

MATH 203: PRINCIPLES OF STATISTICS 1

FALL 2024

HYPOTHESIS TESTING: ONE SAMPLE PROBLEMS

We will test hypotheses regarding the parameters μ or p : for the **null hypothesis**

$$H_0 : \mu = \mu_0 \quad \text{or} \quad H_0 : p = p_0,$$

and for the **alternative hypothesis**, one of

$$H_a : \mu > \mu_0 \quad \text{or} \quad H_a : p > p_0$$

$$H_a : \mu < \mu_0 \quad \text{or} \quad H_a : p < p_0$$

$$H_a : \mu \neq \mu_0 \quad \text{or} \quad H_a : p \neq p_0$$

In this specification, μ_0 and p_0 represent specific **numerical** values that represent the hypothesized null values of the data generating parameters. The steps are

1. Determine H_0 and H_a from the question (null hypothesis and alternative hypothesis)
2. Determine an appropriate **test statistic** T : T is a random variable whose value is determined by the individual sample random variables.
3. Identify the **null distribution** of T assuming H_0 is true.
4. Determine the **Rejection Region**: this will be a set of possible values of the Test Statistic T for which we will reject H_0 in favour of H_a
5. Compute the **observed value**, T_{obs} , of T using the observed data
6. Compare the observed value of the T to the Rejection Region
7. Decide and conclude “**Reject H_0 in favour of H_a** ” or “**Do not reject H_0** ”

We pre-specify a level of the test, α , to determine the Rejection Region. We have that α is the **probability of a Type 1 error**

$$\alpha = P_0(T \text{ lies in Rejection Region})$$

where P_0 indicates that this probability is computed **using the null distribution**.

Key quantities in setting up the test are

- $\bar{X}$: sample mean random variable
- S^2 : sample variance random variable

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

- t_ν : t -distribution with ν degrees of freedom
- z_a : upper a point of the standard Normal distribution
- $t_{a,\nu}$: upper a point of the t -distribution with ν degrees of freedom

and the key quantities to complete the test are the **sample quantities** $\bar{x}$ and s^2 .

Example: A random sample of size $n = 15$ is available from a Normal population. We wish to test the hypotheses

$$H_0 : \mu = 6.0$$

$$H_a : \mu > 6.0$$

with $\alpha = 0.05$. The observed data are

6.32, 9.00, 5.10, 5.45, 6.07, 5.69, 7.94, 4.90, 8.73, 6.59, 5.79, 8.04, 6.24, 8.69, 7.40

Here we are in Case 1, and the test statistic is

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$$

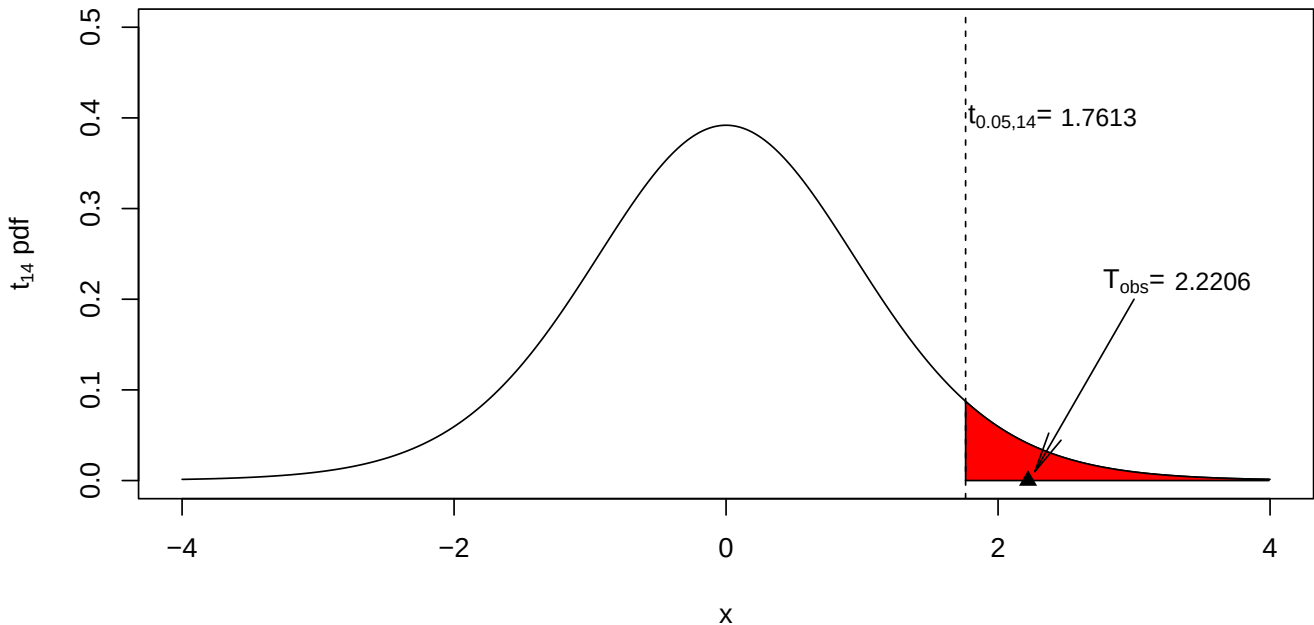
with $\mu_0 = 6.0$. Assuming that H_0 is true, we have that the null distribution has $T \sim t_{n-1} = t_{14}$. The upper $\alpha = 0.05$ point of the t_{14} distribution is $t_{0.05,14} = 1.7613$. Therefore

$$\text{Rejection Region: } \{T \geq 1.7613\}.$$

We compute the observed version of the test statistic: the summary statistics are (carrying full precision) $\bar{x} = 6.7966667$ and $s = 1.3894946$, so the observed value of the test statistic is

$$T_{obs} = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = 2.2205748.$$

We have that $T_{obs} > 1.7613$, T_{obs} lies in the Rejection Region, and so we **reject** H_0 in favour of H_a at $\alpha = 0.05$.



Finally, we can compute the p -value using the cdf function for the t -distribution, pt , as

$$P_0(T \geq T_{obs}) = P_0(T \geq 2.2205748) = 1 - pt(2.2205748, 14) = 1 - 0.978304 = 0.021696.$$

If we only had access to the t -distribution table, then we can observe that for the t_{14} distribution, the $\alpha = 0.025$ and $\alpha = 0.02$ upper points are 2.1448 and 2.2638 respectively, so the p -value is between 0.025 and 0.02.

Example: A random sample of size $n = 83$ is available, and the summary statistics are recorded as $\bar{x} = 9.333$ and $s = 3.040$. We wish to test the hypotheses

$$H_0 : \mu = 10.0$$

$$H_a : \mu < 10.0$$

with $\alpha = 0.01$. Here we are in Case 3, and the test statistic is

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$$

