analysis: statistical summary

- GROUP A:
  - Sample size: $n_A = 10$
  - Sample mean: $\overline{x}_A = 2236.8$
  - Sample variance: $s_A^2 = 61190.4$
- GROUP B:
  - Sample size: $n_B = 7$
  - Sample mean: $\overline{x}_B = 3023.429$
  - Sample variance: $s_B^2 = 198679.6$

Notes

Recall, for data $x_1, \ldots, x_n$

$$\overline{x} = \frac{x_1 + \cdots + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$$

$\overline{x}$ measures the "average" of sample
s^2 measures the amount of variability around the average.

Notes

In the birthweight example

$$\overline{x}_A = 2236.8 \qquad \overline{x}_B = 3023.429$$

so it appears that Group B birthweights are higher....

... BUT ARE THEY SIGNIFICANTLY HIGHER ?

i.e. is the difference due to chance alone

- sample sizes quite small
- birthweights quite variable

Notes

Statistical Testing

We adopt the following procedure to assess the "significance" of the difference between $\overline{x}_A$ and $\overline{x}_B$.

1. Define a *test statistic*, T, that permits comparison of the two groups
2. *Predict* how T will behave assuming that the two groups are **not** significantly different.
3. Compare the *prediction* with what was actually *observed*.

Notes

Formally, we

- assume a **Normal distribution** for the data in the two groups

  i.e. $x_{A1}, \ldots, x_{An_A}$ are drawn from a population of birthweights that is well-modelled by a

  $$Normal(\mu_A, \sigma_A^2)$$

  distribution.

  Similarly

  $$x_{B1}, \ldots, x_{Bn_B} \sim Normal(\mu_B, \sigma_B^2)$$

  We might initially assume that

  $$\sigma_A^2 = \sigma_B^2$$

Notes

- consider the two hypotheses

$$H_0: \quad \mu_A = \mu_B$$
$$H_a: \quad \mu_A \neq \mu_B$$

H_0 is the **NULL HYPOTHESIS**
H_a is the **ALTERNATIVE HYPOTHESIS**

- define the test statistic

$$t = \frac{\overline{x}_A - \overline{x}_B}{s\sqrt{\frac{1}{n_A} + \frac{1}{n_B}}}$$

where

$$s^2 = \frac{(n_A - 1)s_A^2 + (n_B - 1)s_B^2}{n_A + n_B - 2}$$

Notes

s^2 is the estimate of the common population variance

$$\sigma^2 = \sigma_A^2 = \sigma_B^2$$

Here

$$s^2 = \frac{(10 - 1)61190.4 + (7 - 1)198679.6}{10 + 7 - 2} = 116186.1$$

so that

$$s = 340.8608.$$

Thus

$$t = \frac{2236.8 - 3023.429}{340.8608\sqrt{\frac{1}{10} + \frac{1}{7}}} = -4.683$$

Notes

Now, if the null hypothesis was **true**, so that

$$\mu_A = \mu_B$$

the test statistic t should look like an observation from a

Student-t

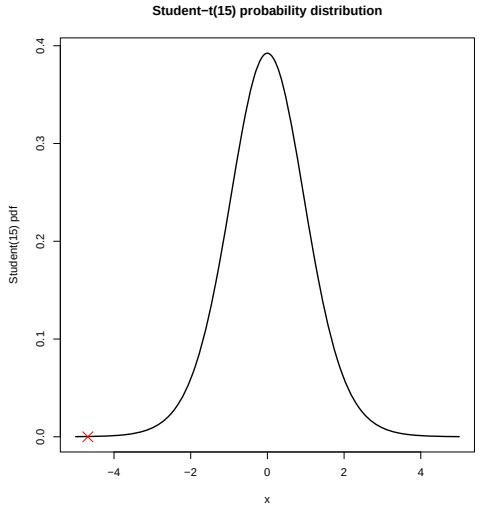
distribution with

$$n_A + n_B - 2 = 15$$

"degrees of freedom".

Notes

i.e. t should lie somewhere in the "high-probability region" of the Student-t(15) probability distribution



Clearly, in this case, t does not lie in a high probability region.

Notes

i.e. we are surprised to see t so far away from zero.

The predicted behaviour of t, under the assumption that H_0 is TRUE

DOES NOT MATCH THE OBSERVED BEHAVIOUR !

Therefore, the assumption that H_0 is true MUST BE INCORRECT and we

REJECT H_0

Notes

How do we quantify the "statistical significance" ?

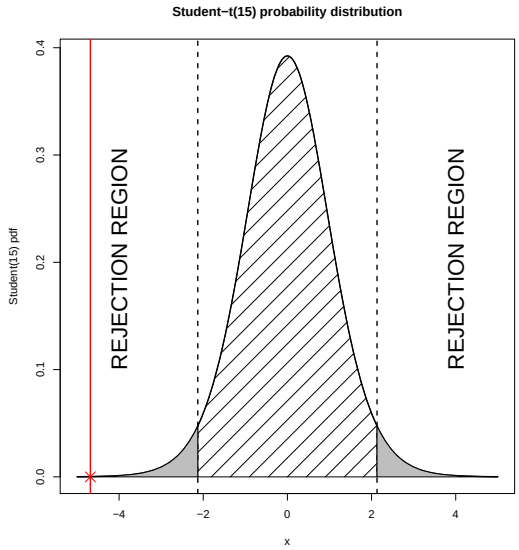
Two approaches:

1. Define the "high-probability" region, and reject H_0 if t does not lie in this region.

2. Compute the level of "surprise" at observing t under the assumption that H_0 is TRUE.

Notes

For 1: Set *significance level* α, with $0 < \alpha < 1$, and find the central $1 - \alpha$ "high-probability" region, between the two values $-C_R$ and C_R (marked by dotted lines).



Notes

If $t < -C_R$ or $t > C_R$, REJECT H_0.

Typically, $\alpha = 0.05$ (or 0.01), so for the Student-t(15) distribution

$$C_R = 2.131 \qquad \text{(or } 2.947\text{)}$$

The regions $(-\infty, -C_R)$ and (C_R, ∞) form the CRITICAL REGION or REJECTION REGION.

If t lies in the critical region, we reject H_0.

Notes

For 2: To compute the level of "surprise", we evaluate the probability of observing a "more extreme" test statistic under the assumption that H_0 is TRUE.

Here, this probability is

$$p = 0.00029.$$

This probability is very small, so we are **very surprised** by the observed result.

p is termed the *p-value* or *observed significance level*.

If $p < \alpha = 0.05$ (or 0.01), we reject H_0.

Notes

Some questions:

- How do we choose the test statistic ?
- How do we choose α ?
- Why is the distribution of T (and t) a Student-t(15) distribution ?
- How do we compute C_R and p ?

Notes

Equal Variances ?

Is the assumption of equal population variances

$$\sigma_A = \sigma_B$$

fair in this case ?

$$\begin{aligned} s_A^2 &= 61190.4 \\ s_B^2 &= 198679.6 \end{aligned}$$

so that

$$\frac{s_A^2}{s_B^2} = 0.3080.$$

Can we test $\sigma_A = \sigma_B$ formally ?

Notes

Yes:

$$\begin{aligned} H_0 &: \quad \sigma_A = \sigma_B \\ H_a &: \quad \sigma_A \neq \sigma_B \end{aligned}$$

Test statistic is

$$F = \frac{s_A^2}{s_B^2} = 0.3080$$

If H_0 is true, F should look like an observation from a

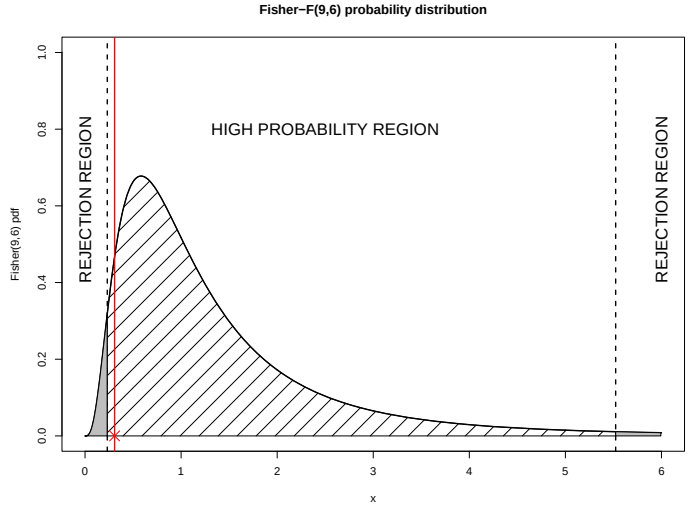
Fisher-F

distribution with

$$(n_A - 1, n_B - 1)$$

"degrees of freedom".

Notes



From tables, for $\alpha = 0.05$,

$$C_{R_1} = 0.231 \qquad C_{R_2} = 5.523$$

so the observed value of F **does** lie in the high probability region, and there is no reason to reject H_0 at $\alpha = 0.05$.

Notes

Can also compute a 95 % *confidence interval* for $\mu_A - \mu_B$

$$(\overline{x}_A - \overline{x}_B) \pm t_{n_A+n_B-2}(0.975)s\sqrt{\frac{1}{n_A} + \frac{1}{n_B}}$$

where

$$t_{n_A+n_B-2}(0.975) = 2.131$$

that is, the 0.975 probability point of the $Student - t(15)$ distribution.

Hence the 95 % confidence interval is

$$(-1144.59, -428.67)$$

- does not contain zero !

Notes

NOTE: Significance level α.

$$\begin{aligned} \alpha &= P[H_0 \text{ is rejected, given that } H_0 \text{ is TRUE}] \\ &= P[H_0 \text{ rejected} | H_0 \text{ is TRUE}] \end{aligned}$$

If

- T is the test statistic *random variable*
- $\mathcal{R}$ is the rejection region

then

$$\alpha = P[T \text{ lies in } \mathcal{R} | H_0 \text{ TRUE}] = P[T \in \mathcal{R} | H_0 \text{ TRUE}]$$

that is, α is the probability of committing a

TYPE I ERROR

Notes

R Code: Course data sets

```
> load("204DataSets.rda")
> ls()
 [1] "AIDS"          "Batteries"     "Birthweight"
 [4] "Butterfat"     "Coleman"       "Diabetes"
 [7] "DZT"           "Gravel"        "Haemo"
[10] "HairEyes"      "Hooker"        "Hypnosis"
[13] "Lyrics"        "Memory"        "Milk"
[16] "MilkBac"       "Muco"          "PEFRAsthma"
[19] "Potato"        "PTSD"          "Pull"
[22] "RBDExample"    "Sign1"         "Sign2"
[25] "Soil"          "TaskDistract"  "Treadmill"
[28] "Viscosity"
```


Notes

R Code: Birthweight data

```
> str(Birthweight)
'data.frame':   17 obs. of  3 variables:
 $ Group: Factor w/ 2 levels "A","B": 1 1 1 1 ...
 $ gp   : int  1 1 1 1 1 1 1 1 1 ...
 $ BW   : int  2164 2600 2184 2080 1820 2496 ...

> table(Birthweight$Group)
 A  B
10  7

> tapply(Birthweight$BW,Birthweight$Group,mean)
       A        B
2236.800 3023.429

> tapply(Birthweight$BW,Birthweight$Group,var)
       A        B
 61190.4 198679.6
```


Notes

R Code: Two-sample t-test

```
> t.test(BW ~ Group,data=Birthweight,var.equal=T)

        Two Sample t-test

data:  BW by Group
t = -4.6829, df = 15, p-value = 0.0002945

alternative hypothesis: true difference in means
is not equal to 0

95 percent confidence interval:
 -1144.6653  -428.5919
sample estimates:
mean in group A mean in group B
       2236.800        3023.429
```


Notes

R Code: Test for equal variances

```
> var.test(BW ~ Group,data=Birthweight)

        F test to compare two variances

data:  BW by Group
F = 0.308, num df = 9, denom df = 6,
p-value = 0.1106

alternative hypothesis: true ratio of variances
is not equal to 1

95 percent confidence interval:
 0.05576002 1.33041078

sample estimates:
ratio of variances
         0.3079853
```


Notes

Notes

Part I

Analysis of Variance

In this section

- introduction to the terminology of *designed experiments*
- extension of statistical testing theory to comparison of more than two population means
- THE ANALYSIS OF VARIANCE (ANOVA) F-TEST

Notes

1.1 DESIGNED EXPERIMENTS

Data collection studies typically fall into one of two categories:

(i) Observational studies: the experimenter has no control over the variables under study, and can only measure outcomes.

- The IQ of MAC and PC users
- The relationship between environmental exposure to toxins and health status.

i.e. The experimenter does not control the exposure to variables that may cause changes in the outcome of interest.

This type of study is common in medicine and epidemiology as it is relatively cheap to carry out.

Common type of observational study:

CASE-CONTROL STUDY

Notes

Example (Smoking and Lung Cancer)

A study (Doll and Hill, 1950) investigated 649 lung cancer cases and 649 matched healthy controls, both drawn from a population of men in the UK. They found out what proportion in each group were smokers.

Neither health status nor smoking status were controlled by the experimenter, but were merely observed.

	Smokers	Non-smokers	Total
Lung cancer	647	2	649
Controls	622	27	649

Notes

This type of study can be unreliable, and cannot uncover all the relationships of interest.

A preferred approach involves the experimenter controlling the variables that cause variation in the other variables.

Note that this may not be ethical in a smoking/lung cancer study.

Notes

(ii) Designed experiments: the experimenter does control the levels of variables that may affect the variable of interest.

For designed experiments, sometimes called experimental studies, the experimenter intervenes to assign the levels of influential variables such as

- dose levels of therapeutic treatments
- types of treatment

Notes

Example (Birthweight study)

GROUP A : 5 or fewer visits
GROUP B : More than 5 visits.

at the control of

(a) Mothers $\longrightarrow$ OBSERVATIONAL STUDY
(b) Doctors $\longrightarrow$ DESIGNED EXPERIMENT

- after each mother is recruited to take part in the study, they are RANDOMLY assigned to either GROUP A or GROUP B. This is termed a

RANDOMIZED EXPERIMENTAL STUDY

This type of study is preferable, but can be more difficult to implement.

Notes

Terminology

- **Response variable** (dependent variable): the variable of interest in the study
- **Factors** : the variables that may have an effect of the response variable
  - quantitative if measured on a numerical scale
  - qualitative otherwise
- **Factor Levels**: the values of the factors utilized in the experiment
- **Treatments**: the factor-level combinations utilized.
- **Experimental Units** (subjects): the objects on which the factors are measured or observed.

Notes

Therefore:

- A *designed experiment* is one for which the analyst or experimenter **controls** the specification of treatments and the method of assigning units to treatments.
- An *observational experiment* or study is one for which the analyst simply **observes** the treatments and response on a sample of experimental units.

Notes

Example (Birthweight study)

- **Response:** Birthweight (g)
- **Factor:** Pre-natal treatment group
- **Factor levels:** GROUP A or GROUP B

that is, we have a single factor with two factor levels.

Notes

Example (SAT scores)

The SAT scores of female and male students in four schools are to be compared.

- **Response:** SAT score
- **Factors:** SEX and SCHOOL (both qualitative)
- **Factor levels:**
  - SEX: Female and Male
  - SCHOOL: A,B,C,D

that is, we have a two factors, SEX with two factor levels and SCHOOL with four factor levels. There are 8 possible treatments:

$$(F, A), (F, B), (F, C), (F, D), (M, A), (M, B), (M, C), (M, D)$$

Notes

Example (Pain Relief)

Different pain relief remedies are to be compared : factors are

- REMEDY (quantitative/qualitative, 3 levels)
  - Dose level 0
  - Dose level 1
  - Dose level 2
- AGE GROUP (quantitative/qualitative, 4 levels)
  - 0-16 years
  - 17-40 years
  - 41-65 years
  - 66 years and over
- SEX (qualitative, 2 levels)
  - Female
  - Male

A total of $3 \times 4 \times 2 = 24$ possible treatment combinations; REMEDY is the only factor that can be assigned by the analyst.

Notes

Completely Randomized Design

A *completely randomized design* (CRD) is a design for which treatments are randomly assigned to experimental units, or in which random samples of experimental units are selected for each treatment.

The term can be applied to both experimental and observational studies. For example,

- if the treatments are FEMALE/MALE for the factor SEX, a CRD draws independent samples of FEMALES and MALES for the two treatment groups.
- if the treatments are DOSE 0/DOSE 1, a CRD assigns experimental units independently to the two treatment groups at random.

Notes

Statistical Objectives

The experimental units assigned to different treatments (factor-level combinations) form

independent samples

from

different populations

in a CRD.

We wish to **compare** treatments: specifically, we wish to compare the treatment means.

A Multiple Group Comparison of Means !

Notes

Suppose that there are k treatments:

TMT 1	Mean μ_1
TMT 2	Mean μ_2
$\vdots$	$\vdots$
TMT k	Mean μ_k

We wish to test the hypotheses

H_0 : $\mu_1 = \mu_2 = \cdots = \mu_k$

H_a : At least two of the k treatment means are different

How do we do this ?

What is the relevant test statistic ?

Notes

Comparing k Treatments

Suppose

TMT 1	has n_1 experimental units
TMT 2	has n_2 experimental units
$\vdots$	$\vdots$
TMT k	has n_k experimental units

Denote by x_{ij} the response for unit j in treatment group i, for $j = 1, \ldots, n_i$ and $i = 1, \ldots, k$.

Notes

Let

$$\overline{x}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij}$$

denote the sample mean for treatment i, and

$$s_i^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} (x_{ij} - \overline{x}_i)^2$$

denote the sample variance for treatment i.

Now we consider pooling, that is, combining all units into a single group.

Notes

Define

- the total sample size

$$n = n_1 + \cdots + n_k = \sum_{i=1}^{k} n_i$$

- the overall sample mean

$$\overline{x} = \frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij}$$

- the overall sample variance

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{ij} - \overline{x})^2$$

Notes

Finally, consider the pooled sample variance

$$s_P^2 = \frac{1}{n-k} \sum_{i=1}^{k} (n_i - 1) s_i^2$$

- the extension of the pooled estimate of the population variance in a two-sample t-test.

Using these quantities, we can derive a test statistic for multiple group comparison.

Notes

We wish to compare how much variation is due to the

A DIFFERENCE BETWEEN TREATMENTS

and how much is due to

B RANDOM VARIATION WITHIN TREATMENTS

We measure A using the statistic

$$SST = \sum_{i=1}^{k} n_i(\overline{x}_i - \overline{x})^2$$

SST - Sum of Squares for Treatments

Notes

We measure B using the statistic

$$\begin{aligned} SSE &= \sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij} - \overline{x}_i)^2 \\ &= (n-k)s_P^2 \end{aligned}$$

SSE - Sum of Squares for Error

NOTE: This measure of random or error variability implicitly assumes that the variability **within** the treatment groups is the **same for each group**. That is, population variances

$$\sigma_1^2, \ldots, \sigma_k^2$$

are equal.

In practice this assumption must be checked.

Notes

Finally, we define the test statistic using the mean levels of variability

- MST - Mean Square for Treatments

$$MST = \frac{SST}{k-1} = \frac{1}{k-1}\sum_{i=1}^{k} n_i(\overline{x}_i - \overline{x})^2$$

- MSE - Mean Square for Error

$$MSE = \frac{SSE}{n-k} = \frac{1}{n-k}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij} - \overline{x}_i)^2 = s_P^2$$

Notes

Then the test statistic is

$$F = \frac{MST}{MSE} = \frac{\text{Average Variation due to Treatments}}{\text{Average Variation due to Errors}}$$

F large $\Longrightarrow$ Treatments Different !
F small $\Longrightarrow$ Treatments Similar !

The behaviour of F is given by the following Theorem

Notes

Theorem (ANOVA F-test to compare k treatments in a Completely Randomized Design)

To test the hypothesis of ***equal treatment means****,*

H_0 : $\mu_1 = \mu_2 = \cdots = \mu_k$

H_a : *At least two of the k treatment means are different*

the test statistic is

$$F = \frac{MST}{MSE}$$

If H_0 is ***TRUE****, then*

$$F \sim \textit{Fisher-F}(k-1, n-k)$$

and the rejection region for a test at significance level α is the region to the right of the $1-\alpha$ probability point of this Fisher-F distribution, C_R.

Notes

NOTE: If

$$SS = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{ij} - \overline{x})^2$$

is the *overall* or *total* sum of squares, then

$$SS = SST + SSE$$

so we can decompose the overall variation (SS) into the variation due to treatments (SST) and the variation due to the errors (SSE).

Notes

Assumptions behind the ANOVA F-test

1. The samples are randomly selected in an independent manner from the k treatment populations.
   (This is satisfied in a CRD)
2. All k treatment groups have distributions that are approximately normal.
3. The k treatment group variances (around the group mean) are equal:
$$\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2.$$

Notes

Example (Milk Quality Data)

The impact on milk protein level of three different diets is being studied.

Data: Measurements of milk protein levels for $n = 1337$ samples.

- **Response:** Milk Protein Level (%)
- **Factor:** DIET
- **Factor levels:** $k = 3$
  - 1: Barley
  - 2: Barley + Lupins
  - 3: Lupins

Notes

	TMT 1	TMT 2	TMT 3
n_i	425	459	453
$\overline{x}i$	3.532	3.430	2.312
s_i^2	0.102	0.091	0.114

$$\begin{aligned} SST &= 10.606 \\ SSE &= 136.432 \\ SS &= 147.038 \end{aligned}$$

$$\begin{aligned} k-1 &= 2 \\ n-k &= 1334 \end{aligned}$$

Notes

Therefore

$$MST = \frac{SST}{k-1} = \frac{10.606}{2} = 5.303$$

$$MSE = \frac{SSE}{n-k} = \frac{136.432}{1334} = 0.102$$

and

$$F = \frac{MST}{MSE} = 51.851$$

If H_0 is true, that is,

$$\mu_1 = \mu_2 = \mu_3$$

then F should look like an observation from a

$$\text{Fisher-F}(k-1, n-k)$$

distribution.

Notes

Here we are dealing with the

$$\text{Fisher-F}(2, 1334)$$

distribution. From tables, we discover that if $\alpha = 0.05$, then

$$F_\alpha(2, 1334) = 3.002$$

and thus we

$$\text{Reject } H_0$$

and conclude that there is a significant impact on milk protein level due to diet.

Notes

Note: Tables in McClave and Sincich only give

$$\begin{aligned} F_{0.05}(2, 120) &= 3.07 \\ F_{0.05}(2, \infty) &= 3.00 \end{aligned}$$

so we cannot look up $F_{0.05}(2, 1334)$. However, we know that

$$3.00 < F_{0.05}(2, 1334) < 3.07$$

and here the test statistic is $F = 51.851$.

Notes

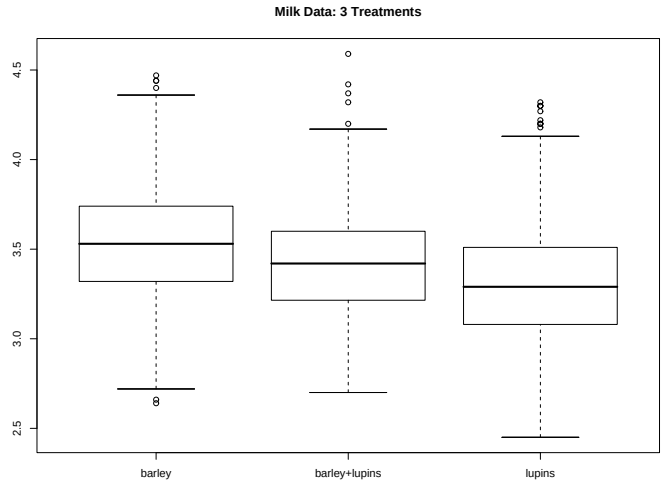
Are the assumptions met ?

1. **Independent samples** : Not possible to tell with current information. In fact, data comprise repeated measurements on 79 cows - potentially not independent, as observations on the same cow are likely to be more similar.
2. **Normal Distributions** : Visual inspection of boxplots indicates that this may be valid.
3. **Equal variances** :

$$s_1^2 = 0.102 \qquad s_2^2 = 0.091 \qquad s_3^2 = 0.114$$

so assumption appears to be valid
- can we test this formally ?

Notes



Notes

R Code: Milk data

Summary statistics by treatment group:

```
> tapply(Milk$protein,Milk$Diet,length)
       barley barley+lupins        lupins
          425           459           453

> tapply(Milk$protein,Milk$Diet,mean)
       barley barley+lupins        lupins
     3.531929      3.429695      3.312384

> tapply(Milk$protein,Milk$Diet,var)
       barley barley+lupins        lupins
   0.10189721    0.09141126    0.11363103
```


Notes

R Code: Milk data

Compute the **total sum of squares**:

```
> summary(aov(protein ~ 1,data=Milk))
              Df Sum Sq Mean Sq F value Pr(>F)
Residuals   1336    147  0.1101

> anova(lm(protein ~ 1,data=Milk))
Analysis of Variance Table

Response: protein
            Df Sum Sq Mean Sq F value Pr(>F)
Residuals 1336 147.04 0.11006
```


Notes

R Code: Milk data

Compute the **fitted sum of squares** and **residual sum of squares**:

```
> summary(aov(protein ~ Diet, data=Milk))
              Df Sum Sq Mean Sq F value Pr(>F)
Diet           2  10.61   5.303   51.85 <2e-16 ***
Residuals   1334 136.43   0.102
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The result of the Fisher-F test is also reported

Notes

R Code: Milk data

Look up the critical value using the `qf` function, and compute the p-value using the `pf` function.

```
>qf(0.95,2,1334)
[1] 3.00247

> 1-pf(51.85,2,1334)
[1] 0
```

Note that the `pf` function computes the cumulative probability (that is, in the **left-tail** of the distribution); for the p-value, we need the **right-tail** probability.

Notes

Example (Anxiety Response Treatment)

In a study of Alzheimer's disease and care of its sufferers, a medication designed to improve anxiety relief has been developed.

In a lab experiment, $n = 20$ rats were assigned to one of four ($k = 4$) treatment groups corresponding to dose-level of the medication.

A measure of response to a "flee stimulus" was recorded.

- **Response:** Pull response to stimulus (units of force)
- **Factor:** DOSE-LEVEL
- **Factor levels:** $k = 4$
  - Dose 0 (zero units)
  - Dose 1 (one unit)
  - Dose 2 (two units)
  - Dose 3 (three units)

Notes

0	1	2	2
27.0	22.8	21.9	23.5
26.2	23.1	23.4	19.6
28.8	27.7	20.1	23.7
33.5	27.6	27.8	20.8
28.8	24.0	19.3	23.9

We find that

$$SST = 140.094 \qquad SSE = 116.324 \qquad SS = 256.418$$

$$MST = 46.698 \qquad MSE = 7.270$$

and

$$F = 6.423$$

which we need to compare with the Fisher-F$(3, 16)$ distribution.

Notes

For $\alpha = 0.05$, from McClave and Sincich tables

$$F_{0.05}(3, 16) = 3.24$$

and so we

$$\text{Reject } H_0$$

at $\alpha = 0.05$ and conclude that there is a significant difference between treatment groups.

p-value is 0.0046.

Notes

R Code: Anxiety response data

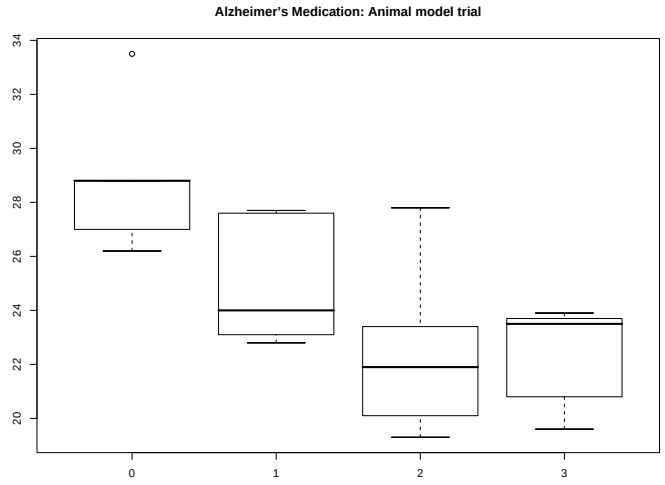
```
> summary(aov(Pull ~ Dose,data=Pull))
            Df Sum Sq Mean Sq F value  Pr(>F)
Dose         3  140.1   46.70   6.423 0.00462 **
Residuals   16  116.3    7.27
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05

> qf(0.95,3,16)
[1] 3.238872

> 1-pf(6.423,3,16)
[1] 0.004623413

> tapply(Pull$Pull,Pull$Dose,var)
 0Unit  1Unit  2Unit  3Unit
 8.018  5.873 11.315  3.875
```


Notes



Notes

Note: Here

	DOSE 0	DOSE 1	DOSE 2	DOSE 3
s_i^2	8.018	5.873	11.315	3.875

so we might suspect that the treatment variances $\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2$ are not equal. We may test this formally using

LEVENE'S TEST.

Note: Visual inspection can give an idea of whether the equal variance assumption is valid, or whether the populations are normal. **But the sample sizes may be small, so that visual inspection or testing may not detect deviations from these assumptions**.

Ideally we would like to be able to relax these assumptions.

Notes

Levene's Test

To test

$$\begin{aligned} H_0 &: \sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2 \\ H_1 &: \text{At least one pair of } \sigma^2 \text{ different.} \end{aligned}$$

Test statistic

$$W = \frac{(n-k)}{(k-1)} \frac{SST_Z}{SSE_Z} = \frac{MST_Z}{MSE_Z}$$

where SST_Z and SSE_Z are the usual sums of squares evaluated for the new data z_{ij} where

$$z_{ij} = |x_{ij} - \overline{x}_i|.$$

If H_0 is true

$$W \sim \text{Fisher-F}(k-1, n-k).$$

Notes

R Code: Anxiety response data - Levene's Test

```
> library(car)
> leveneTest(Pull ~ Dose,center='mean',data=Pull)
Levene's Test for Homogeneity of Variance
(center = "mean")

      Df F value Pr(>F)
group  3  0.2948 0.8286
      16

> leveneTest(Pull ~ Dose,center='median',data=Pull)
Levene's Test for Homogeneity of Variance
(center = "median")

      Df F value Pr(>F)
group  3  0.1955 0.8979
      16
```


Notes

The ANOVA Table

For a completely randomized design, we may report the results of the ANOVA F-test in a stylized form, the **ANOVA Table**

SOURCE	DF	SS	MS	F
TREATMENTS	$k-1$	SST	$MST = \frac{SST}{(k-1)}$	$F = \frac{MST}{MSE}$
ERROR	$n-k$	SSE	$MSE = \frac{SSE}{(n-k)}$	
TOTAL	$n-1$	SS		

Note

(i) $(k-1)+(n-k)=(n-1)$

(ii) $SST + SSE = SS$

i.e. we can fill in missing values if they are not given.

Notes

Sometimes an extra column is added at the right of the table to give the p-value of the ANOVA F-test.

SOURCE	DF	SS	MS	F	p
TMT	$k-1$	SST	MST	$F = \frac{MST}{MSE}$	p-val
ERROR	$n-k$	SSE	MSE		
TOTAL	$n-1$	SS			

where p-val solves

$$\frac{MST}{MSE} = F_{p\text{-val}}(k-1, n-k)$$

and $F_\alpha(\nu_1, \nu_2)$ is the $(1-\alpha)$ probability point of the Fisher-F distribution.

Notes

Handout: Examples and R code

- **DIET**: milk-protein level example
- **DOSE-LEVEL**: pull-strength in Alzheimer's example
- **DIAGNOSIS**:
  - RESPONSE: gut permeability of drug mannitol in AIDS/HIV patients
  - FACTOR: AIDS/HIV Status
  - FACTOR LEVELS: $k = 4$
    - AIDS - Full AIDS
    - ARC - AIDS-related conditions
    - HIV+ - HIV positive
    - HIV- - HIV negative

Notes

Handout: Examples

- **BATCH NUMBER**: bacteria level (per mill.) in different batches of milk
  - RESPONSE: Bacteria level count per million
  - FACTOR: Batch number
  - FACTOR LEVELS: 1,2,3,4,5 ($k = 5$)
- **TREATMENT GROUP**: Post-traumatic stress disorder (PTSD) score in different treatment groups.
  - RESPONSE: PTSD score
  - FACTOR: Therapeutic treatment method
  - FACTOR LEVELS: $k = 4$
    - SIT - "Stress Innoculation Therapy"
    - RE - "Relive Experience"
    - SC - "Standard Counselling"
    - WL - "Waiting List" (Control)

Notes

Example (PTSD Example (see handout))

$n = 45, k = 4$.

F-statistic	$F = 3.046$
Critical Value	$F_{0.05}(3, 41) \doteq 2.84$
	$F_{0.025}(3, 41) \doteq 3.46$
	$F_{0.01}(3, 41) \doteq 4.31$

Tables give $F_\alpha(3, 40)$.

$$\Longrightarrow \text{Reject } H_0 \text{ at } \alpha = 0.05 \quad (p = 0.039).$$

BUT Levene's Test suggests that the assumption of equal variances is **NOT** valid.

Notes

Why do we need the three assumptions ?

- independence
- Normality
- equal variances

- so that we can predict (under H_0) that

$$F \sim \text{Fisher-F}(k-1, n-k)$$

and complete the test (compute p-values and the rejection region).

But our hypothesis of interest is

$$H_0 \ : \text{ No difference between treatments}$$

Notes

Under this hypothesis, the treatment labels

SHOULD NOT MATTER !

i.e. we should be able to exchange the labels, and not notice any major difference in the test statistic.

This leads us to consider **permutation** or **randomization** tests.

i.e. we compute the test statistic for all possible relabellings consistent with H_0, retaining the group sample sizes, and use these values to compute the rejection region.

Notes

Randomization/Permutation Tests

Suppose that there are N possible relabellings that give rise to test statistics

$$F_1, F_2, \ldots, F_N$$

Then the rejection region for significance level α is the interval to the right of

$$N(1-\alpha)\text{th largest of the values } F_1, F_2, \ldots, F_N$$

and the p-value is

$$\frac{\text{Number of F}_1, F_2, \ldots, F_N \geq F}{N}$$

where

$$F = \frac{MST}{MSE}$$

is the true test statistic.

Notes

If the group sample sizes are $n_1, n_2, \ldots, n_k$ then

$$N = \frac{n!}{n_1! n_2! \ldots n_k!}$$

where

$$n! = n(n-1)(n-2)\ldots 3.2.1$$

("n factorial") - potentially very large.

Notes

Notes

Example (PTSD Example)

$$k = 4, n = 45 \qquad (n_1 = 14, n_2 = 10, n_3 = 11, n_4 = 10)$$

There are

$$\frac{45!}{14!10!11!10!} = 2.610 \times 10^{24}$$

possible relabellings: a very big number.

We compute $F = \frac{MST}{MSE}$ for each relabelling. For the real data, $F = 3.046$.

Notes

Example (PTSD Example (continued))

Using this approach, we compute for $\alpha = 0.05$

$$\begin{aligned} \text{CRITICAL VALUE} &: C_R = 2.844 \\ p\text{-VALUE} &: p = 0.040 \end{aligned}$$

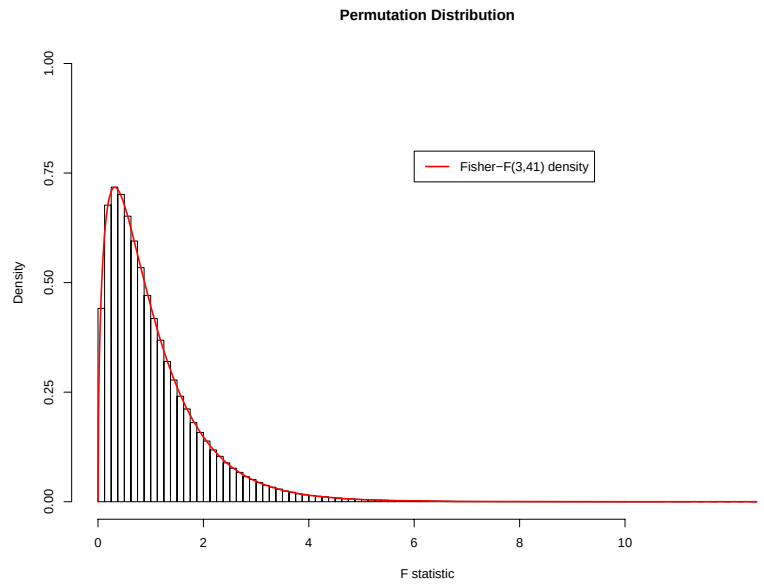
Compare this with the ANOVA F-test values

$$\begin{aligned} \text{CRITICAL VALUE} &: C_R = 2.833 \\ p\text{-VALUE} &: p = 0.039 \end{aligned}$$

(using the Fisher-F(3,41) distribution.

Thus we obtain virtually identical results; **but the randomization test does not need the assumptions of normality or equal variances.**

Notes



Notes

Example (PTSD Example (continued))

Thus the null hypothesis (of equal means) is

REJECTED

under both procedures at the $\alpha = 0.05$ significance level.

In this case, the computations give similar conclusions. Here the truth or otherwise of the normality/equal variance assumptions **does not matter**.

Notes

Final Note on ANOVA F-test for a CRD

If $k = 2$, consider $F = MST/MSE$;

$$\begin{aligned} MST &= \frac{1}{k-1}\sum_{i=1}^{k} n_i(\overline{x}_i - \overline{x})^2 = n_1(\overline{x}_1 - \overline{x})^2 + n_2(\overline{x}_2 - \overline{x})^2 \\ &= \frac{n_1 n_2}{n_1 + n_2}(\overline{x}_1 - \overline{x}_2)^2 \end{aligned}$$

$$\begin{aligned} MSE &= \frac{1}{n-k}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij} - \overline{x}_i)^2 = s_P^2 \\ &= \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1 + n_2 - 2} \end{aligned}$$

Notes

Therefore

$$F = \frac{\left(\frac{n_1 n_2}{n_1 + n_2}\right)(\overline{x}_1 - \overline{x}_2)^2}{s_P^2} = \left(\frac{(\overline{x}_1 - \overline{x}_2)}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}\right)^2$$

Thus $F = t^2$, where t is the two-sample t-test statistic.

Thus if $k = 2$, the ANOVA F-test and the two sample t-test are **EQUIVALENT**

$$\begin{aligned} t &\sim \text{Student-t}(n-2) \\ F &\sim \text{Fisher-F}(1, n-2) \end{aligned}$$

and we must get the same conclusion (to reject H_0 or otherwise) using either statistic.

Notes

Summary

If the assumptions

- independence (holds by design in a CRD)
- Normal populations
- equal variances

hold, use

ANOVA F-test

If the assumptions **do not hold**

- use Randomization/Permutation test
- use Non-parametric test (see Section 3)

Notes

1.3 Multiple Comparison of Means

If the ANOVA F-test null hypothesis

$$H_0 : \mu_1 = \cdots = \mu_k$$

is **rejected**, then it is of interest to discover which of the means are different. For k groups, there are $c = k(k-1)/2$ pairs of group means that can be compared.

Consider a "family" of hypothesis tests - a collection of tests of different hypotheses carried out independently on different data sets. For each test in the family, we consider testing the hypothesis at significance level α.

Notes

Notation

Label the tests $i = 1, \ldots, c$, and for each i, label

- the null hypotheses H_{0i}
- the test statistics T_i
- the rejection regions $\mathcal{R}_i$

that are potentially different for each i.

We specify for each i,

$$\alpha = P[T_i \in \mathcal{R}_i | H_{0i} \text{ is } \mathbf{TRUE}]$$

which implicitly defines $\mathcal{R}_i$. Note that α is the

"Test Type-I Error Rate" or "Comparisonwise Error Rate"

Now consider the results of all tests in the family; what is the "Familywise" Type-I error rate ?

Notes

Using the laws of probability

$$P[T_i \in \mathcal{R}_i | H_{0i} \text{ is } \mathbf{TRUE}] = \alpha$$

means that

$$P[T_i \notin \mathcal{R}_i | H_{0i} \text{ is } \mathbf{TRUE}] = 1 - \alpha$$

giving the probability that the test **does not reject** H_{0i}, if H_{0i} is in fact true, is $1 - \alpha$.

Now we consider all tests together;

$$P[\text{Each } T_i \notin \mathcal{R}_i | \text{Each } H_{0i} \text{ is } \mathbf{TRUE}] = (1 - \alpha)^c$$

This is the probability that each test results in the null hypothesis **not** being rejected, that is, the probability that we **never** commit a Type-I error.

Notes

Therefore the probability of **at least one** Type-I error is

$$\alpha_F = 1 - (1 - \alpha)^c$$

α_F is the **Familywise Error Rate**.

	$\alpha = 0.05$	$\alpha = 0.01$
c	α_F	α_F
5	0.226	0.049
10	0.401	0.096
50	0.923	0.395
100	0.994	0.634

Therefore, whenever we carry out a "family" of tests, we should not use the traditional choices of $\alpha = 0.05$ or 0.01 on each test.

Notes

The Bonferroni Method

To fix $\alpha_F = 0.05$, say, we need to use α on each test where

$$\alpha_F = 1 - (1-\alpha)^c \quad \Longleftrightarrow \quad \alpha = 1 - (1-\alpha_F)^{1/c}$$

For example, if $\alpha_F = 0.05$ and $c = 10$, use

$$\alpha = 1 - (1 - 0.05)^{1/10} = 0.0051$$

It can be shown that

$$1 - (1-\alpha)^c \approx c\alpha$$

Therefore, if α_F is the required **familywise error rate**, we must set the **comparisonwise error** rate to be $\alpha = \alpha_F/c$.

α_F/c is known as the Bonferroni Correction.

Notes

Confidence Intervals

For the $k = 2$ group comparison of means, a $100(1-\alpha)\%$ confidence interval for $\mu_1 - \mu_2$ is

$$(\overline{x}_1 - \overline{x}_2) \pm t_{\alpha/2}(n_1 + n_2 - 2) s_P \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

where $t_\alpha(\nu_1)$ is the $1-\alpha$ probability point of the Student-t distribution with ν_1 degrees of freedom (under the assumptions of independence, Normality and equal group variances).

If we move to a family of c tests, to get simultaneous confidence intervals for the differences in means $\mu_i - \mu_j$ for all pairs of i and j, we should adjust α to α_F when computing the $100(1-\alpha)\%$ confidence interval.

Notes

There are many different methods for correcting the confidence interval for use in different experimental situations. For example

- *planned* comparisons $\mu_1 = \mu_3$, $\mu_7 = \mu_{10}$ etc.
- *all* comparisons

Three methods are recommended:

- **Bonferroni's** Method
- **Tukey's** HSD Method for all pairwise comparisons of group means;
- **Scheffé's** Method for all possible contrast comparisons.

Having selected a multiple comparison correction method, we compute simultaneous confidence intervals for each comparison of means, and identify

- which means are significantly different
- the ranking of differences $\mu_i - \mu_j$ in terms of magnitude.

Notes

1.4 Randomized Block Designs

A **randomized block design** used **matched** experimental units organized into sets known as **blocks** and assigns one member from the set to each treatment.

For k treatments

1. Compile b blocks of k experimental units, with each block comprising units that are similar.
2. Assign one unit from each block to each treatment at random.

Then there are a total of $n = bk$ measured responses.

Notes

We wish to **compare treatments** whilst acknowledging that there may be **differences between the blocks**.

That is, the observed variation is due to

TREATMENTS and BLOCKS and ERROR

rather than merely

TREATMENTS and ERROR

as in the CRD.

Notes

Example (SAT Scores)

- **Response** : Measured SAT Score
- **Factor** : Sex
- **Factor-levels** : $k = 2$ (Female/Male)
- **Blocks** : $b = 5$ (Previous GPA, within same school)

i.e. $k = 2, b = 5 \therefore n = 10$.

	Block	Female SAT	Male SAT
1	A: 2.75	540	530
2	B: 3.00	570	550
3	C: 3.25	590	580
4	D: 3.50	640	620
5	E: 3.75	690	690

Notes

Example (SAT Scores (continued))

This design recognizes that GPA score and school are likely to explain some variation in SAT Score, **but that neither is directly related to the "treatment" of interest (SEX - Female/Male).**

i.e. the **blocking** variable removes systematic variation in response that is not of primary interest.

We pick one Female and one Male in each school/GPA category, and pair them.

Notes

Example (Treatment for Hypertension)

- **Response** : Blood Pressure (mgHg)
- **Factor** : Drug Type
- **Factor-levels** : $k = 3$ (Drug 1, Drug 2, Drug 3)
- **Blocks** : $b = 4$ Age/Sex combinations
  - Female/Under 50
  - Male/Under 50
  - Female/Over 50
  - Male/Over 50

i.e. $k = 3, b = 4 \therefore n = 12$.

Notes

Testing for Equal Treatment Means

An ANOVA F-test can be constructed for a RBD. Let

- $i = 1, \ldots, k$ index **treatments**
- $j = 1, \ldots, b$ index **blocks**

i.e. x_{ij} is the response for the ith treatment in the jth block.
Let

- $\overline{x}_i$ be the ith **treatment** mean
- $\overline{x_j^{(\mathrm{B})}}$ be the jth **block** mean
- $\overline{x}$ be the **overall** mean

Notes

Let

$$\begin{aligned} SST &= \sum_{i=1}^{k} b(\overline{x}_i - \overline{x})^2 \\ SSB &= \sum_{j=1}^{b} k(\overline{x_j^{(\mathrm{B})}} - \overline{x})^2 \\ SS &= \sum_{i=1}^{k}\sum_{j=1}^{b} (x_{ij} - \overline{x})^2 \end{aligned}$$

SST: Sum of Squares for **Treatments**
SSB: Sum of Squares for **Blocks**
SS: **Total** Sum of Squares

Notes

Finally

$$SS = SST + SSB + SSE \quad \therefore \quad SSE = SS - SST - SSB$$

SSE: Sum of Squares for **Errors**

Test statistic is

$$F = \frac{MST}{MSE}$$

where

$$MST = \frac{SST}{k-1} \qquad MSE = \frac{SSE}{n-b-k+1}$$

Notes

ANOVA F-test to compare treatment means in a randomized block design

Theorem (ANOVA F-test for a RBD)

To test

H_0 : $\mu_1 = \cdots = \mu_k$

H_a : *At least one pair of treatment means different.*

use the test statistic

$$F = \frac{MST}{MSE}$$

*If H_0 is **TRUE***

$$F \sim \textit{Fisher-F}(k-1, n-b-k+1)$$

- this defines the rejection region for significance level α, and the p-value, in the usual way.

Notes

Assumptions:

1. Experimental units (between blocks) are independent, and treatments are allocated at random (within blocks).
2. Normality
3. bk block/treatment combinations correspond to populations with equal variances.

ANOVA Table

SOURCE	DF	SS	MS	F
TMTS	$k-1$	SST	MST	F = MST/MSE
BLOCKS	$b-1$	SSB	MSB	
ERROR	$n-k-b+1$	SSE	MSE	
TOTAL	$n-1$	SS		

Notes

After the ANOVA test is complete, and the hypothesis

$$H_0 : \mu_1 = \cdots = \mu_k$$

is **rejected**, we can proceed with the "post-hoc" tests of hypotheses $\mu_i = \mu_j$ for $i \neq j$.

Notes:

1. In a RBD, it is not (in general) possible to estimate individual treatment means, that is, $\overline{x}_i$ does not estimate μ_i as it is an average across blocks, which are believed to be different in terms of response.

Notes

2. **Testing the Block Means** However, we can test whether the block means $\mu_1^{(B)}, \ldots, \mu_b^{(B)}$ are significantly different. For

$$H_0 : \mu_1^{(B)} = \cdots = \mu_b^{(B)}$$

we use the F statistic

$$F = \frac{MSB}{MSE}$$

where

$$MSB = \frac{SSB}{b-1}$$

If H_0 is **TRUE**

$$F \sim \text{Fisher-F}(b-1, n-k-b+1)$$

That is, we treat the blocks as levels of another factor, and test to see whether this factor affects response.

Notes

Example (Soil Analysis (see handout))

Results of two ANOVA F-tests:

Test of	F	p	Conclusion
SOLVENT	0.673	0.585	No Difference
SOIL	10.568	0.001	Difference

Here SOLVENT is the **treatment** variable, SOIL is the **blocking** variable.

3. Remember to check the assumptions (independence, normality, equal variances in each treatment/block combination)

   Equal variances may be hard to check as we only have one observation per treatment/block comparison.

Notes

Comment: The "sum of squares" decompositions

$$CRD: SS = [SST] + SSE$$

$$RBD: SS = [SST + SSB] + SSE$$

are both of the form

$$\begin{array}{ccccc} \text{TOTAL} & = & \text{SYSTEMATIC} & + & \text{RANDOM} \\ \text{VARIATION} & & \text{VARIATION} & & \text{VARIATION} \end{array}$$

"SYSTEMATIC" $\begin{cases} \text{For the CRD:} & SST \\ \text{For the RBD:} & SST + SSB \end{cases}$

"RANDOM" For both: SSE

Notes

We have studied the

Randomized **Complete** Block Design

where each block/treatment combination has one experimental unit.

An incomplete design could also be considered, where some block/treatment combinations are omitted. However, this design does not lead to straightforward ANOVA analysis.

Notes

1.5 Factorial Experiments

Designs studied so far:

- CRD - one factor
- RBD - one factor, plus one blocking variable, so two factors in total, where one (the blocking variable) is a known source of systematic variation.

However, in the RBD, we must assume that the treatments behave in a similar way across blocks.

Notes

Let i index treatments ($1 \leq i \leq k$) and consider block j, and two treatment (factor levels) i_1 and i_2.

In an RBD, we assume that

$$\mathbb{E}[X_{i_1 j} - X_{i_2 j}] = \mu_{i_1} - \mu_{i_2}$$

which does **NOT** depend on j.

That is, the expected difference in response due to the two treatments does not depend on the block.

But perhaps the difference **does** depend on block; perhaps we have **INTERACTION**.

In the current RBD, we do not have enough data to look for this. We now seek to extend the RBD to allow for tests for interaction; we do this by using **replication**.

Notes

RBD with Balanced Replication

Suppose we have r observations per block/treatment combination (termed *replicates*), so that we have $n = bkr$ experimental units in total.

Balanced designs have equal numbers of replicates in each block/treatment combination.

In this design, all the quantities

$$SST, SSB, SSE, SS$$
$$MST, MSB, MSE$$

can be defined, and an ANOVA F-test can be carried out - the only difference is that $n = bkr$.

Notes

- Sum of Squares for Treatments (SST)

$$SST = \sum_{i=1}^{k} br(\overline{x}_i - \overline{x})^2$$

- Sum of Squares for Blocks (SSB)

$$SSB = \sum_{j=1}^{b} kr(\overline{x_j^{(\mathrm{B})}} - \overline{x})^2$$

- Overall Sum of Squares (SS)

$$SS = \sum_{i=1}^{k}\sum_{j=1}^{b}\sum_{t=1}^{r}(x_{ijt} - \overline{x})^2$$

and $SSE = SS - SST - SSB$

Third index t indexes the replicates.

Notes

The RBD with replication does allow the investigation of interaction. The new test is based on the decomposition

$$SS = SST + SSB + SSI + SSE$$

where SSI is the sum of squares for Interaction.

We have SST, SSB and SS as before, and

$$SSI = \sum_{i=1}^{k}\sum_{j=1}^{b} r(\overline{x}_{ij} - \overline{x}_i - \overline{x_j^{(\mathrm{B})}} + \overline{x})^2$$

where

$$\overline{x}_{ij} = \frac{1}{r}\sum_{t=1}^{r} x_{ijt} \qquad i = 1, \ldots, k,\ j = 1, \ldots, b$$

is the sample mean for replicates in (i, j)th treatment/block combination.

Notes

Testing in the RBD with Replication

The three F statistics

$$F = \frac{MST}{MSE} \quad F = \frac{MSB}{MSE} \quad F = \frac{MSI}{MSE}$$

can be used to test for significant Treatment, Block and Interaction effects respectively.

Now

$$MSE = \frac{SSE}{\text{Error d.f.}}$$

But what is "Error d.f." ? It is a constant that dictates how large SSE should be on average.

The general rule for computing the error d.f. for any model is

$$\text{Error d.f.} = n - p$$

where n is the total **sample size** and p is the total **number of parameters fitted**.

Notes

How many parameters do we fit ?

- No Interaction

$$p = 1 + (b-1) + (k-1)$$

that is, the overall mean μ, plus the $b-1$ differences from μ due to the blocks, plus the $k-1$ differences from μ due to the treatments.

- Interaction

$$p = bk$$

that is, one parameter in each cell of the two-way table of blocks by treatments.

Notes

Thus

- No Interaction

$$p = 1 + (b-1) + (k-1) = b + k - 1$$

parameters, so

$$\text{Error d.f.} = n - p = n - b - k + 1$$

- Interaction: we fit $p = bk$ parameters, so

$$\text{Error d.f.} = n - p = n - bk$$

Notes

It transpires that if

$$MSI = \frac{SSI}{(b-1)(k-1)}$$

is the Mean Square for Interaction, then

$$F = \frac{MSI}{MSE}$$

yields a test statistic suitable for testing interaction. If there is **no interaction**, then

$$F \sim \text{Fisher-F}((b-1)(k-1), n-bk)$$

where $n = bkr$.

Why $(b-1)(k-1)$? This is the number of **extra** parameters we fit to include the interaction.

Notes

For the CRD:

H_a		H_0
FULL MODEL	$\longrightarrow$	NULL MODEL
k parameters	$\longrightarrow$	1 parameter

so there are $(k-1)$ extra parameters, and SST varies on $(k-1)$ degrees of freedom.

Notes

For the RBD: the (i, j)th treatment/block combination has mean

$$\mu_i + \mu_j^B$$

so for testing for a TREATMENT effect

H_a		H_0
FULL MODEL	$\longrightarrow$	NULL MODEL
k parameters	$\longrightarrow$	1 parameter

so there are $(k-1)$ extra parameters, and SST varies on $(k-1)$ degrees of freedom.

$$\mu_1, \ldots, \mu_k \longrightarrow \mu$$

Notes

For testing for a BLOCK effect

H_a		H_0
FULL MODEL	$\longrightarrow$	NULL MODEL
b parameters	$\longrightarrow$	1 parameter

so there are $(b-1)$ extra parameters, and SSB varies on $(b-1)$ degrees of freedom.

$$\mu_1^{(B)}, \ldots, \mu_k^{(B)} \longrightarrow \mu^{(B)}$$

These models and tests can be fitted and carried out even if we do not have replication.

Notes

With replication, we can investigate the interaction, that is the model where the (i, j)th treatment/block combination has mean

$$\mu_i + \mu_j^B + \mu_{ij}$$

rather than the model where

$$\mu_i + \mu_j^B$$

that is, we wish to test

$$\begin{array}{lcll} H_0 & : & \mu_{ij} = 0 & \text{for all } i \text{ and } j \\ H_a & : & \mu_{ij} \neq 0 & \end{array}$$

Notes

In the **full interaction** model: we fit bk parameters

In the **restricted**, **no interaction** model: we fit

$$1 + (b-1) + (k-1) = b + k - 1$$

parameters. Therefore the differences is

$$bk - (b + k - 1) = bk - b - k + 1 = (b-1)(k-1)$$

and SSI varies on $(b-1)(k-1)$ degrees of freedom.

Notes

ANOVA Table

SOURCE	DF	SS	MS	F
TMTS	$k-1$	SST	MST	F_T
BLOCKS	$b-1$	SSB	MSB	F_B
INTERACTION	$(b-1)(k-1)$	SSI	MSI	F_I
ERROR	$(n-bk)$	SSE	MSE	
TOTAL	$n-1$	SS		

where

$$MST = \frac{SST}{k-1} \qquad MSB = \frac{SSB}{b-1}$$

$$MSI = \frac{SSI}{(b-1)(k-1)} \qquad MSE = \frac{SSE}{n-bk}$$

and

$$F_T = \frac{MST}{MSE} \qquad F_B = \frac{MSB}{MSE} \qquad F_I = \frac{MSI}{MSE}$$

Notes

Example: Batteries Data (see handout)

```
> summary(aov(life ~ temp+material+temp:material,
                      data=Batteries))

             Df Sum Sq Mean Sq F value  Pr(>F)
temp          2  39083   19542 29.344 1.69e-07 ***
material      2  10633    5317  7.983  0.00189 **
temp:material 4   9438    2359  3.543  0.01897 *
Residuals    27  17981     666
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1
```

For $\alpha = 0.05$, there is a significant **temp** effect ($p < 0.001$), and a significant **material** effect ($p < 0.002$), and a significant interaction ($p < 0.019$)

Notes

NB: If we do not have replication, we CANNOT fit the interaction. Recall that

$$\text{Error d.f.} = n - bk$$

but if $r = 1$, $n = rbk = bk$, so the error d.f. is zero.

In fact, $SSE = 0$ also, so the MSE is not defined.

Notes

We now study multifactor designs, to assess the effects and interactions of several factors simultaneously.

We consider all possible combinations of

$$\begin{array}{lll} \text{FACTOR A} & \text{with} & a \text{ levels} \\ \text{FACTOR B} & \text{with} & b \text{ levels} \\ \text{FACTOR C} & \text{with} & c \text{ levels} \\ & \vdots & \end{array}$$

to define the treatments in a factorial design.

Notes

Factorial Experiments

A complete factorial experiment is one in which every combination of a number of factors is utilized.

i.e. the number of treatments is equal to the total number of factor-level combinations.

We focus on two factor experiments

FACTOR A	with	a levels
FACTOR B	with	b levels

so there are ab treatments in total.

Notes

A two-way layout with $a = 3$ and $b = 5$.

		Factor B				
		1	2	3	4	5
Factor A	1					
	2					
	3					

This design is very similar to the RBD, but now the second factor is not a blocking factor;

- that is, the ab treatment populations are constructed independently from the same base population, or from populations not necessarily believed to be systematically different.
- individuals from the same base population are assigned at random to one of the ab treatments.

Notes

In this design we can study the effect of Factor A and Factor B (**main effects**) as well as the **interaction** provided we have (balanced) replication.

We construct ANOVA F-tests based on the decomposition

$$SS = SST_A + SST_B + SSI_{AB} + SSE$$

- Sum of Squares for Treatments due to factor A (SST_A)

$$SST_A = \sum_{i=1}^{a} br(\overline{x}_{i.} - \overline{x}_{..})^2$$

- Sum of Squares for Treatments due to factor B (SST_B)

$$SST_B = \sum_{j=1}^{b} ar(\overline{x}_{.j} - \overline{x}_{..})^2$$

- Sum of Squares for Interaction (SSI_AB)

$$SSI_{AB} = \sum_{i=1}^{a}\sum_{j=1}^{b} r(\overline{x}_{ij} - \overline{x}_{i.} - \overline{x}_{.j} + \overline{x}_{..})^2$$

Notes

New notation:

- sample mean for Factor A level i

$$\overline{x}_{i.} = \frac{1}{br}\sum_{j=1}^{b}\sum_{t=1}^{r} x_{ijt} \qquad i = 1, \ldots, a$$

- sample mean for Factor B level j

$$\overline{x}_{.j} = \frac{1}{ar}\sum_{i=1}^{a}\sum_{t=1}^{r} x_{ijt} \qquad j = 1, \ldots, b$$

- sample mean for replicates in (i, j)th factor combination

$$\overline{x}_{ij} = \frac{1}{r}\sum_{t=1}^{r} x_{ijt} \qquad i = 1, \ldots, a,\ j = 1, \ldots, b$$

- overall sample mean

$$\overline{x}_{..} = \frac{1}{n}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{t=1}^{r} x_{ijt}$$

Notes

These allow computation of

$$SST_A, SST_B, SSI_{AB}, SS, SSE$$

$$MST_A, MST_B, MSI_{AB}, MSE$$

using the degrees of freedom identical to those in the RBD with replication.

Tests for

- significant effect for Factor A
- significant effect for Factor B
- significant interaction

will be carried out as before using an ANOVA table.

Notes

ANOVA Table

SOURCE	DF	SS	MS	F
FACTOR A	$a-1$	SST_A	MST_A	F_A
FACTOR B	$b-1$	SST_B	MST_B	F_B
INTERACTION	$(a-1)(b-1)$	SSI_{AB}	MSI_{AB}	F_{AB}
ERROR	$(n-ab)$	SSE	MSE	
TOTAL	$n-1$	SS		

If Factor A is not influential (H_0 specifying no difference between responses at different levels of factor A), then

$$F_A \sim \text{Fisher-F}(a-1, n-ab)$$

Similarly,

No effect of Factor B : $F_B \sim \text{Fisher-F}(b-1, n-ab)$

No Interaction : $F_{AB} \sim \text{Fisher-F}((a-1)(b-1), n-ab)$

SEE EXAMPLES HANDOUT

Notes

Note: For two factors A and B, the **main effects plus interaction** model can be written

$$A + B + A.B$$

whereas the **main effects only** can be written

$$A + B$$

The models

$$A + A.B \qquad B + A.B$$

do not make sense.

Notes

For a two factor design, the only models that should be considered and or reported are

MODEL	FACTOR	INTERACTION
NULL	NONE	NONE
A	A	NONE
B	B	NONE
A+B	A,B	NONE
A+B+A.B	A,B	YES

that is, if the interaction is significant, the only model you should report is

$$A + B + A.B$$

Notes

Note: ANOVA analysis for the RBD and FD (both with replication) are identical. The only difference lies in the interpretation of the factors

- RBD: one blocking, one treatment factor
- FD: two treatment factors

"Blocking" factors are known or strongly believed to have a significant effect on the response.

Notes

Estimating Effect Size

In multifactor designs, parameter estimation can be carried out in different parameterizations

For the CRD (one-way layout):

- Natural parameters: $\mu_1, \ldots, \mu_k$
- Contrast parameters: $\beta_1, \beta_2, \ldots, \beta_{k-1}$ where

$$\beta_1 = \mu_1 \qquad \beta_i = \mu_i - \mu_1, \quad i = 1, \ldots, k-1$$

that is, differences from baseline.

For the two-factor designs (RBD/FD): In the two-way layout, with cells (i, j), $i = 1, \ldots, a$, $j = 1, \ldots, b$. The cell means are m_{ij}, where

$$m_{ij} = \mu_{i\cdot} + \mu_{\cdot j} + \mu_{ij}$$

where $\mu_{i\cdot}$ gives the Factor A contribution, $\mu_{\cdot j}$ gives the Factor B contribution, and μ_{ij} gives the interaction.

Notes

The default parameterization used by R is the contrast parameterization:

$$\begin{aligned} m_{ij} &= \beta_1 && i=1, j=1 \\ &= \beta_1 + \beta_i^{(A)} && i=2,\ldots,a, j=1 \\ &= \beta_1 + \beta_j^{(B)} && i=1, j=2,\ldots,b \\ &= \beta_1 + \beta_i^{(A)} + \beta_j^{(B)} + \gamma_{ij}^{(AB)} && \begin{array}{l} i=2,\ldots,a \\ j=2,\ldots,b \end{array} \end{aligned}$$

where

$$\begin{aligned} \beta_i^{(A)} &: \text{contrasts for factor A} \\ \beta_j^{(B)} &: \text{contrasts for factor B} \\ \gamma_{ij}^{(AB)} &: \text{interaction} \end{aligned}$$

Notes

By default, R takes the first level of each factor as the baseline level, and looks at differences compared to this baseline.

The ab parameters are

β_1	1
$\beta_2^{(A)}, \ldots, \beta_a^{(A)}$	$(a-1)$
$\beta_2^{(B)}, \ldots, \beta_b^{(B)}$	$(b-1)$
$\gamma_{ij}^{(AB)}, i=2,\ldots,a, j=2,\ldots,b$	$(a-1)(b-1)$
Total	ab

Notes

For example: $a = 3, b = 4$.

	Factor B			
Factor A	1	2	3	4
1	β_1	$\beta_1 + \beta_1^{(B)}$	$\beta_1 + \beta_2^{(B)}$	$\beta_1 + \beta_3^{(B)}$
2	$\beta_1 + \beta_1^{(A)}$	①	②	③
3	$\beta_1 + \beta_2^{(A)}$	④	⑤	⑥

where

$$① = \beta_1 + \beta_2^{(A)} + \beta_2^{(B)} + \gamma_{22}^{(AB)}$$

$$⑥ = \beta_1 + \beta_3^{(A)} + \beta_4^{(B)} + \gamma_{34}^{(AB)}$$

and so on.

Notes

Estimation is still straightforward:

PARAMETER	ESTIMATE
β_1	$\overline{x}_{11}$
$\beta_i^{(A)}$	$\overline{x}_{i.} - \overline{x}_{11}$
$\beta_j^{(B)}$	$\overline{x}_{.j} - \overline{x}_{11}$
$\gamma_{ij}^{(AB)}$	$\overline{x}_{ij} - \overline{x}_{i.} - \overline{x}_{.j} + \overline{x}_{11}$

for $i = 2, \ldots, a$, $j = 2, \ldots, b$.

Other contrast parameterizations can be used.

Notes

Example

For example: $a = 4, b = 3$.

- Factor A: levels $1, 2, \ldots, a$
- Factor B: levels $1, 2, \ldots, b$

Most complicated model: Main Effects plus Interaction

$$\text{A} + \text{B} + \text{A.B}$$

that is, we have

- a baseline mean: β_1
- an effect for each level of Factor A: $\beta_i^{(A)}$
- an effect for each level of Factor B: $\beta_j^{(B)}$
- an interaction that modifies the effect of changing levels of Factor A at each level of Factor B: $\gamma_{ij}^{(AB)}$

Notes

Parameterization: contrasts from the baseline group.

In R, the baseline group is the one where Factor A has level 1 and Factor B has level 1, but this choice is **arbitrary**; changing this assumption should have no effect on the results we obtain.

Thus we adopt the following modelling strategy:

- Establish a baseline
- Look for changes from baseline introduced by Factor A
- Look for changes from baseline introduced by Factor B
- Look for changes from baseline introduced by Factor A and Factor B **additively**, so that the effect of changing the level of Factor A is **identical** in each level of Factor B, and *vice versa*).
- Look for changes from baseline introduced by Factor A and Factor B **additively with interaction**, so that the effect of changing the level of Factor A is **different** in each level of Factor B, and *vice versa*).

Notes

Two-way table: 4×3

.

Factor B

Factor A	1	2	3
1			
2			
3			
4			

Notes

Null Model : Baseline Mean Only

Null Model: cell entries are means for data for each treatment.

Factor B

Factor A	1	2	3
1	β_1	β_1	β_1
2	β_1	β_1	β_1
3	β_1	β_1	β_1
4	β_1	β_1	β_1

Notes

Effect of Factor A only

Main Effect Only: A

Factor B

Factor A	1	2	3
1	β_1	β_1	β_1
2	$\beta_1 + \beta_2^{(A)}$	$\beta_1 + \beta_2^{(A)}$	$\beta_1 + \beta_2^{(A)}$
3	$\beta_1 + \beta_3^{(A)}$	$\beta_1 + \beta_3^{(A)}$	$\beta_1 + \beta_3^{(A)}$
4	$\beta_1 + \beta_4^{(A)}$	$\beta_1 + \beta_4^{(A)}$	$\beta_1 + \beta_4^{(A)}$

Notes

Effect of Factor B only

Main Effect Only: B

Factor B

Factor A	1	2	3
1	β_1	$\beta_1 + \beta_2^{(B)}$	$\beta_1 + \beta_3^{(B)}$
2	β_1	$\beta_1 + \beta_2^{(B)}$	$\beta_1 + \beta_3^{(B)}$
3	β_1	$\beta_1 + \beta_2^{(B)}$	$\beta_1 + \beta_3^{(B)}$
4	β_1	$\beta_1 + \beta_2^{(B)}$	$\beta_1 + \beta_3^{(B)}$

Notes

Effect of Factor A plus Effect of Factor B

Main Effects Only: A + B

Factor B

Factor A	1	2	3
1	β_1	$\beta_1 + \beta_2^{(B)}$	$\beta_1 + \beta_3^{(B)}$
2	$\beta_1 + \beta_2^{(A)}$	$\beta_1 + \beta_2^{(A)} + \beta_2^{(B)}$	$\beta_1 + \beta_2^{(A)} + \beta_3^{(B)}$
3	$\beta_1 + \beta_3^{(A)}$	$\beta_1 + \beta_3^{(A)} + \beta_2^{(B)}$	$\beta_1 + \beta_3^{(A)} + \beta_3^{(B)}$
4	$\beta_1 + \beta_4^{(A)}$	$\beta_1 + \beta_4^{(A)} + \beta_2^{(B)}$	$\beta_1 + \beta_4^{(A)} + \beta_3^{(B)}$

Notes

Main effects plus Interaction between A and B

Main Effects Plus Interaction: A + B + A.B

Factor B

Factor A	1	2	3
1	β_1	$\beta_1 + \beta_2^{(B)}$	$\beta_1 + \beta_3^{(B)}$
2	$\beta_1 + \beta_2^{(A)}$	$\beta_1 + \beta_2^{(A)} + \beta_2^{(B)} + \gamma_{22}^{(AB)}$	$\beta_1 + \beta_2^{(A)} + \beta_3^{(B)} + \gamma_{23}^{(AB)}$
3	$\beta_1 + \beta_3^{(A)}$	$\beta_1 + \beta_3^{(A)} + \beta_2^{(B)} + \gamma_{32}^{(AB)}$	$\beta_1 + \beta_3^{(A)} + \beta_3^{(B)} + \gamma_{33}^{(AB)}$
4	$\beta_1 + \beta_4^{(A)}$	$\beta_1 + \beta_4^{(A)} + \beta_2^{(B)} + \gamma_{42}^{(AB)}$	$\beta_1 + \beta_4^{(A)} + \beta_3^{(B)} + \gamma_{43}^{(AB)}$

Notes

Null Model : Baseline Mean Only

Null Model: cell entries are means for data for each treatment.

Factor B

Factor A	1	2	3
1	10	10	10
2	10	10	10
3	10	10	10
4	10	10	10

Notes

Effect of Factor A only

Main Effect Only: A

Factor B

Factor A	1	2	3
1	10	10	10
2	10 + 2	10 + 2	10 + 2
3	10 + 5	10 + 5	10 + 5
4	10 + 8	10 + 8	10 + 8

Notes

Effect of Factor B only

Main Effect Only: B

Factor B

Factor A	1	2	3
1	10	10 + (-1)	10 + (-4)
2	10	10 + (-1)	10 + (-4)
3	10	10 + (-1)	10 + (-4)
4	10	10 + (-1)	10 + (-4)

Notes

Main effects of factors A and B only

Main Effects Only: A + B

Factor B

Factor A	1	2	3
1	10	10 + (-1)	10 + (-4)
2	10 + 2	10 + 2 + (-1)	10 + 2 + (-4)
3	10 + 5	10 + 5 + (-1)	10 + 5 + (-4)
4	10 + 8	10 + 8 + (-1)	10 + 8 + (-4)

Notes

Main effects plus Interaction between A and B

Main Effects Plus Interaction: A + B + A.B

Factor B

Factor A	1	2	3
1	10	10 + (-1)	10 + (-4)
2	10 + 2	10 + 2 + (-1) + (-2)	10 + 2 + (-4) + 1
3	10 + 5	10 + 5 + (-1) + (-5)	10 + 5 + (-4) + 8
4	10 + 8	10 + 8 + (-1) + 2	10 + 8 + (-4) + 7

Notes

Q. Why are the following models

- A.B
- A + A.B
- B + A.B

not considered ?

A. Because they make specific and perhaps **unrealistic** assumptions about the data, and they imply that the levels of the factors are **not arbitrarily labelled**.

Notes

Q. Why are the following models

- A.B
- A + A.B
- B + A.B

not considered ?

A. Because they make specific and perhaps **unrealistic** assumptions about the data, and they imply that the levels of the factors are **not arbitrarily labelled**.

R will not fit such models, although it appears that it does !

Notes

Recall the definition of interaction:

- Variation in the effect of changing levels of one factor at the different levels of the other factor.
- For example, the effect on the response mean of moving from level 1 to level 2 for Factor B is **different** at different levels of Factor A.

Consider the model

A.B

this model implies that all parameters apart from the **baseline** and the **interaction** parameters are zero.

Notes

Interaction between A and B only

Interaction: A.B

Factor B

Factor A	1	2	3
1	β_1	$\beta_1 + 0$	$\beta_1 + 0$
2	$\beta_1 + 0$	$\beta_1 + 0 + 0 + \gamma_{22}^{(AB)}$	$\beta_1 + 0 + 0 + \gamma_{23}^{(AB)}$
3	$\beta_1 + 0$	$\beta_1 + 0 + 0 + \gamma_{32}^{(AB)}$	$\beta_1 + 0 + 0 + \gamma_{33}^{(AB)}$
4	$\beta_1 + 0$	$\beta_1 + 0 + 0 + \gamma_{42}^{(AB)}$	$\beta_1 + 0 + 0 + \gamma_{43}^{(AB)}$

Notes

In this set-up,

- for Factor A, Level 1: the effect of moving from Level 1 to Level 2 of factor B is **zero**
- for Factor A, Level 3: the effect of moving from Level 1 to Level 2 of factor B is $\gamma_{32}^{(AB)}$.

Therefore, there is a **fundamental difference** between the way that we regard the levels of Factor A.

Notes

Main Effect of A plus Interaction between A and B only

Main Effects of Factor A Plus Interaction: A + A.B

Factor B

Factor A	1	2	3
1	β_1	$\beta_1 + 0$	$\beta_1 + 0$
2	$\beta_1 + \beta_2^{(A)}$	$\beta_1 + \beta_2^{(A)} + 0 + \gamma_{22}^{(AB)}$	$\beta_1 + \beta_2^{(A)} + 0 + \gamma_{23}^{(AB)}$
3	$\beta_1 + \beta_3^{(A)}$	$\beta_1 + \beta_3^{(A)} + 0 + \gamma_{32}^{(AB)}$	$\beta_1 + \beta_3^{(A)} + 0 + \gamma_{33}^{(AB)}$
4	$\beta_1 + \beta_4^{(A)}$	$\beta_1 + \beta_4^{(A)} + 0 + \gamma_{42}^{(AB)}$	$\beta_1 + \beta_4^{(A)} + 0 + \gamma_{43}^{(AB)}$

Notes

In this set-up,

- for Factor A, Level 1: the effect of moving from Level 1 to Level 2 of Factor B is **zero**
- for Factor A, Level 3: the effect of moving from Level 1 to Level 2 of Factor B is $\gamma_{32}^{(AB)}$.

Notes

Therefore, there is a **fundamental difference** between the way that we regard the levels of Factor A. If we rearrange the labels of the levels of Factor A

we may get a different result.

Therefore, although it is possible **in general** to fit such models, it is no longer possible to talk of the effect of "Factor A".

Notes

How does R Handle Such Models ?

It is possible to use the models

A + A.B B + A.B A.B

in an ANOVA in R. For example, for the model A+A.B, we would use the specification

```
summary(aov(Y ~ A+A:B))
```


Notes

How does R Handle Such Models ?

This produces the usual ANOVA table, with terms including

Factor A

and

Factor A:Factor B

However, in fact the model

A + B + A.B

has been fitted !

- The results are just reported differently
- The terms B and A.B are reported together !

Notes

Example: Batteries Data

A - Material
B - Temperature
Model A + B + A.B

```
> summary(aov(life~material*temp,data=Batteries))
             Df Sum Sq Mean Sq F value   Pr(>F)
material      2  10633    5317   7.983  0.00189 **
temp          2  39083   19542  29.344 1.69e-07 ***
material:temp 4   9438    2359   3.543  0.01897 *
Residuals    27  17981     666
```

$$SS = SST_A + SST_B + SSI_{AB} + SSE$$

Notes

Example: Batteries Data

A - Material
B - Temperature
Model A + A.B

```
> summary(aov(life ~ material+material:temp,
                        data=Batteries))
              Df Sum Sq Mean Sq F value  Pr(>F)
material       2  10633    5317   7.983 0.00189 **
material:temp  6  48521    8087  12.143 1.35e-06 ***
Residuals     27  17981     666
```

$$SS = SST_A + SSI_{B:AB} + SSE$$

where

$$SSI_{B:AB} = SST_B + SSI_{AB}$$

Notes

Example: Batteries Data

A - Material
B - Temperature
Model B + A.B

```
> summary(aov(life ~ temp+material:temp,
                      data=Batteries))
              Df Sum Sq Mean Sq F value  Pr(>F)
temp           2  39083   19542  29.344 1.69e-07 ***
temp:material  6  20071    3345   5.023 0.00142 **
Residuals     27  17981     666
```

$$SS = SST_B + SSI_{A:AB} + SSE$$

where

$$SSI_{A:AB} = SST_A + SSI_{AB}$$

Notes

Final Note on ANOVA

We have studied the simplest design scenarios: extension to

- *incomplete*
- *unbalanced*
- *nested*
- *random effect*

designs are possible.

Furthermore R has greater functionality: for example, it has the capability to carry out ANOVA-like analyses even for the case of non-equal variances (when Levene's test rejects the hypothesis of equal variances).

Notes

Part II

Linear Regression Modelling

Notes

2. Linear regression Modelling

In the previous section, we attempted to explain the variation in an observed response variable by fitting models with one or more factors.

Factors are **discrete** variables taking different levels; in this section we will now utilize **continuous** variables that can similarly explain variation in an observed response.

Notes

2.1 Simple Linear Regression

We will investigate models relating two quantities x and y through equations of the form

$$y = ax + b$$

where a and b are constants (that is, a straight-line).

Variables x and y will not be treated exchangeably - we will regard y as being a function of x.

Such models are deterministic, that is, if we know x (and the values of the constants), we can compute y exactly without error.

A more useful model allows for the possibility that the system is not observed perfectly, that is, we do not observe (x, y) pairs that are always consistent with a simple functional relationship.

Notes

Example (Pharmacokinetic Model)

If a dose of drug is taken at time $x = 0$, the amount (concentration) of drug still in the bloodstream at time x is often well-modelled by a simple equation. Let

- D denote the amount of drug taken at $x = 0$
- x time
- $y^\star$ is the amount (concentration per unit volume) in the bloodstream.

Then

$$y^\star = \frac{D}{V} \exp\{-\lambda x\}$$

where

- λ is the elimination rate
- V is the volume of bloodstream.

Notes

Example (Pharmacokinetic Model (continued))

Taking logs of both sides, setting $y = \log y^\star$, then

$$y = -\lambda x + \log(D/V) = -\lambda x + (\log D - \log V)$$

that is, $y = ax + b$ where

- $a = -\lambda$
- $b = (\log D - \log V)$

However, in practice, when we measure concentration, we do so with random error.

Notes

2.1.1 Probabilistic Models

In a **probabilistic** model, we allow for the possibility that y is observed with random error, that is,

$$y = ax + b + ERROR$$

where $ERROR$ is a random term that is present due to imperfect observation of the system due to (i) measurement error or (ii) missing information.

Note that we do not treat x and y exchangeably; x is a fixed observed variable that is measured *without error*, whereas y is an observed variable that is measured *with random error*.

We model the variation in y as a function of x. We observe pairs $(x_i, y_i), i = 1, \ldots, n$.

Notes

A Basic Probabilistic Model

Terminology:

- y - *Dependent variable* or *response variable*
- x - *Independent variable*, or *predictor*, or *covariate*

The model we study takes the form

$$y = \beta_0 + \beta_1 x + \epsilon$$

where ϵ is a random error term, a random variable with mean zero and finite variance ($\mathbb{E}[\epsilon] = 0, Var[\epsilon] = \sigma^2$); it represents the error present in the measurement of y.

- β_0 - *Intercept* parameter
- β_1 - *Slop*e parameter

Notes

- $\beta_1 > 0$ - increasing y with increasing x
- $\beta_1 < 0$ - decreasing y with increasing x
- $\beta_1 = 0$ - no relationship between x and y

Note:

$$\mathbb{E}[Y|x] = \beta_0 + \beta_1 x$$

where $\mathbb{E}[Y|x]$ is the expected value of Y for fixed value of x.

Recall the notation

- Y - a random variable with a probability distribution
- y - a fixed value that the variable Y can take.

Fundamental Problem: If we believe the straight-line model with error is correct, how do we find the values of parameters β_0 and β_1. We only have the observed data $\{(x_i, y_i), i = 1, \ldots, n\}$.

Notes

2.1.2 Least Squares Fitting

We select the best values of β_0 and β_1 by minimizing the *error in fit*. For two data points (x_1, y_1) and (x_2, y_2), the errors in fit are

$$\begin{aligned} e_1 &= y_1 - (\beta_0 + \beta_1 x_1) \\ e_2 &= y_2 - (\beta_0 + \beta_1 x_2) \end{aligned}$$

respectively. But note that, potentially, $e_1 > 0$ and $e_2 < 0$ so there is a possibility that these fitting errors cancel each other out. Therefore we look at **squared** errors (as a large negative error is as bad as a large positive error)

$$\begin{aligned} e_1^2 &= (y_1 - (\beta_0 + \beta_1 x_1))^2 \\ e_2^2 &= (y_2 - (\beta_0 + \beta_1 x_2))^2 \end{aligned}$$

Notes

For n data, we obtain n misfit squared errors

$$e_1^2, \ldots, e_n^2$$

We select β_0 and β_1 as the values of the parameters that minimize SSE, where

$$SSE = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - (\beta_0 + \beta_1 x_i))^2$$

We wish to make the total misfit squared error as small as possible.

SSE - sum of squared errors - is similar to the SSE for ANOVA. We could write

$$SSE = SSE(\beta_0, \beta_1)$$

to show the dependence of SSE on the parameters.

Minimization of $SSE(\beta_0, \beta_1)$ is achieved **analytically**.

Notes

Two routes: (i) calculus and (ii) geometric methods. It follows that the best parameters $\widehat{\beta}_0$ and $\widehat{\beta}_1$ are given by

$$\widehat{\beta}_1 = \frac{SS_{xy}}{SS_{xx}} \qquad \widehat{\beta}_0 = \overline{y} - \widehat{\beta}_1 \overline{x}$$

where

- Sum of Squares SS_{xx}:

$$SS_{xx} = \sum_{i=1}^{n} (x_i - \overline{x})^2$$

- Sum of Squares SS_{xy}:

$$SS_{xy} = \sum_{i=1}^{n} (x_i - \overline{x})(y_i - \overline{y})$$

Notes

$\widehat{\beta}_0$ and $\widehat{\beta}_1$ are the **least-squares estimates**

$$y = \widehat{\beta}_0 + \widehat{\beta}_1 x$$

is the **least-squares line of best fit**. The **fitted-values** are

$$\widehat{y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 x_i \qquad i = 1, \ldots, n$$

and the **residuals** or **residual errors** are

$$\widehat{e}_i = y_i - \widehat{y}_i = y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_i \qquad i = 1, \ldots, n$$

Notes

2.1.3 Model Assumptions for Least-Squares

To utilize least-squares for the probabilistic model

$$Y = \beta_0 + \beta_1 x + \epsilon$$

we make the following assumptions

1. The expected error $\mathbb{E}[\epsilon]$ is zero so that

$$\mathbb{E}[Y] = \beta_0 + \beta_1 x$$

2. The variance of the error, $Var[\epsilon]$, is constant and does not depend on x.
3. The probability distribution of ϵ is a symmetric distribution about zero (a stronger assumption is that ϵ is Normally distributed).
4. The errors for two different measured responses are independent, i.e. the error ϵ_1 in measuring y_1 at x_1 is independent of the error ϵ_2 in measuring y_2 at x_2.

Notes

2.1.4 Parameter Estimation: Estimating σ^2

Using the LS procedure, we can construct an estimate of the *error* or *residual error* variance

Recall that

$$Var[\epsilon] = \sigma^2$$

An estimate of σ^2 is

$$\widehat{\sigma}^2 = \frac{SSE(\widehat{\beta}_0, \widehat{\beta}_1)}{n-2} = s^2$$

say.

Notes

Note that the denominator $n-2$ is again a *degrees of freedom* parameter of the form

TOTAL NUMBER OF DATA − NUMBER OF PARAMETERS ESTIMATED

or $n-p$, where in the simple linear regression, $p=2$ ($\widehat{\beta}_0$ and $\widehat{\beta}_1$). Note also that

$$SSE(\widehat{\beta}_0, \widehat{\beta}_1) = \sum_{i=1}^{n}(y_i - \widehat{y}_i)^2 = SS_{yy} - \widehat{\beta}_1 SS_{xy}$$

where

$$SS_{yy} = \sum_{i=1}^{n}(y_i - \overline{y})^2$$

Notes

Estimation and Testing for Slope

In the model where

$$\mathbb{E}[Y] = \beta_0 + \beta_1 x$$

it is of interest to test the hypothesis

$$\begin{aligned} H_0 &: \beta_1 = 0 \\ H_a &: \beta_1 \neq 0 \end{aligned}$$

i.e. H_0 implies that there is no systematic contribution of x to the variation of y.

Notes

To test H_0 vs H_a we us the test statistic

$$t = \frac{\widehat{\beta}_1}{\text{e.s.e}(\widehat{\beta}_1)} = \frac{\widehat{\beta}_1}{s_{\widehat{\beta}_1}}$$

where e.s.e$(\widehat{\beta}_1)$ is the *Estimated Standard Error* of $\widehat{\beta}_1$, computed as

$$\text{e.s.e}(\widehat{\beta}_1) = \frac{s}{\sqrt{SS_{xx}}}$$

where s is the estimate of σ defined previously.

If H_0 is true, and $\beta_1 = 0$, then

$$t = \frac{\widehat{\beta}_1}{s/\sqrt{SS_{xx}}} \sim \text{Student}(n-2)$$

so we can carry out a significance test at level α in the usual way (use a p-value, or construct the rejection region).

Notes

Note: we might also consider a one-sided test, where $H_a : \beta_1 > 0$, say.

- If $H_a : \beta_1 \neq 0$, we use the *two-sided* rejection region, with critical values
$$C_R = \pm t_{n-2}(\alpha/2)$$
- If $H_a : \beta_1 > 0$, we use the *one-sided* rejection region, with critical value
$$C_R = +t_{n-2}(\alpha)$$
- If $H_a : \beta_1 < 0$, we use the *one-sided* rejection region, with critical value
$$C_R = -t_{n-2}(\alpha)$$

Notes

Note: To test

$$\begin{aligned} H_0 &: \beta_1 = b \\ H_a &: \beta_1 \neq b \end{aligned}$$

for any b, the test statistic is

$$t = \frac{\widehat{\beta}_1 - b}{s/\sqrt{SS_{xx}}}$$

(for example, $b = 1$ may be of interest). If H_0 is true

$$t \sim \text{Student}(n-2)$$

Notes

Confidence Interval

A $100(1-\alpha)\%$ confidence interval for β_1 is

$$\widehat{\beta}_1 \pm t_{n-2}(\alpha/2) \times s_{\widehat{\beta}_1}$$

where

$t_{n-2}(\alpha/2)$: $\alpha/2$ prob. point of Student$(n-2)$ distn.

$s_{\widehat{\beta}_1}$: Estimated standard error of $\widehat{\beta}_1$

Note: we could perform a similar analysis for β_0, but this is generally of less interest.

The only quantity that needs attention is the estimated standard error of $\widehat{\beta}_0$. It can be shown that

$$\text{e.s.e.}(\widehat{\beta}_0) = s_{\widehat{\beta}_0} = \sqrt{\frac{1}{n}\left(1 + \frac{n\overline{x}^2}{SS_{xx}}\right)}$$

Notes

2.1.5 The Coefficient of Correlation

To measure the *strength of association* between the two variables x and y we can use the

Pearson Product Moment Coefficient Of Correlation

or *correlation coefficient* which measures the strength of the **linear** relationship between x and y.

The coefficient, r, is defined by

$$r = \frac{SS_{xy}}{\sqrt{SS_{xx}SS_{yy}}}$$

where

$$SS_{xx} = \sum_{i=1}^{n}(x_i - \overline{x})^2 \quad SS_{yy} = \sum_{i=1}^{n}(y_i - \overline{y})^2$$

$$SS_{xy} = \sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})$$

Notes

Note: $-1 \leq r \leq 1$.

- If r is close to 1, there is a strong linear relationship between x and y where y **increases** with x.
- If r is close to -1, there is a strong linear relationship between x and y where y **decreases** with x.

Note: In the model

$$y = \beta_0 + \beta_1 x$$

$\beta_1 = 0 \Longrightarrow r \approx 0$, so tests for $\beta_1 = 0$ can also be used to deduce a lack of correlation between the variables.

Notes

Notes

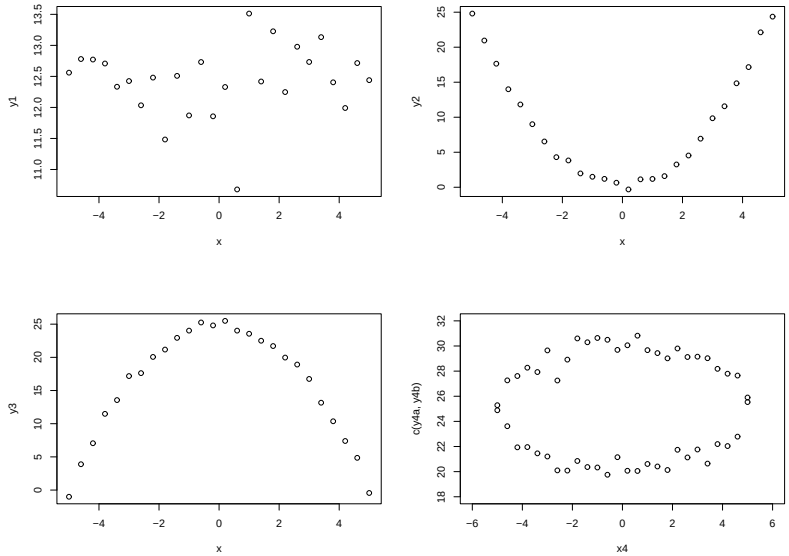
Notes

1. A strong linear relationship is not necessarily a **causal** relationship, that is, just because $r \approx 1$ does not mean that x **causes** changes in y (we may have a *spurious* correlation).

2. Just because $r \approx 0$ does not mean that that x and y are unrelated, merely that they are **uncorrelated**. That is, it is possible to construct examples where x and y have a strong functional relationship, but where $r = 0$.

Notes

Examples where $r \approx 0$.



Notes

Testing Correlation

We use ρ to denote the **true** correlation between X and Y.

We can test the hypothesis that $\rho = 0$ (that is, that X and Y are uncorrelated using r. For testing

$$\begin{aligned} H_0 &: \rho = 0 \\ H_a &: \rho \neq 0 \end{aligned}$$

we can use the test statistic

$$t = \frac{r}{\sqrt{(1-r^2)/(n-2)}}$$

If H_0 is true, then approximately

$$t \sim \text{Student}(n-2)$$

Notes

Alternately, we could use

$$z = \frac{1}{2}\log\left(\frac{1+r}{1-r}\right)$$

and then, if H_0 is true, as (approximately)

$$Z \sim N\left(\frac{1}{2}\log\left(\frac{1+\rho}{1-\rho}\right), \frac{1}{n-3}\right)$$

when $\rho = 0$, so that (approximately)

$$\sqrt{n-3}\, Z \sim N(0,1)$$

Notes

A related quantity is the

Coefficient of Determination

or $\boldsymbol{R^2}$ **Statistic**

$$r^2 = \frac{SS_{yy} - SSE}{SS_{yy}} = 1 - \frac{SSE}{SS_{yy}}$$

Note that the *total variation* in y is recorded via

$$SS_{yy} = \sum_{i=1}^{n}(y_i - \overline{y})^2$$

and the *random variation* is recorded via

$$SSE = \sum_{i=1}^{n}(y_i - \widehat{y}_i)^2$$

Notes

Therefore the **variation explained by the linear regression** is

$$SSR = SS_{yy} - SSE \qquad \text{as} \qquad SS_{yy} = SSR + SSE$$

Thus

$$r^2 = \frac{SSR}{SS_{yy}} = \frac{\text{Variation explained by Regression}}{\text{Total Variation}}$$

r^2 is a measure of model adequacy, that is, if $r^2 \approx 1$, then the linear model is a **good fit**.

Notes

Example (Blood Viscosity vs PCV)

We have

- $n = 32$
- $r = 0.879$
- $R^2 = r^2 = (0.879)^2 = 0.772$

Test of $\rho = 0$:

$$t = \frac{r}{\sqrt{(1 - r^2)/(n - 2)}} = 10.087$$

We compare with a Student$(n - 2) \equiv$ Student(30) distribution; the p-value is 3.73×10^{-11}, so there is strong evidence that $\rho \neq 0$.

Notes

2.1.6 Prediction

After the linear model is fitted, it can be used for **forecasting** or **prediction**. That is, given a new x value we can predict the corresponding y.

As before, we see that at any value of x_p, the prediction $\widehat{y}_p$ is

$$\widehat{y}_p = \widehat{\beta}_0 + \widehat{\beta}_1 x_p$$

This is the best predictor of y at this x value.

Notes

We can also compute the standard error of this prediction; if the value of the random error variance σ^2 is known, then

$$\text{s.e.}(\widehat{y}_p) = \sigma\sqrt{\frac{1}{n} + \frac{(x_p - \overline{x})^2}{SS_{xx}}}$$

If σ is unknown, we estimate σ by $\widehat{\sigma} = s$ as defined previously

$$s^2 = \frac{SSE(\widehat{\beta}_0, \widehat{\beta}_1)}{n-2}$$

so that

$$\text{e.s.e.}(\widehat{y}_p) = s\sqrt{\frac{1}{n} + \frac{(x_p - \overline{x})^2}{SS_{xx}}}$$

Notes

Note: This prediction is the expected value of y at $x = x_p$.
That is, we have worked out

$$Var[\widehat{Y}_p] = Var[\widehat{\beta}_0 + \widehat{\beta}_1 x_p]$$

to compute the s.e. for $\widehat{Y}_p$.

But we can actually predict an **error corrupted** version of $\widehat{Y}_p$, $\widehat{Y}_p^\star$ say, where

$$\widehat{Y}_p^\star = \widehat{Y}_p + \epsilon_p$$

where ϵ_p is a new random error.

Notes

But

$$Var[\widehat{Y}_p^\star] = Var[\widehat{Y}_p + \epsilon_p] = Var[\widehat{Y}_p] + Var[\epsilon_p] = Var[\widehat{Y}_p] + \sigma^2$$

that is, there is an **extra** piece of variation due to ϵ_p.

Thus

$$\text{e.s.e.}(\widehat{y}_p^\star) = s\sqrt{1 + \frac{1}{n} + \frac{(x_p - \overline{x})^2}{SS_{xx}}} > \text{e.s.e.}(\widehat{y}_p)$$

Notes

Prediction Intervals

A $100(1-\alpha)\%$ prediction interval for the **mean** value at $x = x_p$ is

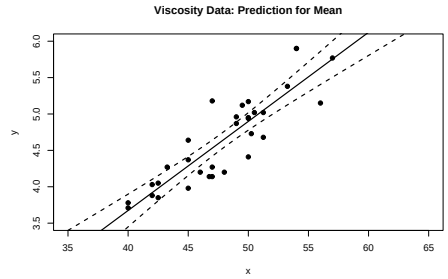
$$\widehat{y}_p \pm t_{n-2}(\alpha/2)s\sqrt{\frac{1}{n} + \frac{(x_p - \overline{x})^2}{SS_{xx}}}$$

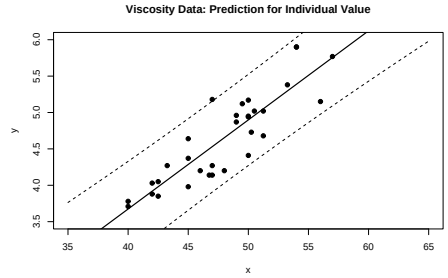
whereas for an individual new value (predicted with error) at $x = x_p$ is

$$\widehat{y}_p \pm t_{n-2}(\alpha/2)s\sqrt{1 + \frac{1}{n} + \frac{(x_p - \overline{x})^2}{SS_{xx}}}$$

Notes

Prediction Intervals





Notes

ANOVA-F test in Regression

An ANOVA-F test can be constructed to test overall (*global*) fit of the linear regression model.

The decomposition of sums of squares for regression takes the form

$$SS = SSR + SSE$$

where

- $SS = SS_{yy}$: overall or total sum of squares
- SSR: sum of squares due to Regression
- SSE: sum of squares due to Error

Notes

$$\begin{aligned} SS &= \sum_{i=1}^{n}(y_i - \overline{y})^2 \\ SSR &= \sum_{i=1}^{n}(\widehat{y}_i - \overline{y})^2 \\ SSE &= \sum_{i=1}^{n}(y_i - \widehat{y}_i)^2 \end{aligned}$$

where

$$\widehat{y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 x_i \qquad i = 1, \ldots, n$$

Degrees of Freedom

- TOTAL: $n - 1$
- REGRESSION: 1
- ERROR: $n - 2$

(error d.f. is $n - p$, here $p = 2$).

Notes

The ANOVA Table

SOURCE	DF	SS	MS	F
REGRESSION	1	SSR	$MSR = \frac{SSR}{1}$	$F = \frac{MSR}{MSE}$
ERROR	$n-2$	SSE	$MSE = \frac{SSE}{(n-2)}$	
TOTAL	$n-1$	SS		

The test of the hypothesis

$$\begin{aligned} H_0 &: \mathbb{E}[Y] = \beta_0 \\ H_a &: \mathbb{E}[Y] = \beta_0 + \beta_1 x \end{aligned}$$

can be completed by using the test statistic

$$F = \frac{MSR}{MSE}$$

If H_0 is true

$$F \sim \text{Fisher-F}(1, n-2)$$

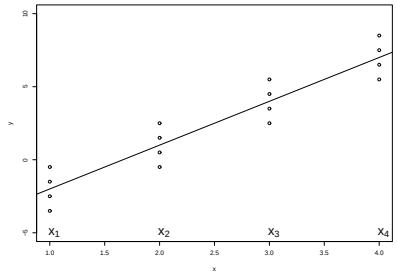
Notes

This is just like the ANOVA in the one-way layout (CRD) with n groups, but where

$$\mu_i = \beta_0 + \beta_1 x_i$$

That is, the group means are **structured**, that is, we have a formula relating the μ_i quantities.

Consider four replicates at x values (x_1, x_2, x_3, x_4) in a regression;



Then for group i, $\mu_i = \beta_0 + \beta_1 x_i$, $i = 1, 2, 3, 4$.

Notes

Checking the Local Fit

A plot of the *residuals*

$$\widehat{e}_i = y_i - \widehat{y}_i$$

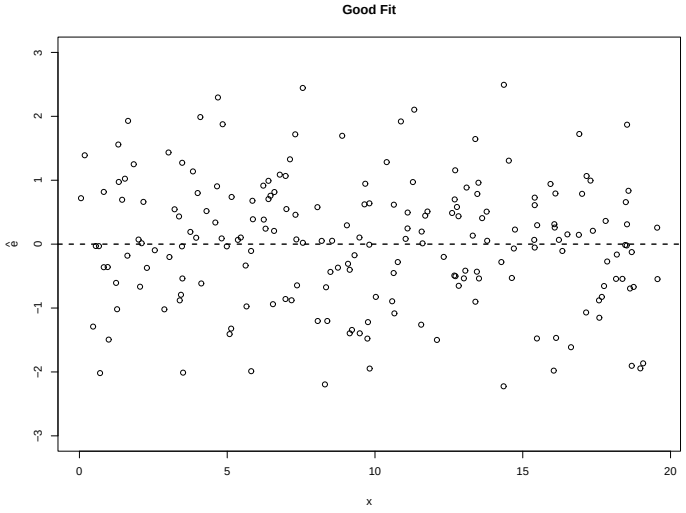
can reveal model inadequacies. We should observe that in plots of

- x vs $\widehat{e}$
- $\widehat{y}$ vs $\widehat{e}$

there is no discernible pattern

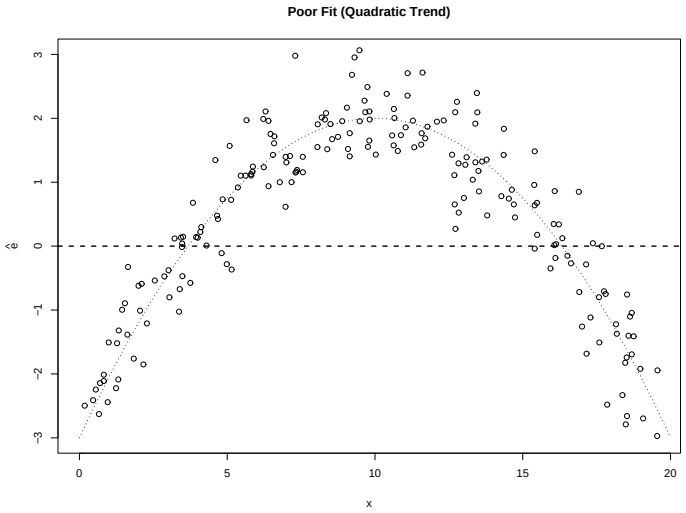
Notes

Checking the Local Fit: Good Fit



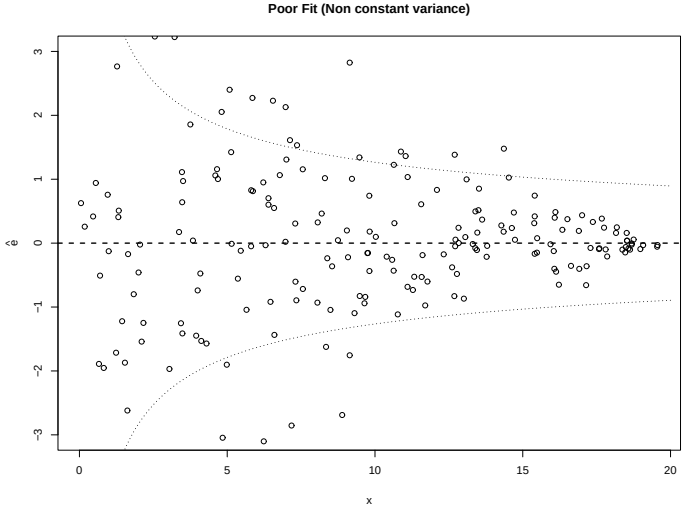
Notes

Checking the Local Fit: Poor Fit



Notes

Checking the Local Fit: Poor Fit



Notes

R^2 and adjusted R^2

Model assessment can be made using either the R^2 statistic

$$R^2 = 1 - \frac{SSE}{SS}$$

and the **adjusted** R^2 statistic

$$R^2 = 1 - \frac{SSE/EDF}{SS/TDF}$$

where

- EDF = error degrees of freedom = $n - 2$
- TDF = total degrees of freedom = $n - 1$

Notes

2.1.7 Polynomial Regression

In many practical situations, the simple straight line

$$y = \beta_0 + \beta_1 x$$

is not appropriate. Instead, a model including powers of x

$$x^2, x^3, \ldots, x^k$$

should be considered. For example

$$y = \beta_0 + \sum_{j=1}^{k} \beta_j x^j = \beta_0 + \beta_1 x + \cdots + \beta_k x^k$$

Notes

The **Polynomial Regression Model**

$$Y = \beta_0 + \beta_1 x + \cdots + \beta_k x^k + \epsilon$$

where ϵ is a random error term as before can be used to model data.

Two immediate problems:

1. How to choose k
2. How to carry out inference
   - estimation
   - testing
   - prediction

We begin by addressing 2. The estimation of parameters can be again carried out using **Least Squares** provided that the model assumptions listed before are valid.

Notes

$k = 2$: quadratic regression

We choose $\underset{\sim}{\beta} = (\beta_0, \beta_1, \beta_2)^\top$ to minimize the **sum of squared errors**

$$SSE(\underset{\sim}{\beta}) = \sum_{i=1}^{n} (y_i - \widehat{y}_i)^2 = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i - \beta_2 x_i^2)^2$$

that is the fitted values for parameters $\underset{\sim}{\beta}$ are

$$\widehat{y}_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2$$

$\widehat{\underset{\sim}{\beta}}$ can be found to minimize SSE using calculus techniques (differentiating with respect to the elements of $\underset{\sim}{\beta}$) to give the minimum SSE

$$SSE(\underset{\sim}{\beta}) = \sum_{i=1}^{n} (y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_i - \widehat{\beta}_2 x_i^2)^2$$

Notes

We can also compute the estimated **standard errors**

$$s_{\widehat{\beta}_0}, s_{\widehat{\beta}_1}, s_{\widehat{\beta}_2}$$

which allow tests of parameters to be carried out, and confidence intervals calculated.

We can also compute prediction intervals.

The best estimate of the residual error variance σ^2 is

$$\widehat{\sigma}^2 = \frac{SSE(\widehat{\underset{\sim}{\beta}})}{n-3}$$

p is the number of parameters estimated equal to three, so we divide by $n-3$.

Notes

We can also compute

- Residuals
  - can be used to assess the fit of the model.
  - the residuals should be *patternless* if the model fit is good.
- R^2, Adjusted R^2 statistics
  - used to assess the global fit of the model.
  - used to compare the quality of fit with other models.

Notes

Example (Hooker Pressure Data)

For the Hooker pressure data, a **quadratic** polynomial ($k = 2$) might be suitable.

$$Y = \beta_0 + \beta_1 x + \beta_2 x^2 + \epsilon$$

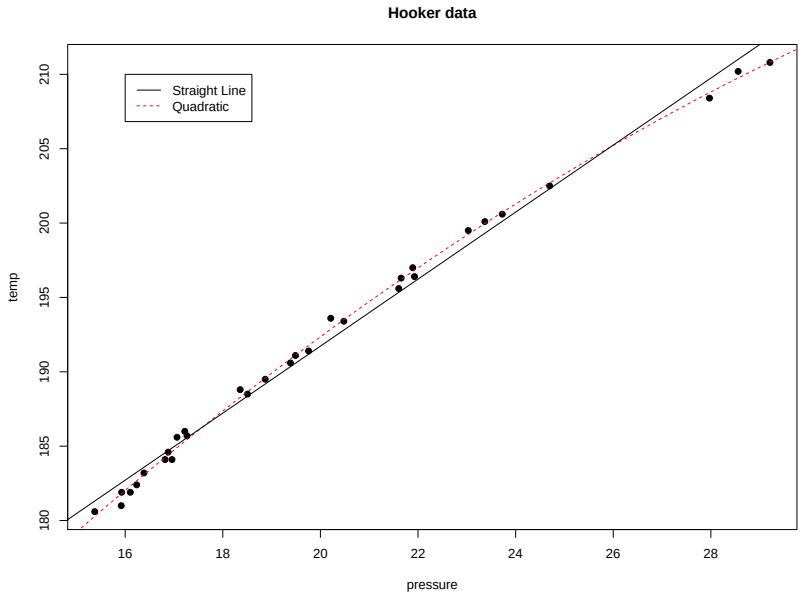
We need to estimate β_0, β_1 and β_2 for these data to see if the model fits better than the straight line model we fitted previously.

It transpires that the quadratic model produces a set of residuals that are patternless, which the straight line model when fitted does not.

See Handout for full details.

Notes

Hooker data: straight-line and quadratic fits



Notes

Note: It is common to use the **Standardized Residuals**

$$\widehat{z}_i = \frac{\widehat{e}_i}{\widehat{\sigma}} = \frac{y_i - \widehat{y}_i}{\widehat{\sigma}}$$

where $\widehat{\sigma}^2$ is the estimate of σ^2 defined previously, as

$$\text{Var}[\widehat{z}_i] \approx 1$$

if the model fit is good, whereas

$$\text{Var}[\widehat{e}_i] \approx \sigma^2$$

which clearly depends on σ. This makes it hard to compare $\widehat{e}_i$ across different models when inspecting residuals.

Notes

Note: Although the model based on

$$y = \beta_0 + \beta_1 x + \beta_2 x^2$$

is **not** linear in x, it **is** linear in the parameters. Because of this, we still term this a *linear model*. It is this fact that makes the least-squares solutions easy to find.

This model is no more difficult to fit than the model

$$y = \beta_0 + \beta_1 \frac{x}{1+x} + \beta_2(1 - e^{-x})$$

say - it is still a *linear in the parameters model*. It is in the general class of models

$$y = \beta_0 + \beta_1 g_1(x) + \beta_2 g_2(x)$$

where $g_1(x)$ and $g_2(x)$ are general functions of x.

Notes

In fact, any model of the form

$$y = \sum_{j=0}^{k} \beta_j g_j(x) + \epsilon \qquad (1)$$

can be fitted routinely using least-squares; if we know x, then we can compute

$$g_0(x), g_1(x), \ldots, g_k(x)$$

and plug those values into the formula (1).

Notes

Example (Harmonic Regression)

Let

$$\begin{aligned} g_0(x) &= 1 \\ g_1(x) &= \begin{cases} \cos(\lambda_j x) & j \text{ odd} \\ \sin(\lambda_j x) & j \text{ even} \end{cases} \end{aligned}$$

where k is an even number, $k = 2p$ say.

$\lambda_j, j = 1, 2, \ldots, p$ are constants

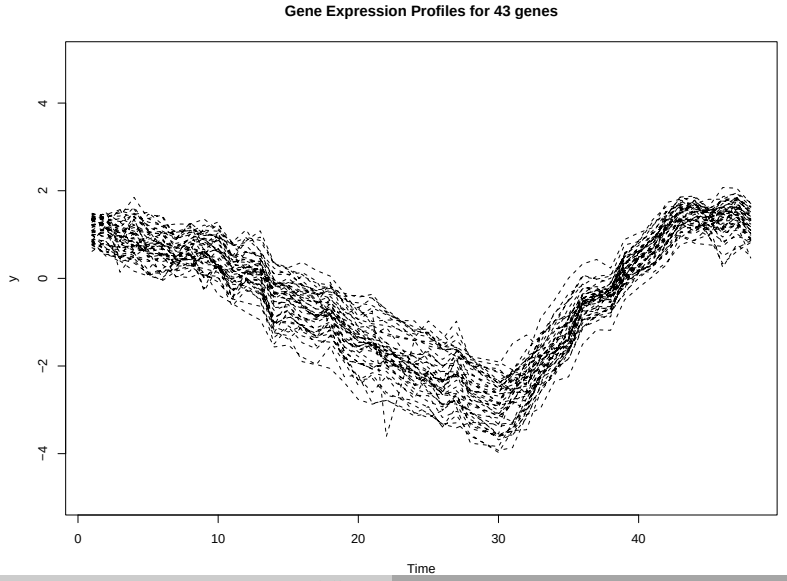
$$\lambda_1 < \lambda_2 < \cdots < \lambda_p$$

For fixed x, $\cos(\lambda_j x)$ and $\sin(\lambda_j x)$ are also fixed, known values.

Notes

Gene Expression Data Example

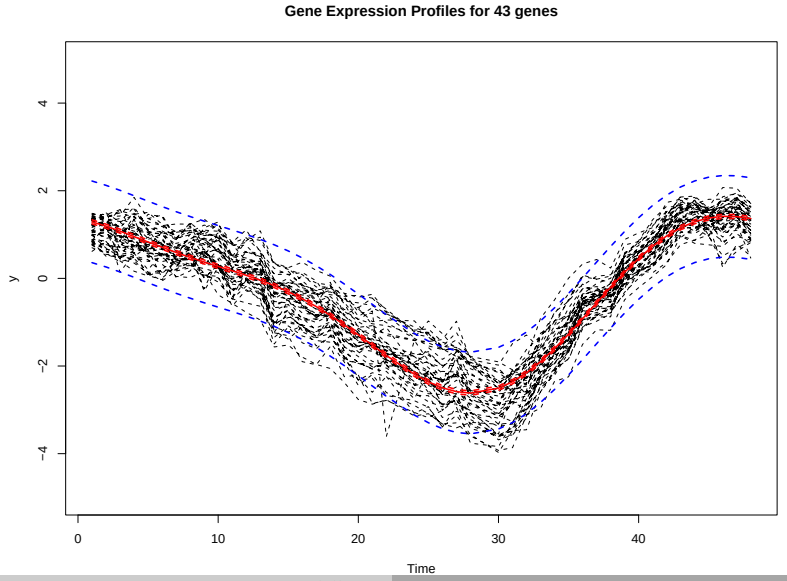
Harmonic Regression Fit with $p = 2$.



Notes

Gene Expression Data Example

Harmonic Regression Fit with $p = 2$.



Notes

Why are things so straightforward ?

- because the system of equations based on the derivatives

$$\frac{\partial}{\partial \beta_j}\left\{SSE(\underset{\sim}{\beta})\right\} = 0 \qquad j = 0, 1, \ldots, k$$

can always be solved routinely, so we can always find $\widehat{\underset{\sim}{\beta}}$.

In the general model (1), simple formulae for

- $\widehat{\underset{\sim}{\beta}}$
- $s.e.(\widehat{\underset{\sim}{\beta}})$
- $\widehat{\sigma}^2$

can be found using a matrix formulation.

Notes

Note: One-way ANOVA can be formulated in the form of model (1). Recall

- k independent groups
- means $\mu_1, \ldots, \mu_k$
- y_{ij} is the jth observation in the ith group

Let

$$\begin{aligned} \beta_0 &= \mu_1 \\ \beta_t &= \mu_t - \mu_k \qquad t = 2, \ldots, k. \end{aligned}$$

Define new data $x_{ij}(t)$ where

$$x_{ij}(t) = \begin{cases} 1 & \text{if } t = i \\ 0 & \text{if } t \neq i \end{cases}$$

Notes

Then, using the linear regression formulation

$$y_{ij} = \beta_0 + \sum_{t=1}^{k-1} \beta_t x_{ij}(t) + \epsilon_{ij}.$$

For any i, j, $x_{ij}(t)$ is non-zero for only one value of t, when $t = i$.

We term this a regression on a *factor predictor*; it is clear that $\beta_0, \beta_1, \ldots, \beta_{k-1}$ can be estimated using least-squares.

This clarifies the link between

ANOVA and Linear Modelling

- they are essentially the SAME MODEL formulation.

This link extends to **ALL ANOVA** models.

Notes

2.2 Multiple Linear Regression

Multiple linear regression models model the variation in response y as a function of **more than one** independent variable.

Suppose we have variables

$$X_1, X_2, \ldots, X_k$$

recording different features of the experimental units. We wish to model response Y as a function of $X_1, X_2, \ldots, X_k$.

Notes

2.2.1 Multiple Linear Regression Models

Consider the model for datum i

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_k x_{ik} + \epsilon_i$$

where x_{ij} is the measured value of *covariate* j on experimental unit i. That is

$$y_i = \beta_0 + \sum_{j=1}^{k} \beta_j x_{ij} + \epsilon_i$$

where the first two terms on the right hand side are the *systematic* or *deterministic* components, and the final term ϵ_i is the *random* component.

Notes

Example ($k = 2$)

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i$$

A three parameter model.

Note: We can also include *interaction* terms

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_{12}(x_{i1} \,.\, x_{i2}) + \epsilon_i$$

where

- The first two terms in x_{i1} and x_{i2} are **main effects**
- The third term in $(x_{i1} \,.\, x_{i2})$ is an **interaction**

This is a four parameter model.

Notes

Multiple Linear Regression Examples

SEE HANDOUT

- Multiple regression: Viscosity Example
- Factor Regression:
- Interaction
- Residuals

Notes

Subgroup analysis, with a factor predictor and a continuous covariate, is a form of interaction modelling; the factor predictor *interacts* with the covariate to modify the slope across the subgroups, for example.

We can describe the models using the notation previously introduced for ANOVA; consider the single binary factor predictor and single covariate case;

MODEL 0	Single horizontal straight line	1
MODEL 1	Two parallel horizontal straight lines	X_2
MODEL 2	Single straight line, non-zero slope	X_1
MODEL 3	Two parallel straight lines, non-zero slope	$X_1 + X_2$
MODEL 4	Two non-parallel straight lines	$X_1 + X_2 + X_1.X_2$

Notes

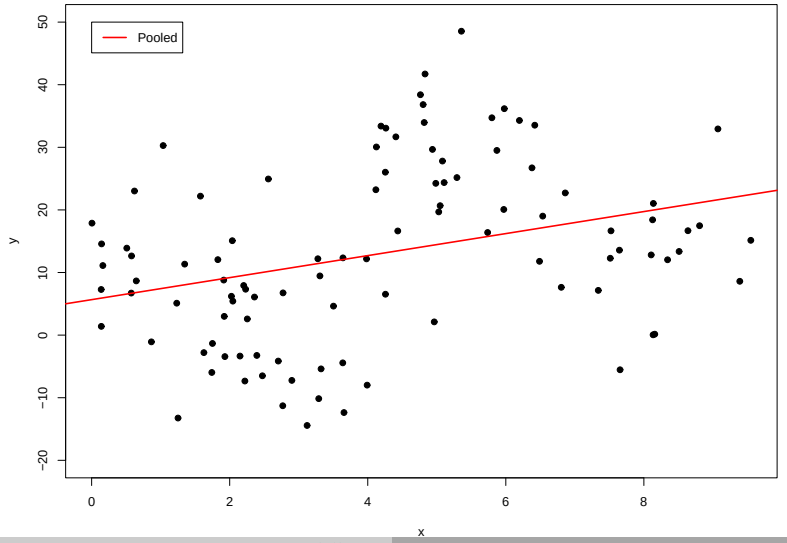
Note: Always be on the lookout for *lurking* subgroups (subgroups determined by the levels of an unnoticed factor predictor)

Inferences can change radically when the lurking factor is included in the model

- positive association can be converted into negative association with the continuous covariate.

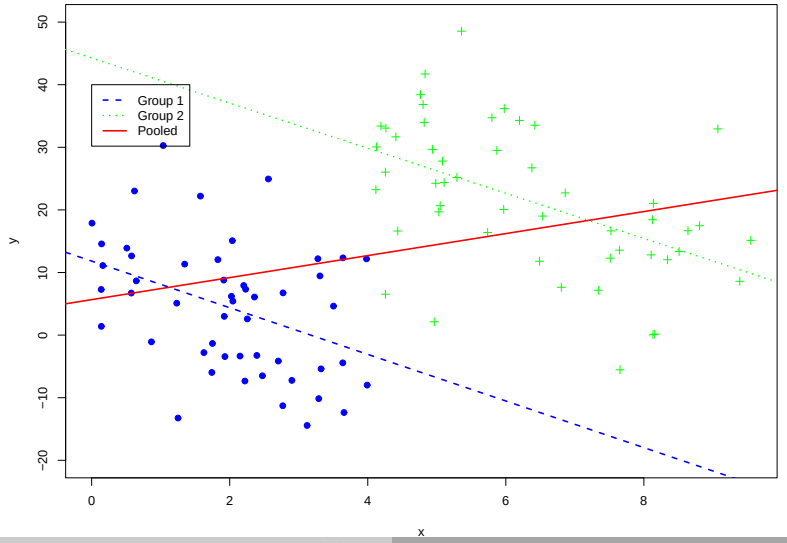
Notes

For example, for factor predictor X_2 taking two levels and continuous covariate X_1. When the pooled data are examined, a **positive association** between Y and X_1 is revealed.



Notes

When the pooled data are separated into subgroups, a **negative association** between Y and X_1 in each subgroup is revealed. $X_2 = 1$ (●) and $X_2 = 2$ (+).



Notes

i.e. increasing X_1 decreases response in subgroup 1, and decreases response in subgroup 2, but appears to increase response overall.

This is known as **Simpson's Paradox in Regression**. It illustrates that pooling data over subgroups must be carried out with care !

- you must fit the factor predictor in the model if you suspect subgroup differences exist.

In the example, the problem arises due to **dependence** between X_1 and X_2; all the group with $X_2 = 0$ have **low** values of X_1, whereas all the group with $X_2 = 1$ have **high** values of X_1

Dependence between covariates and factor predictors makes model fitting and results interpretation complicated.

Notes

Recap: we can build general models

$$y_i = \beta_0 + \sum_{j=1}^{k} \beta_j x_{ij} + \epsilon_i$$

to explain the variation of y in terms of covariates and factor predictors $x_1, \ldots, x_k$.

- Simple Linear Regression
- Polynomial Regression
- Multiple Regression
- Factor Predictor Regression
- Interaction Models

Notes

We can fit each of these models easily using least-squares to obtain

- estimates $\underset{\sim}{\widehat{\beta}} = (\widehat{\beta}_1, \widehat{\beta}_2, \ldots, \widehat{\beta}_k)^\top$
- standard errors
- goodness of fit measures R^2 and Adjusted R^2
- residuals for model checking
- predictions

Notes

Interpreting $\widehat{\beta}_j$

$\widehat{\beta}_j$ can be interpreted as the amount of increase in response y when x_j increases by one unit when the other predictors

$$x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k$$

are held fixed.

We can test the hypothesis

$$\begin{aligned} H_0 &: \beta_j = 0 \\ H_0 &: \beta_j \neq 0 \end{aligned}$$

using the usual hypothesis testing approach.

Notes

Test statistic:

$$t_j = \frac{\widehat{\beta}_j}{s_{\widehat{\beta}_j}} = \frac{\text{ESTIMATE}}{\text{STANDARD ERROR}}$$

If H_0 is **true**,

$$t_j \sim Student(n - k - 1)$$

as we are estimating $k + 1$ parameters overall.

Note: In multiple regression, when testing each of

$$\widehat{\beta}_0, \widehat{\beta}_1, \ldots, \widehat{\beta}_k$$

we should strictly use a **multiple testing correction** (as in post-hoc tests in ANOVA)

Notes

2.2.2 Model Checking

Using the General Linear Model approach to regression, we can fit models with different numbers of predictors, and

- assess whether any individual covariate is influential in the model (look at $\widehat{\beta}, s_{\widehat{\beta}}$ and t-statistics).
- assess whether there is any explanatory power in the variables combined (look at ANOVA statistics)

Notes

ANOVA table for multiple regression

SOURCE	DF	SS	MS	F
REGRESSION	k	SSR	MSR	$F = \frac{MSR}{MSE}$
ERROR	$n-k-1$	SSE	MSE	
TOTAL	$n-1$	SS		

where

$$MSR = \frac{SSR}{k} \qquad MSE = \frac{SSE}{n-k-1}$$

the F statistic is

$$F = \frac{MSR}{MSE}$$

and if H_0 is true

$$F \sim \text{Fisher-F}(k, n-k-1)$$

Notes

Here

$$\begin{aligned} H_0 &: \beta_1 = \beta_2 = \cdots = \beta_k = 0 \\ H_a &: \text{At least one } \beta_j \neq 0 \end{aligned}$$

The model for H_0 has one parameter β_0.
The model for H_a has $k+1$ parameters

$$\beta_0, \beta_1, \beta_2, \ldots, \beta_k$$

Therefore the number of extra parameters for model H_a is

$$(k+1) - 1 = k$$

i.e. to obtain model H_0 from model H_a we constrain k parameters to be zero.

Notes

Because we can constrain model H_a by setting some parameters equal to zero to obtain model H_0, we say that

Model H_0 is nested inside Model H_a

The number, k, of constraints $\beta_1 = \beta_2 = \cdots = \beta_k = 0$ explains why the ANOVA table Regression degrees of freedom is k

- the multiple regression brings in k extra parameters.

Notes

In addition, we can use the R^2 or Adjusted R^2 statistic to check overall model adequacy

$$R^2 = 1 - \frac{SSE}{SS_{yy}} = \frac{SS_{yy} - SSE}{SS_{yy}} = \frac{SSR}{SS}$$

which is equal to

$$\frac{\text{VARIATION EXPLAINED BY THE REGRESSION}}{\text{TOTAL VARIATION}}$$

Also

$$\text{Adj. } R^2 = 1 - \frac{SSE/(n-k-1)}{SS/(n-1)}$$

$R^2 > 0.7$ implies that the model is a good fit, that is, most of the variation observed is explained by the regression model.

Notes

We can now fit completely general models in the form of the General Linear Model; if y is the response, and $x_1, \ldots, x_k$ are the covariates or factor predictors, we can include combinations of

- Polynomial Main Effects : $x_j, x_j^2, x_j^3, \ldots$
- Two-way Interactions: $x_{j_1} \,.\, x_{j_2}$
- Three-way Interactions: $x_{j_1} \,.\, x_{j_2} \,.\, x_{j_3}$

etc.

Notes

Dummy Variables

Recall that we can fit the factor predictor using the Linear Regression pulldown if we create **dummy variables**.

For example, if factor predictor X has L levels, we create L **new** binary predictors $X_1, \ldots, X_L$, where, for $l = 1, \ldots, L$

$$X_l = \begin{cases} 1 & \text{whenever } X = l \\ 0 & \text{otherwise} \end{cases}$$

We can then include $X_1, \ldots, X_L$ in the regression model.

Notes

Example ($L = 4$)

X	X_1	X_2	X_3	X_4
3	0	0	1	0
1	1	0	0	0
3	0	0	1	0
4	0	0	0	1
2	0	1	0	0
2	0	1	0	0

See McClave and Sincich, Section 12.7.

Notes

2.2.3 Stepwise Model Selection

We seek a method that allows us to compare nested models.

Suppose we want to compare

$$\begin{aligned} \text{MODEL 1} &: y = \beta_0 + \beta_1 x + \beta_2 x^2 \\ \text{MODEL 2} &: y = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 \end{aligned}$$

Model 1 is nested inside Model 2 as if we set $\beta_3 = 0$ in Model 2, we get Model 1.

Notes

If

$$\begin{aligned} \text{MODEL 1} &: y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 \\ \text{MODEL 2} &: y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12}(x_1.x_2) \end{aligned}$$

we can set $\beta_{12} = 0$ in Model 2 to obtain Model 1, so again the models are nested.

We can set up a hypothesis test to assess whether the simplification of Model 2 to Model 1 (by setting one or more parameters equal to zero) is justified by the data.

Notes

ANOVA tests for Comparing Nested Models

Terminology

- *Complete Model*

$$\mathbb{E}[Y] = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k$$

- *Reduced Model*

$$\mathbb{E}[Y] = \beta_0 + \beta_1 x_1 + \cdots + \beta_g x_g$$

where $g < k$. The reduced model is obtained from the complete model by setting

$$\beta_{g+1} = \beta_{g+2} = \cdots = \beta_k = 0$$

Notes

The reduced model is nested inside the complete model.

We wish to test the hypothesis

$$\begin{aligned} H_0 &: \beta_{g+1} = \beta_{g+2} = \cdots = \beta_k = 0 \\ H_a &: \text{At least one of these } \beta_j \neq 0 \end{aligned}$$

We can test this hypothesis by fitting both models, and combining the results; we focus on the sums of squares quantities.

Notes

Method

1. Fit the **complete model** and obtain the sum of squared errors, SSE_C, available from the ANOVA table.
2. Fit the **reduced model** and obtain the sum of squared errors, SSE_R, available from the ANOVA table.
3. Form the test statistic

$$F = \frac{(SSE_R - SSE_C)/(k-g)}{SSE_C/(n-k-1)}$$

If H_0 is **true**, then $F \sim$ Fisher-F$(k-g, n-k-1)$

Note: $k-g$ is the number of parameters we set equal to zero when moving from complete to reduced model.

Using this F statistic, we can assess whether there is evidence to support the reduced model over the complete model.

Notes

Complete Model ANOVA table:

SOURCE	DF	SS	MS	F
COMPLETE MODEL	k	SSR_C	MSR_C	F_C
ERROR$_C$	$n-k-1$	SSE_C	MSE_C	
TOTAL	$n-1$	SS		

Reduced Model ANOVA table:

SOURCE	DF	SS	MS	F
REDUCED MODEL	g	SSR_R	MSR_R	F_R
ERROR$_R$	$n-g-1$	SSE_R	MSE_R	
TOTAL	$n-1$	SS		

Notes

The result holds for comparing any two nested models where the standard model assumptions hold:

- ϵ uncorrelated
- ϵ independent of $x_1, \ldots, x_k$
- ϵ has constant variance
- $\epsilon \sim N(0, \sigma^2)$

Note: It does not allow us to compare non-nested models; for example

$$\begin{aligned} \text{MODEL 1} &: y = \beta_0 + \beta_1 x_1 + \epsilon \\ \text{MODEL 2} &: y = \beta_0 + \beta_2 x_2 + \epsilon \end{aligned}$$

- **NOT NESTED !**

Notes

$$F = \frac{(SSE_R - SSE_C)/(k-g)}{SSE_C/(n-k-1)} = \frac{①/②}{③/④}$$

① - $SSE_R - SSE_C$: this is the improvement in fit when the reduced model is extended to the complete model

② - $k - g$: this is the number of extra parameters needed to extend the reduced model to the complete model

③ - SSE_C
④ - $n - k - 1$

③/④ - this is the best guess we have at the true value of σ^2, that is, the estimate of σ^2 constructed using as much information as possible, once the effects of

$$x_1, \ldots, x_k$$

have been accounted for.

Notes

Example (Hooker's Data)

We consider the two models:

$$\begin{aligned} \text{MODEL 1} &: y = \beta_0 + \beta_1 x + \epsilon \\ \text{MODEL 2} &: y = \beta_0 + \beta_1 x + \beta_2 x^2 + \epsilon \end{aligned}$$

Here

- MODEL 1: Reduced Model
- MODEL 2: Complete Model

$k = 2$, $g = 1$.

IS THE QUADRATIC TERM NEEDED ?

Notes

Example (Hooker's Data)

COMPLETE MODEL	SSR_C	2286.933
	SSE_C	4.382
REDUCED MODEL	SSR_R	2272.474
	SSE_R	18.840

with $n = 31, k = 2, g = 1$

$$\implies k - g = 1, n - k - 1 = 28$$

So

$$F = \frac{(SSE_R - SSE_C)/(k-g)}{SSE_C/(n-k-1)} = \frac{(18.840 - 4.382)/1}{4.382/28} = 92.383$$

Notes

Example (Hooker's Data)

We compare F with the

$$\text{Fisher-F}(k-g, n-k-1) \equiv \text{Fisher-F}(1, 28)$$

distribution.

$$F_{0.05}(1, 28) = 4.20$$

Thus

$$92.383 = F > F_{0.05}(1, 28) = 4.20$$

and $H_0 : \mathbb{E}[Y] = \beta_0 + \beta_1 x$ is **REJECTED** in favour of $H_a : \mathbb{E}[Y] = \beta_0 + \beta_1 x + \beta_2 x^2$.

i.e. the **quadratic model** fits better than the straight-line model.

Notes

NOTE: From the original ANOVA tables, we already know that Model 1 and Model 2 both fit better than the null model

$$\begin{aligned} \text{MODEL 0} \qquad \mathbb{E}[Y] &= \beta_0 \\ y &= \beta_0 + \epsilon \end{aligned}$$

where there is no dependence on x.

We have now confirmed that Model 2 fits better than Model 1.

Notes

Example (Diabetes Data)

Factor Predictor: **group** (X_2)
Continuous Covariate: **loggluf** (X_1)
Response: **logglut** (Y)

We have five models to confirm:

$$\begin{aligned}
\text{MODEL 0} &: 1 \\
\text{MODEL 1} &: X_2 \\
\text{MODEL 2} &: X_1 \\
\text{MODEL 3} &: X_1 + X_2 \\
\text{MODEL 4} &: X_1 + X_2 + X_1.X_2
\end{aligned}$$

Notes

Example (Diabetes Data)

MODEL 4 us the most complex model with 6 parameters

$$\beta_{10}, \beta_{11}, \beta_{20}, \beta_{21}, \beta_{30}, \beta_{31}$$

MODEL 4:

$$\mathbb{E}[Y] = \begin{cases} \beta_{10} + \beta_{11} x_1 & \text{GROUP 1} \\ \beta_{20} + \beta_{21} x_1 & \text{GROUP 2} \\ \beta_{30} + \beta_{31} x_1 & \text{GROUP 3} \end{cases}$$

All of the other models are nested inside Model 4; we can obtain them all by setting parameters equal to zero.

Notes

Example (Diabetes Data)

In the default parameterization:

β_{10}, β_{11}	Group 1 Intercept and Slope
$\beta_{20} = \beta_{10} + \delta_{20}$ $\beta_{30} = \beta_{10} + \delta_{30}$	Changes in the Intercepts in groups 2 and 3 are δ_{20} and δ_{30}
$\beta_{21} = \beta_{11} + \delta_{21}$ $\beta_{31} = \beta_{11} + \delta_{31}$	Changes in the Slopes in groups 2 and 3 are δ_{21} and δ_{31}

Thus the six new parameters are

$$\beta_{10}, \beta_{11}, \delta_{20}, \delta_{30}, \delta_{21}, \delta_{31}$$

Notes

MODEL 0 $\beta_{11} = 0$
$\delta_{20} = \delta_{30} = \delta_{21} = \delta_{31} = 0$

MODEL 1 $\beta_{11} = \delta_{21} = \delta_{31} = 0$

MODEL 2 $\delta_{20} = \delta_{30} = \delta_{21} = \delta_{31} = 0$

MODEL 3 $\delta_{21} = \delta_{31} = 0$

Note: $\beta_{11} = 0 \Longrightarrow \delta_{21} = \delta_{31} = 0$, as X_1 is not included in the model.

Notes

Counting Parameters

- Whenever we remove a **continuous covariate**, from a model, we set **one** parameter to zero.
- Whenever we remove a **factor predictor** with L levels from a model, we set $L-1$ parameters to zero.
- Whenever we remove a two-way interaction between these variables from a model, we set $1.(L-1) = L-1$ parameters to zero.

Notes

Models 0,1,2,3 are nested inside Model 4.

Two approaches to finding the best model are used:

1. Start with Model 0 and try to add terms that improve the model fit (**Forward Selection**)
2. Start with Model 4 and try to remove terms that improve the model fit (**Backward Selection**)

Note:

- Models 0,1 and 2 are nested inside Model 3.
- Model 0 is nested inside Models 1 and 2.

Therefore we can begin with Model 4, or Model 3 or Model 1 or 2, and simplify to a nested model.

Notes

Example (Diabetes Data)

Here $n = 144$. From output handouts:

Model	Description	SSE	p
0	1	28.504	1
1	X_2	4.160	3
2	X_1	3.738	2
3	$X_1 + X_2$	1.472	4
4	$X_1 + X_2 + X_1.X_2$	1.318	6

p is the number of non-zero parameters; k or g is always $p - 1$ in the following analysis.

Notes

Backward Selection:

Complete Model : Model 4
Reduced Model : Model 3

Here $k = 5, g = 3$ so $k - g = 2$, and

$$n - k - 1 = 144 - 5 - 1 = 138.$$

We have

$$F = \frac{(SSE_R - SSE_C)/(k-g)}{SSE_C/(n-k-1)} = \frac{(1.472 - 1.318)/2}{1.318/138} = 8.062$$

We compare this with the

$$\text{Fisher-F}(k - g, n - k - 1) = \text{Fisher-F}(2, 138)$$

distribution; we have $F_\alpha(2, 138) = 3.061$, so we

$$\text{Reject } H_0 \text{ at } \alpha = 0.05$$

Notes

i.e. Model 4

$$X_1 + X_2 + X_1.X_2$$

fits **significantly better** than Model 3

$$X_1 + X_2.$$

- we cannot simplify the complete model to the reduced model without the loss of significant explanatory power.

The Interaction is Necessary in the Model

Backward selection stops here; we cannot simplify further from the complete model.

Notes

Forward Selection: we start with Model 0 and build up.

Model 1 vs Model 0 $F = 412.568$

Model 2 vs Model 0 $F = 940.846$

It seems that Model 2 is the better improvement, so we try the selection path

$$\text{Model 0} \longrightarrow \text{Model 2} \longrightarrow \text{Model 3} \longrightarrow \text{Model 4}$$

Model	SSE	$SSE_R - SSE_C$
0	28.504	-
2	3.738	24.766
3	1.472	2.266
4	1.318	0.154

ie we work down the table, $28.504 - 3.738 = 24.766$ etc.

Notes

Comparison	k	g	SSE_C	$SSE_R - SSE_C$	F
2 vs 0	1	0	3.738	24.766	940.82
3 vs 2	3	1	1.472	2.266	107.76
4 vs 3	5	3	1.318	0.154	8.06

Recall that $n = 144$, and

$$F = \frac{(SSE_R - SSE_C)/(k-g)}{SSE_C/(n-k-1)}$$

Under each H_0,

$$F \sim \text{Fisher-F}(k-g, n-k-1)$$

Notes

- $F_{0.05}(1, 142) \doteq 3.92 < 940.82$
  Therefore Model 0 is **NOT** an adequate simplification of Model 2
- $F_{0.05}(2, 140) \doteq 3.07 < 107.76$
  Therefore Model 2 is **NOT** an adequate simplification of Model 3
- $F_{0.05}(2, 138) \doteq 3.07 < 8.06$
  Therefore Model 3 is **NOT** an adequate simplification of Model 4

All of the null hypotheses are **rejected**.

Notes

Therefore by both forward and backward selection, we select Model 4

$$X_1 + X_2 + X_1.X_2$$

as the most appropriate model.

Note: In this sequence of hypothesis tests, the convention is **not** to correct for multiple testing (we don't know how many tests we are going to do), although a correction could be used.

Notes

F-tests for Unbalanced Designs

Example (Potato Damage Data)

The damage to potato plants caused by cold temperatures is to be studied.

In this experimental study, three binary factor predictors were used: we label them A, B and C rather than X_1, X_2, X_3 to recall the link with Factorial Designs in ANOVA. Each factor takes two levels:

	Factor	Levels
A	Potato Variety	0- Variety 1, 1- Variety 2
B	Acclimatization Routine	0- Room Temp, 1- Cold Room
C	Preparation Treatment	0- -4C, 1- -8C

Thus we have a $2 \times 2 \times 2$ three-way factorial design.

Notes

F-tests for Unbalanced Designs

However, the design is **unbalanced**; we have different numbers of replicates in each of the 8 factor-level combinations.

This means we cannot use conventional 3-way ANOVA; the lack of balance means that the stated p-values **may be misleading if we perform a standard ANOVA**.

Thus we are forced to use the F-test model comparison approach.

Notes

We begin with the most complex model and do backward selection.

Here the most complex model can be written

$$A + B + C + A.B + A.C + B.C + A.B.C$$

that is,

- all main effects (terms 1,2 and 3)
- all two-way interactions (terms 4,5 and 6)
- all three-way interactions (term 7)

We may write this model

$$A * B * C$$

which is termed the full factorial model.

Notes

Counting the numbers of parameters

Term	Parameters	
A	$(a-1)$	1
B	$(b-1)$	1
C	$(c-1)$	1
$A.B$	$(a-1)(b-1)$	1
$A.C$	$(a-1)(c-1)$	1
$B.C$	$(b-1)(c-1)$	1
$A.B.C$	$(a-1)(b-1)(c-1)$	1
Total		7

where $a = b = c = 2$.

We have 7 parameters in total (excluding the baseline mean) when all terms are considered, so

$$k = 7$$

Notes

In the following tables columns are:

Complete Model
Reduced Model
SSE_C
SSE_R
k
g
F (test statistic)
$F_{0.05}(k-g, n-k-1)$

We denote the critical value by F_α and check whether $F > F_\alpha$.

Notes

Potato Damage Data: ANOVA-F Tests

We compare four models: M_{R_1}, M_{R_2} and M_{R_3} are nested within the complete model M_C.

$$
\begin{array}{lcl}
M_C & : & A + B + C + A.B + A.C + B.C + A.B.C \\
M_{R_1} & : & A + B + C + A.B \\
M_{R_2} & : & A + B + C \\
M_{R_3} & : & A + B + A.B
\end{array}
$$

COMP.	RED.	SSE_C	SSE_R	k	g	F	F_α
M_C	M_{R_1}	4968.876	5093.746	7	4	0.561	2.76
M_{R_1}	M_{R_2}	5093.746	7183.674	4	3	28.721	3.92
M_{R_1}	M_{R_3}	5093.746	6319.640	4	3	16.846	3.92

Note: The quoted F_α values are approximate as the textbook does not tabulate all Fisher-F distributions. We take $\alpha = 0.05$

Notes

Conclusions

Taking the comparisons in order:

1. $M_C \, vs \, M_{R_1} \, : \, F < F_\alpha$. Therefore the result is **not significant**: Model M_{R_1} **is an adequate simplification** of Model M_C, and we choose M_{R_1} over M_C.

   The model M_{R_1} now becomes the complete model.

2. $M_{R_1} \, vs \, M_{R_2} \, : \, F > F_\alpha$. Therefore the result **is significant**: Model M_{R_2} **is not an adequate simplification** of Model M_{R_1}

3. $M_{R_1} \, vs \, M_{R_3} \, : \, F > F_\alpha$. Therefore the result **is significant**: Model M_{R_3} **is not an adequate simplification** of Model M_{R_1}

Notes

Thus the final model is

$$A + B + C + A.B$$

i.e. all main effects, plus the interaction between potato variety and acclimatization routine.

We cannot simplify this model further without significant loss in terms of goodness of fit.

Note: $R^2 = 0.631$ and Adjusted $R^2 = 0.610$, so we have a reasonable fit.

Notes

Task Distraction Data

Example (Task Distraction Data)

In an experimental study, the number of errors made in performing a specified task was recorded. The experiment investigated the influence of various predictors on the numbers of errors made.

There are two factor predictors (A, B) and one continuous covariate (X).

We have a balanced design with 15 people (replicates) in each factor-level subgroup.

Notes

Example (Task Distraction Data)

A	Group	1 : Non-smoker
		2 : Delayed smoker
		3 : Active smoker
B	Task	1 : Pattern Recognition
		2 : Cognitive Task
		3 : Driving Simulation
X	Distraction Level	

Notes

We compare four models with the **complete** model

Complete Model : $A * B * X$

$$A + B + X + A.B + A.X + B.X + A.B.X$$

Number of parameters

Term	Parameters		Tot.
A	$(a-1)$	$= 3-1$	2
B	$(b-1)$	$= 3-1$	2
X	(1)		1
$A.B$	$(a-1)(b-1)$	$= 2 \times 2$	4
$A.X$	$(a-1)(1)$	$= 2 \times 1$	2
$B.X$	$(b-1)(1)$	$= 2 \times 1$	2
$A.B.X$	$(a-1)(b-1)(c-1)$	$= 2 \times 2 \times 1$	4
Total			17

Notes

For illustration we consider the following sequence of models:

- Reduced Model 1: M_{R_1}

$$A + B + X + A.X + B.X$$

- Reduced Model 2: M_{R_2}

$$A + B + X + B.X$$

- Reduced Model 3: M_{R_3}

$$B + X + B.X$$

- Reduced Model 4: M_{R_4}

$$B + X$$

Notes

Task Distraction Data: ANOVA-F Tests

$$\begin{array}{lcl} M_C & : & A + B + X + A.B + A.X + B.X + A.B.X \\ M_{R_1} & : & A + B + X + A.X + B.X \\ M_{R_2} & : & A + B + X + B.X \\ M_{R_3} & : & B + X + B.X \\ M_{R_4} & : & B + X \end{array}$$

COMP.	RED.	SSE_C	SSE_R	k	g	F	F_α
M_C	M_{R_1}	5660.010	7627.479	17	9	5.084	2.02
M_{R_1}	M_{R_2}	7627.479	7971.274	9	7	2.817	3.07
M_{R_2}	M_{R_3}	7971.274	8404.654	7	5	3.452	3.07
M_{R_3}	M_{R_4}	8404.654	11154.715	5	3	21.105	3.07

Notes

Conclusions

Taking the comparisons in order:

1. $M_C \, vs \, M_{R_1} \, : \, F > F_\alpha$. Therefore the result is **significant**: Model M_{R_1} **is not an adequate simplification** of Model M_C
2. $M_{R_1} \, vs \, M_{R_2} \, : \, F < F_\alpha$. Therefore the result **is not significant**: Model M_{R_2} **is an adequate simplification** of Model M_{R_1}
3. $M_{R_2} \, vs \, M_{R_3} \, : \, F > F_\alpha$. Therefore the result **is significant**: Model M_{R_3} **is not an adequate simplification** of Model M_{R_2}
4. $M_{R_3} \, vs \, M_{R_4} \, : \, F > F_\alpha$. Therefore the result **is significant**: Model M_{R_4} **is not an adequate simplification** of Model M_{R_3}

Notes

Follow-up Analysis

In a follow-up analysis, it transpires that the model

$$A + B + X + A.B + A.X + B.X$$

ie selected.

Note: $R^2 = 0.863$ and Adjusted $R^2 = 0.831$, so we have a good fit.

Note: we must take great care with the sequence of models.

Notes

2.2.5 Pitfalls of Regression Modelling

Five issues to bear in mind in ANOVA, Regression and General Linear Modelling.

1. Model assumptions
2. Data transformations
3. Model selection
4. Multicollinearity
5. Predicting beyond the range of the covariates

See Handout.

Notes

Part III

Non-Parametric Statistics

Notes

Non-Parametric Statistics

All of the previous statistical analysis methods studied (t-tests, ANOVA, Regression, General Linear Modelling) have depended heavily on **distributional assumptions**.

i.e. we assume that the data are **Normally distributed**.

We now seek statistical procedures that do not rely on this strong assumption. We term these methods

NON-PARAMETRIC

or

DISTRIBUTION-FREE

They substitute **large sample approximations** for the distributional assumptions.

Notes

3.1 Distribution-free tests for Categorical Data

Categorical data are data in which experimental units are allocated to one of a number of categories according to their characteristics. The categories are defined by one or more factors

Examples:

- Female/Male - two categories
- Smoker/Former Smoker/Non Smoker - three categories.

Notes

Doll and Hill Data

Table 13.11. Smokers and non-smokers among male cancer patients and controls (Doll and Hill 1950)

	Smokers	Non-smokers	Total
Lung cancer	647	2	649
Controls	622	27	649

Notes

Juvenile Delinquency and Spectacle-Wearing

Table 10.14 Spectacle wearing among juvenile delinquen and non-delinquents who failed a vision test (Weindling *al* 1986)

		Juvenile delinquents	Non delinquents	Total
Spectacle wearers	Yes	1	5	6
	No	8	2	10
	Tota	9	7	16

Notes

The data are **counts** of experimental units that fall into each category. Suppose

1. There are n experimental units in the study
2. There are k categories
3. The probabilities of the k outcomes are $p_1, \ldots, p_k$, where

$$p_1 + \cdots + p_k = 1$$

4. The experimental units are **independent**
5. The counts in the k categories are $n_1, \ldots, n_k$, where

$$n_1 + \cdots + n_k = n$$

Notes

The experimental design is termed a **Multinomial Experiment**

Note: The categories can arise as combinations of factor levels; we can have

- **one-way** classification (categories of a single factor, A)
- **two-way** classification (categories defined by combinations of levels of two factors, A and B)

and so on. The counts table is often called a **contingency table** and the entries in the table are called **cells**.

The idea can be extended to larger numbers of factors $(A, B, C, \ldots)$ to produce a multi-way table. We will focus on at most two-way tables, with r rows and c columns, yielding an $r \times c$ table.

Notes

Recap: The Binomial Distribution

Suppose random variable X can take values on the set

$$\{0, 1, \ldots, n\}$$

for some whole number $n \geq 1$. The **Binomial distribution** is a probability distribution for which

$$f(x) = \mathrm{P}[X = x] = \binom{n}{x} p^x (1-p)^{n-x} \qquad 0 \leq x \leq n$$

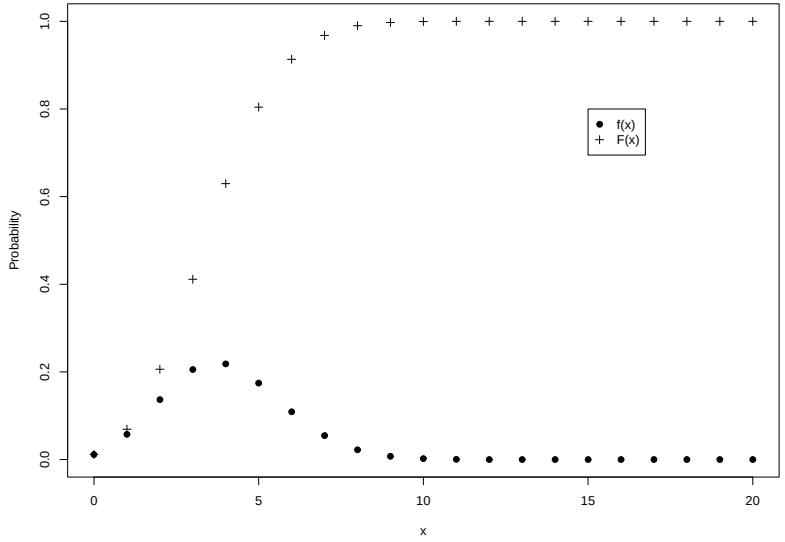
for some probability p.

The **cumulative probability function**, $F(x)$, is defined by

$$F(x) = \mathrm{P}[X \leq x] = \sum_{t=0}^{x} \binom{n}{t} p^t (1-p)^{n-t}$$

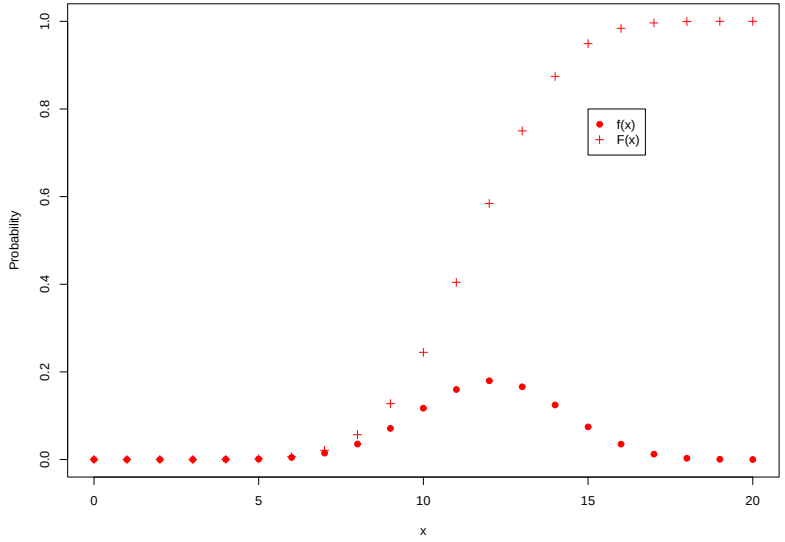
Notes

$n = 20, p = 0.2$



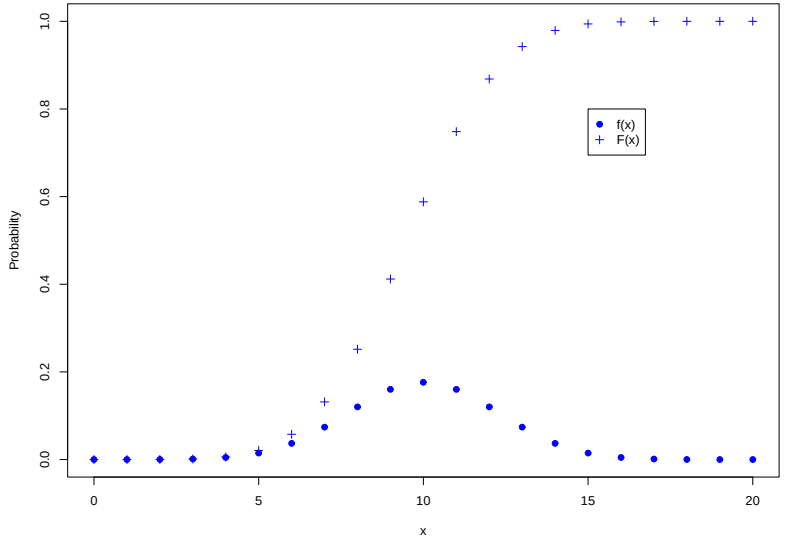
Notes

$n = 20, p = 0.6$



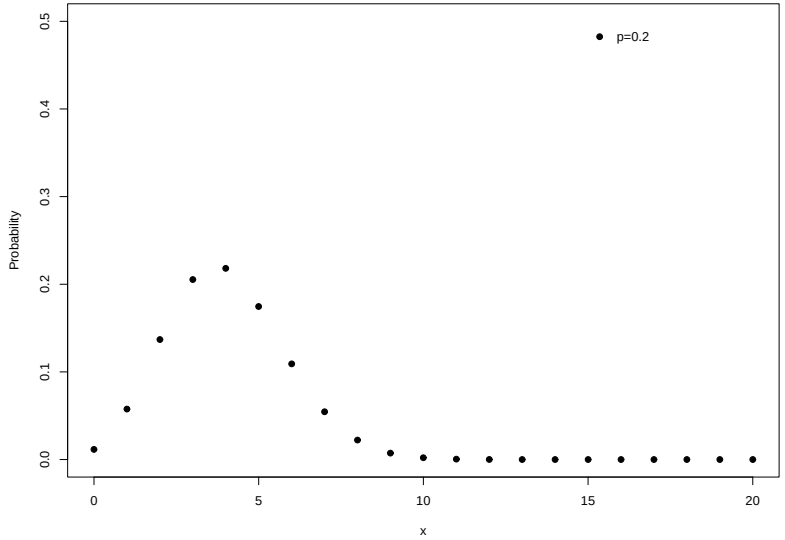
Notes

$n = 20, p = 0.5$



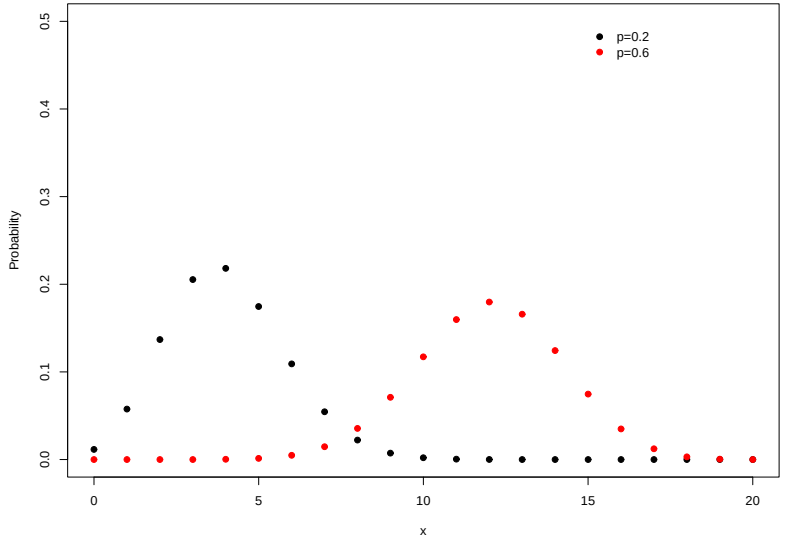
Notes

Comparison



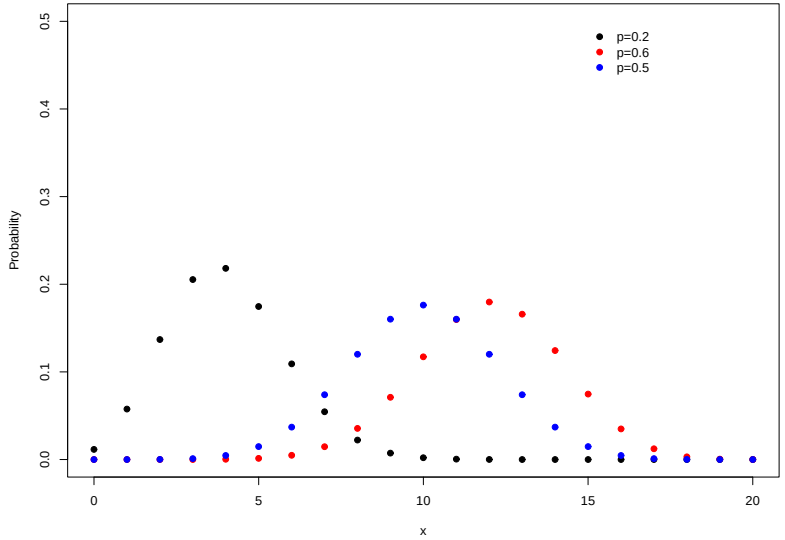
Notes

Comparison



Notes

Comparison



Notes

$f(x)$ and $F(x)$

		0	1	2	3	4	5	6	7	8	9	10
$p = 0.2$	$f(x)$	0.012	0.058	0.137	0.205	0.218	0.175	0.109	0.055	0.022	0.007	0.002
	$F(x)$	0.012	0.069	0.206	0.411	0.630	0.804	0.913	0.968	0.990	0.997	0.999
$p = 0.6$	$f(x)$	0.000	0.000	0.000	0.000	0.000	0.001	0.005	0.015	0.035	0.071	0.117
	$F(x)$	0.000	0.000	0.000	0.000	0.000	0.002	0.006	0.021	0.057	0.128	0.245
$p = 0.5$	$f(x)$	0.000	0.000	0.000	0.001	0.005	0.015	0.037	0.074	0.120	0.160	0.176
	$F(x)$	0.000	0.000	0.000	0.001	0.006	0.021	0.058	0.132	0.252	0.412	0.588

		11	12	13	14	15	16	17	18	19	20
$p = 0.2$	$f(x)$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	$F(x)$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$p = 0.6$	$f(x)$	0.160	0.180	0.166	0.124	0.075	0.035	0.012	0.003	0.000	0.000
	$F(x)$	0.404	0.584	0.750	0.874	0.949	0.984	0.996	0.999	1.000	1.000
$p = 0.5$	$f(x)$	0.160	0.120	0.074	0.037	0.015	0.005	0.001	0.000	0.000	0.000
	$F(x)$	0.748	0.868	0.942	0.979	0.994	0.999	1.000	1.000	1.000	1.000

Notes

Tests for Binomial/Multinomial Data

What kinds of tests can be carried out for such data ?

1. Tests about $p_1, \ldots, p_k$
   - $H_0 : p_1 = \cdots = p_k = 1/k$
   - $H_0 : p_1, \ldots, p_k$ determined by some parametric distribution (Normal, Poisson etc.)
2. Tests about the factors A and B
   - are A and B dependent ?
   - i.e. does classification by A influence classification by B.

Notes

Chi-Squared Test

For one-way tables: suppose that a null hypothesis **completely specifies** $p_1, \ldots, p_k$, that is, we have H_0 of the form

$$H_0 : p_1 = p_1^{(0)}, \ldots, p_k = p_k^{(0)}$$

where $p_1^{(0)}, \ldots, p_k^{(0)}$ are fixed probabilities. For example, for $k = 3$,

$$H_0 : p_1 = p_2 = p_3 = 1/3$$

or

$$H_0 : p_1 = 1/2, p_2 = p_3 = 1/4$$

Notes

To test this hypothesis against the general alternative hypothesis

$$H_a \ : \ H_0 \text{ not true.}$$

we use the test statistic

$$X^2 = \sum_{i=1}^{k} \frac{\left(n_i - np_i^{(0)}\right)^2}{np_i^{(0)}}$$

If H_0 is true,

$$X^2 \approx \text{Chi-squared}(k-1).$$

that is, X^2 is approximately distributed as Chi-squared$(k-1)$.

Notes

In this formula

- n_i is the **observed** count in cell i
- $np_i^{(0)}$ is the **expected** count in cell i if H_0 is **true**.

Sometimes the formula is written

$$X^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

where O_i is the observed count, and E_i is the expected count.

If

$$X^2 > \text{Chisq}_\alpha(k-1)$$

then we reject H_0 at the α significance level, where $\text{Chisq}_\alpha(k-1)$ is the $1-\alpha$ (right-hand) tail critical value of the Chi-squared distribution with $k-1$ degrees of freedom.

Notes

This method can be extended in the one-way case to test **distribution** assumptions, that is, for example

$$H_0 \ : \text{Data Normally distributed}$$

or

$$H_0 \ : \text{Data Poisson distributed}$$

Notes

Worked Example

For the **two-way** table, we can test the hypothesis

H_0 : Factor A and Factor B levels are assigned independently

that is, classification by factor A is independent of classification by factor B. We use the same test statistic that can be rewritten

$$X^2 = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{(n_{ij} - \widehat{n}_{ij})^2}{\widehat{n}_{ij}}$$

where

$$\widehat{n}_{ij} = \frac{n_{i.} n_{.j}}{n} \qquad n_{i.} = \sum_{j=1}^{c} n_{ij} \qquad n_{.j} = \sum_{i=1}^{r} n_{ij}.$$

The terms $n_{i.}$ and $n_{.j}$ are the row and column totals for row i and column j respectively.

Notes

If H_0 is true

$$X^2 \approx \text{Chi-squared}((r-1)(c-1))$$

i.e. the degrees of freedom quantity is $(r-1)(c-1)$. Otherwise the test proceeds as before: we check whether

$$X^2 > \text{Chisq}_\alpha((r-1)(c-1))$$

and if so, we reject H_0.

Notes

Example: 2×3 contingency table

Patients suffering from a chronic condition are recruited to a randomized clinical trial with three treatment groups:

- Drug 1
- Drug 2
- Drug 3

The alleviation of the chronic condition was studied:

- Condition did not improve
- Condition did improve

Notes

Data:

Improved	Drug 1	2	3	Total
No	10	12	10	32
Yes	10	18	15	43
Total	20	30	25	75

Notes

Notation:

Improved	Drug 1	2	3	Total
No	n_{11}	n_{12}	n_{13}	$n_{1.}$
Yes	n_{21}	n_{22}	n_{23}	$n_{2.}$
Total	$n_{.1}$	$n_{.2}$	$n_{.3}$	n

Notes

Scientific question

Is there a difference between the improvement rates across drugs ?

Null Hypothesis:

H_0 : Improvement rate is the same across drugs.

Under the null hypothesis, there is no difference between the drugs, and we would consider the elements in the columns of the table as realizations of three independent Binomial sampling schemes with the same probability of success p.

The best estimate of the common improvement rate p is

$$\widehat{p} = \frac{n_{2.}}{n} = \frac{43}{75} = 0.573$$

Notes

That is

$$\begin{aligned} n_{21} &\sim Binomial(n_{.1}, p) \\ n_{22} &\sim Binomial(n_{.2}, p) \\ n_{23} &\sim Binomial(n_{.3}, p) \end{aligned}$$

and thus, under H_0, we use the fitted values determined by the expectations of these distributions

$$\begin{aligned} \widehat{n}_{21} &= n_{.1}\widehat{p} = \frac{n_{.1}n_{2.}}{n} \\ \widehat{n}_{22} &= n_{.2}\widehat{p} = \frac{n_{.2}n_{2.}}{n} \\ \widehat{n}_{23} &= n_{.3}\widehat{p} = \frac{n_{.3}n_{2.}}{n} \end{aligned}$$

Notes

We can complete the "fitted" or "expected" table by noting that we must have

$$\begin{aligned} n_{.1} &= \widehat{n}_{11} + \widehat{n}_{21} \\ n_{.2} &= \widehat{n}_{12} + \widehat{n}_{22} \\ n_{.3} &= \widehat{n}_{13} + \widehat{n}_{23} \end{aligned}$$

This is sufficient to allow us to complete the "fitted" table

Notes

Fitted table under H_0

	Drug			
Improved	1	2	3	Total
No	8.533	12.800	10.667	32
Yes	11.467	17.200	14.333	43
Total	20.000	30.000	25.000	75

so that the test statistic is

$$X^2 = \sum_{i=1}^{2}\sum_{j=1}^{3} \frac{(n_{ij} - \widehat{n}_{ij})^2}{\widehat{n}_{ij}} = 0.599$$

Notes

Completing the test

Under H_0, we have that

$$X^2 \approx \text{Chi-squared}((r-1)(c-1))$$

Here $r = 2$, $c = 3$, so

$$X^2 \approx \text{Chi-squared}(2)$$

From the tables in McClave and Sincich Table VII, we find that

$$\text{Chisq}_{0.05}(2) = 5.991$$

so we **do not reject** the null hypothesis, and conclude that there is no evidence of a difference between the improvement rates for the three drugs.

Notes

Example: Nose bleeds and Alcohol Consumption

140 patients suffering from nose bleeds, and 113 controls were interviewed about their alcohol consumption, specifically whether they were

- 1. Non-drinkers
- 2. Occasional drinkers
- 3. Regular drinkers

Scientific question

Is there a difference between the proportions of different consumption groups in nose-bleed cases and controls ?

Notes

Data:

Improved	Group 1	2	3	Total
Nose-bleed	47	30	63	140
Control	39	40	34	113
Total	86	70	97	253

Notes

R Code: Nose Bleed data

```
> NoseBleeds<-matrix(c(47,30,63,39,40,3),
                nrow=2,byrow=T)
> chisq.test(NoseBleeds)

        Pearson's Chi-squared test

data:  NoseBleeds
X-squared = 44.61, df = 2, p-value = 2.056e-10
```


Notes

R Code: Nose Bleed data

```
> #Form the fitted values from
> #the row and column totals
> NB.rowtots<-apply(NoseBleeds,1,sum)
> NB.coltots<-apply(NoseBleeds,2,sum)
>(NB.fitted<-outer(NB.rowtots,
                   NB.coltots,"*")/sum(NoseBleeds))
         [,1]     [,2]     [,3]
[1,] 54.23423 44.14414 41.62162
[2,] 31.76577 25.85586 24.37838
>
> (Xsq<-sum((NoseBleeds-NB.fitted)^2/NB.fitted))
[1] 44.61003
> (pval<-1-pchisq(Xsq,2))
[1] 2.056156e-10
```


Notes

Notes

Example (DNA Sequence Data)

Counts of Nucleotides A,C,G,T in a genomic segment related to the breast cancer gene BRCA2.

Example (Eye and Hair Colour Data)

The assignment of hair and eye colour in a sample of humans

See Handout.

Note: For the Chi-squared test to be valid, we need the expected cell counts

$$np_i^{(0)} \qquad i = 1, \ldots, k$$

or

$$\widehat{n}_{ij} \qquad i = 1, \ldots, r, j = 1, \ldots, c$$

to be sufficiently large. The convention is to require the expected value to be greater than **five**.

Notes

Note: If $r = c = 2$ we have a 2×2 table, and another **exact** test can be used which does not rely on the large sample approximation

Fisher's Exact Test

- another test for independence of assignment of the row and column factor levels
- test statistic and null distribution are complicated (based on the **hypergeometric distribution**)
- `R` computes test statistic and p-value.

Notes

Example (Juvenile Delinquency and Spectacle Wearing)

Is there any association between the two factors ?

A : Spectacle Wearing (Yes/No)

B : Juvenile Delinquent (Yes/No)

		Delinquent		
		Yes	No	$n_{i.}$
Spectacles	Yes	1	5	6
	No	8	2	10
	$n_{.j}$	9	7	16

Notes

Example (Juvenile Delinquency and Spectacle Wearing)

Chi-squared Test:

$$X^2 = 6.112$$

Compare with Chi-squared$((r-1)(c-1))$ = Chi-squared(1); we have

$$\text{Chi-squared}_{0.05}(1) = 3.841$$

and a p-value of 0.013. Therefore we **reject** H_0.

Fisher's Exact Test: p-value is 0.035 (1-sided) or 0.024 (2-sided).

Thus we reject H_0 and we have evidence of association between the factors.

Notes

Case-Control Studies

A **case-control** study is an observational study where participants are selected for the study with regard to their **disease status**.

- a sample of **cases** (disease sufferers)
- a sample of **controls** (healthy patients)

We investigate the possible association between disease status and a factor that takes two levels. A 2×2 table of counts is formed for all combinations of disease status/factor level.

Notes

Example (BCG Vaccination and Leprosy)

Disease Status : Leprosy Sufferer (Yes/No)
Factor : Vaccination Scar (Yes/No)

		Disease Status: Case	Disease Status: Control	
		Yes	No	$n_{i.}$
Scar	Yes	101	554	655
	No	159	446	605
	$n_{.j}$	260	1000	1260

Is there an association ? Does vaccination induce leprosy ?

Notes

Odds Ratio

An alternative test statistic is based on the **odds ratio**

$$\text{O.R.} = \frac{n_{11}n_{22}}{n_{12}n_{21}} = \widehat{\psi}$$

say. The test statistic is

$$Z = \frac{\log \widehat{\psi}}{\text{s.e.}(\log \widehat{\psi})}$$

where

$$\text{s.e.}(\log \widehat{\psi}) = \sqrt{\frac{1}{n_{11}} + \frac{1}{n_{12}} + \frac{1}{n_{21}} + \frac{1}{n_{22}}}$$

Notes

That is,

$$Z = \frac{\log n_{11} + \log n_{22} - \log n_{12} - \log n_{21}}{\sqrt{\frac{1}{n_{11}} + \frac{1}{n_{12}} + \frac{1}{n_{21}} + \frac{1}{n_{22}}}}$$

Under

H_0 : No association between factor and disease status

it follows that

$$Z \dot{\sim} N(0, 1)$$

Here log means ln or **natural log**.

Notes

Example (BCG Vaccination and Leprosy)

$$n_{11} = 101, n_{12} = 554, n_{21} = 159, n_{22} = 446$$

Therefore

$$\widehat{\psi} = \frac{n_{11}n_{22}}{n_{12}n_{21}} = 0.511 \qquad \log \widehat{\psi} = -0.671$$

and

$$\text{s.e.}(\log \widehat{\psi}) = \sqrt{\frac{1}{n_{11}} + \frac{1}{n_{12}} + \frac{1}{n_{21}} + \frac{1}{n_{22}}} = 0.142$$

so

$$Z = \frac{-0.671}{0.142} = -4.717$$

For a text at $\alpha = 0.05$, the two-sided critical value is ± 1.96, so we

Reject H_0.

Notes

Example (Smoking and Lung Cancer)

$$n_{11} = 647, n_{12} = 622, n_{21} = 2, n_{22} = 27$$

Therefore

$$\log \widehat{\psi} = \log \frac{647 \times 27}{2 \times 622} = 2.642$$

and

$$\text{s.e.}(\log \widehat{\psi}) = \sqrt{\frac{1}{647} + \frac{1}{2} + \frac{1}{622} + \frac{1}{27}} = 0.735$$

so

$$Z = \frac{2.642}{0.735} = 3.590$$

For a text at $\alpha = 0.05$, the two-sided critical value is ± 1.96, so we

Reject H_0

and report evidence for association.

Notes

We seek non-parametric or distribution-free tests for hypotheses relating to single samples, the equivalents of one-sample Z- or T-tests, which rely on the **normality** of the samples.

Normally these tests are formulated in terms of **ranks** of the data to give

Rank Tests

Notes

Notes

For example, if the data are

0.24 3.16 1.97 2.10 0.92

we sort them into **ascending** order, and assign ranks in order

	0.24	0.92	1.97	2.10	3.16
Rank	1	2	3	4	5

The tests depend on the behaviour of statistics computed in terms of the ranks, and rely on a **large sample** justification.

Rather than test the **mean**, we test the **median**, x_{MED}, where

$$\Pr[\text{Observation } \leq x_{\text{MED}}] = \frac{1}{2}$$

i.e. the halfway point of the distribution.

The **sample median** is the halfway point of the sorted sample.

Let η denote the population median. We wish to test, for example,

$$H_0 \, : \, \eta = \eta_0$$

See Handout

Notes

3.3 Comparing Two Populations : Independent Samples

We seek a non-parametric equivalent to the two-sample t-test.

Instead of testing population **means**,

$$H_0 \, : \, \mu_1 = \mu_2$$

we test population **medians**

$$H_0 \, : \, \eta_1 = \eta_2$$

Notes

One- and Two- sample tests

- In the **one sample** case we use the

  SIGN TEST

  to test hypotheses about η

- In the **two sample** case we use the

  WILCOXON RANK SUM or **MANN-WHITNEY** ***U*** test.

See Handout

Notes

Note: For the MWW test

- **Textbook convention :** Label the samples so that $n_1 > n_2$ (i.e. sample 1 is the one with the larger sample size)

- **SPSS convention :** Label the samples such that

  $$x_{\text{MED}_1} < x_{\text{MED}_2}$$

  (i.e. sample 1 is the one with the smaller median) and only test

  $$H_0 \ : \ \eta_1 = \eta_2$$

Notes

Other two sample tests are available:

- Kolmogorov-Smirnov Test
- Moses Extreme Reactions Test
- Wald-Wolfowitz Runs Test

None make distributional assumptions, all perform best when the sample size is large.

Notes

3.4 Comparing Two Dependent Samples

Suppose we have repeat measurements on the same experimental units.

In this case, the **within-subject** data are **dependent**; we have pairing of observations.

We can use the

Wilcoxon Signed Rank Test

See Handout

Notes

The Wilcoxon Signed Rank Test

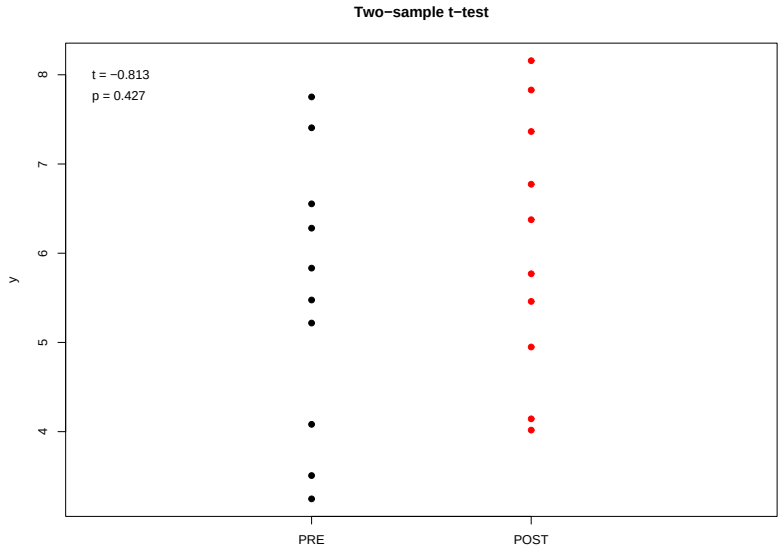
Remember:

- The **Wilcoxon** Test is a one or two sample test for **independent** samples.
- The **Wilcoxon Signed Rank Test** is a test for **paired** data

Notes

Two sample test **without** Pairing

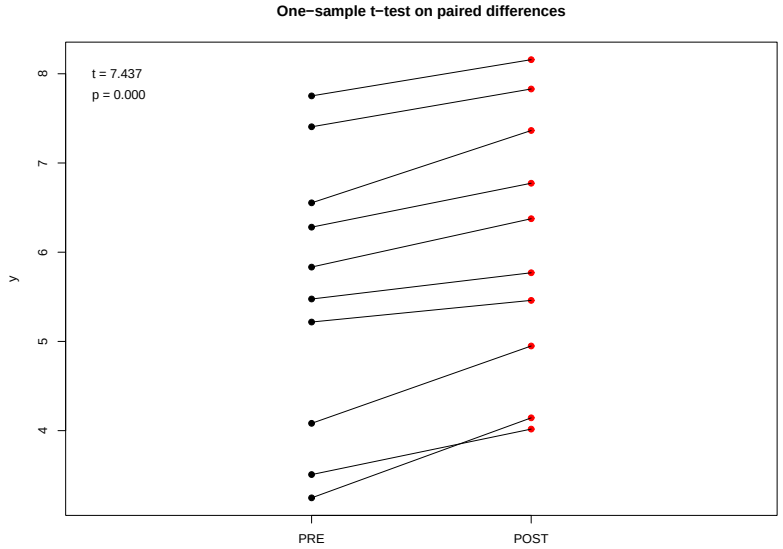
No rejection of the null.



Notes

Two sample test **with** Pairing

Rejection of the null.



Notes

The Wilcoxon Signed Rank Procedure

For paired pre-treatment $(y_{11}, \ldots, y_{n1})$ and post-treatment: $(y_{21}, \ldots, y_{n2})$ data, the procedure is as follows:

1. Form $x_i = y_{i1} - y_{i2}$, for $i = 1, \ldots, n$.
2. Form $s_i = |x_i|$, for $i = 1, \ldots, n$.
3. Drop every $x_i = 0$, leaving a sample size of m.
4. Assign ranks $r_1, \ldots, r_m$ to $s_1, \ldots, s_m$ after sorting into ascending order.
5. Form

$$T_+ = \sum_{i=1}^{m} r_i z_i \qquad T_- = \sum_{i=1}^{m} r_i (1 - z_i)$$

where

$$z_i = \begin{cases} 1 & x_i > 0 \\ 0 & x_i < 0 \end{cases}$$

Notes

Computing the Null Distribution

The behaviour of T_+ and T_- can be predicted under the null hypothesis under the assumption that the original data are generated from **continuous distributions** with the same **median**.

- Under this assumption, we will **never** get $x_i = 0$.
- Under this assumption, if K is the random variable recording the **number** of positive x_i values, then

$$K \sim Binomial(n, 1/2)$$

  as in the **sign test**.

- Under this assumption, given that $K = k$, T_+ is the sum of k numbers (ranks) chosen at random with replacement from the set $\{1, 2, \ldots, n\}$. Similarly, T_- is the sum of the remaining $n - k$ ranks.

Notes

By enumerating all the possible selections, we can compute the probability distribution (conditional on $K = k$) of T_+; denote it

$$p(t|k).$$

Then, using the **Theorem of Total Probability** and the binomial distribution formula, we can obtain the null distribution as

$$\begin{aligned} P[T_+ = t] &= \sum_{k=0}^{n} P[T_+ = t|K = k]\, P[K = k] \\ &= \sum_{k=0}^{n} p(t|k) \binom{n}{k} \left(\frac{1}{2}\right)^n \end{aligned}$$

An equivalent calculation holds for T_-; in fact, under H_0, the distributions of T_+ and T_- are identical.

Notes

Example: $n = 5$

We have to enumerate all possible sums of k numbers chosen from $\{1, \ldots, n\}$, for each $k = 0, \ldots, n$.

k	Possible values of T_+
0	0
1	1,2,. . . ,5
2	1+2=3, 1+3=4, 1+4=5, 2+3=5, 1+5=6, 2+4=6, 2+5=7, 3+4=7, 3+5=8, 4+5=9
3	1+2+3=6, 1+2+4=7, 1+2+5=8, 1+3+4=8, 1+3+5=9, 2+3+4=9, 1+4+5=10, 2+3+5=10, 2+4+5=11, 3+4+5=12
4	1+2+3+4=10, 1+2+3+5=11, 1+2+4+5=12, 1+3+4+5=13, 2+3+4+5=14
5	1+2+3+4+5=15

Notes

Example: $n = 5$

Therefore the probability distribution of T_+ given k is given by

	Possible values of T_+															
k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	0	0	0	0	0	0	0	0	0	0
2	0	0	0	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	0	0	0	0	0	0
3	0	0	0	0	0	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Notes

Example: $n = 5$

Hence, using the previous formula, we can compute $P[T_+ = t]$:

	Possible values of T_+							
t	0	1	2	3	4	5	6	7
$P[T_+ = t]$	0.031	0.031	0.031	0.062	0.062	0.094	0.094	0.094
$P[T_+ \leq t]$	0.031	0.062	0.094	0.156	0.219	0.312	0.406	0.500
t	8	9	10	11	12	13	14	15
$P[T_+ = t]$	0.094	0.094	0.094	0.062	0.062	0.031	0.031	0.031
$P[T_+ \leq t]$	0.594	0.688	0.781	0.844	0.906	0.938	0.969	1.000

Notes

Note: The same calculation WITH zeros

If the $x_i = 0$ data are left in the sample, then the calculation becomes more complicated

- The selection of the test statistic is not straightforward; T_+ and T_- are still the obvious choices that will distinguish H_a from H_0.
- In the presence of zeros the distribution of K is **no longer Binomial**.
- We need to propose a model for the **number** of zeros.
- We would need a **different** table of critical values for each different number of zeros.

Overall, it is more straightforward to **omit** the zeros; it can be shown that this does not compromise the effectiveness of the test.

Notes

3.4 Comparing Three or More Populations

We now seek non-parametric equivalents to ANOVA useful for different designs. We study tests for

(a) the **Completely Randomized Design** (CRD)

(b) the **Randomized Block Design** (RBD)

For (a) we use the

Kruskal-Wallis Test

and for (b) we use the

Friedman Test.

See Handout

Notes

Summary of the Non-Parametric Tests

- **Chi-Squared Test :** Goodness of Fit/independence in contingency tables
- **Sign Test :** One Sample (equivalent of one sample t-test)
- **Mann-Whitney-Wilcoxon :** Two Sample (equivalent of two sample t-test)
- **Wilcoxon Signed Rank :** Paired Data
- **Kruskal-Wallis :** one-way layout, multigroup comparison - equivalent of ANOVA for CRD.
- **Friedman :** two-way blocked layout, equivalent of two-way ANOVA for RBD.

Notes

Pros:

- No distributional assumptions
- Applicable for most sorts of data
- Large sample approximations make them easy to implement

Cons:

- Low power compared to parametric tests (i.e. often do not reject H_0 when they should - prone to Type II Error)
- Small sample null distributions difficult to compute.

Notes

3.6 Rank Correlation

To measure the association between two variables, we previously used the *correlation coefficient*, r; for data $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$,

$$r = \frac{SS_{xy}}{\sqrt{SS_{xx}SS_{yy}}}$$

where

$$SS_{xy} = \sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y}) \quad SS_{xx} = \sum_{i=1}^{n}(x_i - \overline{x})^2 \quad SS_{yy} = \sum_{i=1}^{n}(y_i - \overline{y})^2$$

r is a measure of the linear association between X and Y

Pearson Product Moment Coefficient of Correlation

Notes

A more general measure of association is the

Spearman Rank Correlation Coefficient

We compute this as follows:

1. For each sample separately, compute the **ranks** of the data, denote the ranks for the data $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ by $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ respectively.

2. Compute
$$r_S = \frac{SS_{uv}}{\sqrt{SS_{uu}SS_{vv}}}$$
ie the Pearson correlation between the ranks.

r_S is the **Spearman Correlation**.

Notes

Notes:

1. If there are no ties in the data
$$r_S = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2-1)}$$
where $d_i = u_i - v_i$.

2. r_S is potentially a measure of the **non-linear** association between X and Y.

   The calculation can be applied directly to rank data i.e. $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ can be preference ranks given by two observers.

Notes

Tests for r_S

To test

$$H_0 \ : \ \rho = 0$$

vs

$$\begin{array}{lll} (1) & H_a & : \quad \rho > 0 \\ (2) & H_a & : \quad \rho < 0 \\ (3) & H_a & : \quad \rho \neq 0 \end{array}$$

We may use r_S as a test statistic. The distribution of r_S under H_0 is tabulated in McClave and Sincich.

Notes

If Spearman_α is the α tail quantile of the null distribution, we have the following rejection regions:

$$\begin{array}{lll} (1) & : & \text{Reject } H_0 \text{ if } r_S > \text{Spearman}_\alpha \\ (2) & : & \text{Reject } H_0 \text{ if } r_S < -\text{Spearman}_\alpha \\ (3) & : & \text{Reject } H_0 \text{ if } |r_S| > \text{Spearman}_{\alpha/2} \end{array}$$

Notes

The Role of Randomization and Permutation Tests

Randomization or **Permutation** procedures are useful for computing **exact** null distributions for non-parametric test statistics when sample sizes are small.

We focus first on two sample comparisons; suppose that two data samples $x_1 \ldots, x_{n_1}$ and $y_1 \ldots, y_{n_2}$ (where $n_1 \geq n_2$) have been obtained, and we wish to carry out a comparison of the two populations from which the samples are drawn. The Wilcoxon test statistic, W, is the sum of the ranks for the second sample. The permutation test proceeds as follows:

Notes

1. Let $n = n_1 + n_2$. Assuming that there are no ties, the pooled and ranked samples will have ranks

   $$1 \quad 2 \quad 3 \quad \ldots \quad n$$

2. The test statistic is $W = R_2$, the rank sum for sample two items. For the observed data, W will be the sum of n_2 of the ranks given in the list above.

3. If the null hypothesis

   H_0 : No difference between population 1 and population 2

   were **true**, then we would expect **no pattern** in the arrangements of the group labels when sorted into ascending order. That is, the sorted data would give rise a **random** assortment of group 1 and group 2 labels.

Notes

4. To obtain the exact distribution of W under H_0 (which is what we require for the assessment of statistical significance), we could compute W for all possible permutations of the group labels, and then form the probability distribution of the values of W. We call this the **permutation null distribution**.

5. But W is a rank sum, so we can compute the permutation null distribution simply by tabulating **all possible subsets** of size n_2 of the set of ranks $\{1, 2, 3, \ldots, n\}$.

Notes

6. There are

$$\binom{n}{n_2} = \frac{n!}{n_1!\, n_2!} = N$$

say possible subsets of size n_2. For example, for $n = 6$ and $n_2 = 2$, the number of subsets of size n_2 is

$$\binom{8}{2} = \frac{8!}{6!\, 2!} = 28$$

However, the number of subsets increases dramatically as n increases; for $n_1 = n_2 = 10$, so that $n = 20$, the number of subsets of size n_2 is

$$\binom{20}{10} = \frac{20!}{10!\, 10!} = 184756$$

Notes

7. The exact rejection region and p-value are computed from the permutation null distribution. Let $W_i, i = 1, \ldots, N$ denote the value of the Wilcoxon statistic for the N possible subsets of the ranks of size n_2. The probability that the test statistic, W, is less than or equal to w is

$$\Pr[W \leq w] = \frac{\text{Number of } W_i \leq w}{N}$$

We seek the values of w that give the appropriate rejection region, $\mathcal{R}$, so that

$$\Pr[W \in \mathcal{R}] = \frac{\text{Number of } W_i \in \mathcal{R}}{N} = \alpha$$

It may not be possible to find critical values, and define $\mathcal{R}$, so that this probability is **exactly** α as the distribution of W is **discrete**.

Notes

Simple Example

Suppose $n_1 = 7$ and $n_2 = 3$. There are

$$\binom{10}{3} = \frac{10!}{7!\,3!} = 120$$

subsets of the ranks $\{1, 2, 3, \ldots, 10\}$ of size 3. The subsets are listed below, together with the rank sums.

Notes

Ranks			W	Ranks			W	Ranks			W	Ranks			W
1	2	3	6	1	7	8	16	2	7	10	19	4	6	7	17
1	2	4	7	1	7	9	17	2	8	9	19	4	6	8	18
1	2	5	8	1	7	10	18	2	8	10	20	4	6	9	19
1	2	6	9	1	8	9	18	2	9	10	21	4	6	10	20
1	2	7	10	1	8	10	19	3	4	5	12	4	7	8	19
1	2	8	11	1	9	10	20	3	4	6	13	4	7	9	20
1	2	9	12	2	3	4	9	3	4	7	14	4	7	10	21
1	2	10	13	2	3	5	10	3	4	8	15	4	8	9	21
1	3	4	8	2	3	6	11	3	4	9	16	4	8	10	22
1	3	5	9	2	3	7	12	3	4	10	17	4	9	10	23
1	3	6	10	2	3	8	13	3	5	6	14	5	6	7	18
1	3	7	11	2	3	9	14	3	5	7	15	5	6	8	19
1	3	8	12	2	3	10	15	3	5	8	16	5	6	9	20
1	3	9	13	2	4	5	11	3	5	9	17	5	6	10	21
1	3	10	14	2	4	6	12	3	5	10	18	5	7	8	20
1	4	5	10	2	4	7	13	3	6	7	16	5	7	9	21
1	4	6	11	2	4	8	14	3	6	8	17	5	7	10	22
1	4	7	12	2	4	9	15	3	6	9	18	5	8	9	22
1	4	8	13	2	4	10	16	3	6	10	19	5	8	10	23
1	4	9	14	2	5	6	13	3	7	8	18	5	9	10	24
1	4	10	15	2	5	7	14	3	7	9	19	6	7	8	21
1	5	6	12	2	5	8	15	3	7	10	20	6	7	9	22
1	5	7	13	2	5	9	16	3	8	9	20	6	7	10	23
1	5	8	14	2	5	10	17	3	8	10	21	6	8	9	23
1	5	9	15	2	6	7	15	3	9	10	22	6	8	10	24
1	5	10	16	2	6	8	16	4	5	6	15	6	9	10	25
1	6	7	14	2	6	9	17	4	5	7	16	7	8	9	24
1	6	8	15	2	6	10	18	4	5	8	17	7	8	10	25
1	6	9	16	2	7	8	17	4	5	9	18	7	9	10	26
1	6	10	17	2	7	9	18	4	5	10	19	8	9	10	27

Notes

There are 22 possible rank sums, $\{6, 7, 8, \ldots, 25, 26, 27\}$; the number of times each is observed is displayed in the table below, with the corresponding probabilities and cumulative probabilities.

W	6	7	8	9	10	11	12	13	14	15	16
Frequency	1	1	2	3	4	5	7	8	9	10	10
Prob.	0.008	0.008	0.017	0.025	0.033	0.042	0.058	0.067	0.075	0.083	0.083
Cumulative Prob.	0.008	0.017	0.033	0.058	0.092	0.133	0.192	0.258	0.333	0.417	0.500

W	17	18	**19**	20	21	22	23	24	25	26	27
Frequency	10	10	**9**	8	7	5	4	3	2	1	1
Prob.	0.083	0.083	**0.075**	0.067	0.058	0.042	0.033	0.025	0.017	0.008	0.008
Cumulative Prob.	0.583	0.667	0.742	0.808	0.867	0.908	0.942	0.967	0.983	0.992	1.000

Notes

Thus, for example, the probability that $W = 19$ is 0.075, with a frequency of 9 out of 120. From this table, we deduce that

$$\Pr[8 \leq W \leq 25] = 0.983 - 0.033 = 0.950$$

implying that the two-sided rejection region for $\alpha = 0.05$ is the set $\mathcal{R} = \{6, 7, 26, 27\}$.

Notes

Placenta Permeability Data

Example (Placenta Permeability Data)

Measurements of placenta permeability are made on two groups of subjects.

The data and their ranks for are displayed below:

Group	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2
Obs.	0.73	0.80	0.83	1.04	1.38	1.45	1.46	1.64	1.89	1.91	0.74	0.88	0.9	1.15	1.21
Rank	1	3	4	7	10	11	12	13	14	15	2	5	6	8	9

Notes

Placenta Permeability Data

Example

Thus the Wilcoxon statistic is

$$W = R_2 = 2 + 5 + 6 + 8 + 9 = 30$$

Now, here $n_1 = 10$ and $n_2 = 5$. There are

$$\binom{15}{5} = \frac{15!}{10!\,5!} = 3003$$

subsets of the ranks $\{1, 2, 3, \ldots, 15\}$ of size 5.

In the permutation null distribution, the possible values of W are $\{15, 16, \ldots, 64, 65\}$; the probabilities are given below.

Notes

Placenta Permeability Data

W	15	16	17	18	19	20	21	22	23	24	25	26	27
Frequency	1	1	2	3	5	7	10	13	18	23	30	36	45
Prob.	0.000	0.000	0.001	0.001	0.002	0.002	0.003	0.004	0.006	0.008	0.010	0.012	0.015
Cumulative Prob.	0.000	0.001	0.001	0.002	0.004	0.006	0.010	0.014	0.020	0.028	0.038	0.050	0.065
W	28	29	30	31	32	33	34	35	36	37	38	39	40
Frequency	53	63	72	83	92	103	111	121	127	134	137	141	141
Prob.	0.018	0.021	0.024	0.028	0.031	0.034	0.037	0.040	0.042	0.045	0.046	0.047	0.047
Cumulative Prob.	0.082	0.103	0.127	0.155	0.185	0.220	0.257	0.297	0.339	0.384	0.430	0.477	0.523
W	41	42	43	44	45	46	47	48	49	50	51	52	53
Frequency	141	137	134	127	121	111	103	92	83	72	63	53	45
Prob.	0.047	0.046	0.045	0.042	0.040	0.037	0.034	0.031	0.028	0.024	0.021	0.018	0.015
Cumulative Prob.	0.570	0.616	0.661	0.703	0.743	0.780	0.815	0.845	0.873	0.897	0.918	0.935	0.950
W	54	55	56	57	58	59	60	61	62	63	64	65	
Frequency	36	30	23	18	13	10	7	5	3	2	1	1	
Prob.	0.012	0.010	0.008	0.006	0.004	0.003	0.002	0.002	0.001	0.001	0.000	0.000	
Cumulative Prob.	0.962	0.972	0.980	0.986	0.990	0.994	0.996	0.998	0.999	0.999	1.000	1.000	

Notes

Placenta Permeability Data

Example

By inspection of the table, we see that

$$\Pr[25 \leq W \leq 55] = 0.972 - 0.038 = 0.934$$

and

$$\Pr[24 \leq W \leq 56] = 0.980 - 0.028 = 0.952$$

Notes

Placenta Permeability Data

Example

Thus for a symmetric two-sided interval which contains at most probability 0.95, we take the interval

$$\{25, 26, \ldots, 54, 55\}$$

and hence define the rejection region

$$\mathcal{R} = \{16, 17, \ldots, 23, 24, 56, 57, \ldots, 64, 65\}$$

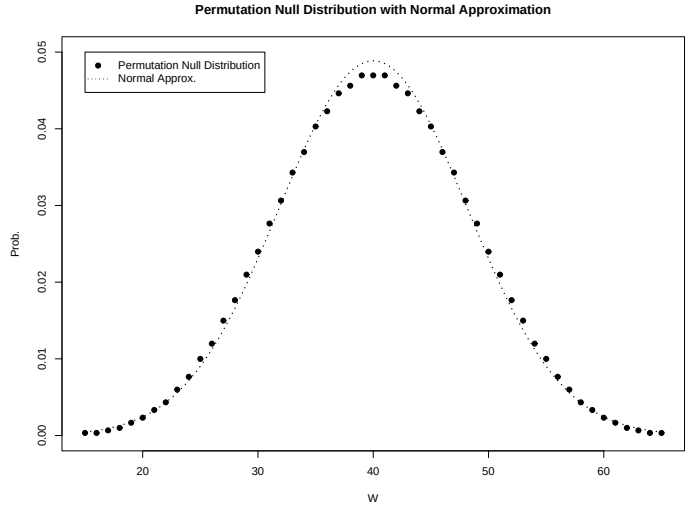
Note that this choice of rejection region ensures that there is at least probability 0.025 in each tail.

Thus in this example we **do not reject** the hypothesis of equal medians.

Notes

Normal Approximation

$$W \dot{\sim} \text{Normal}\left(\frac{n_2(n_1+n_2+1)}{2}, \frac{n_1 n_2(n_1+n_2+1)}{12}\right)$$



Notes

Notes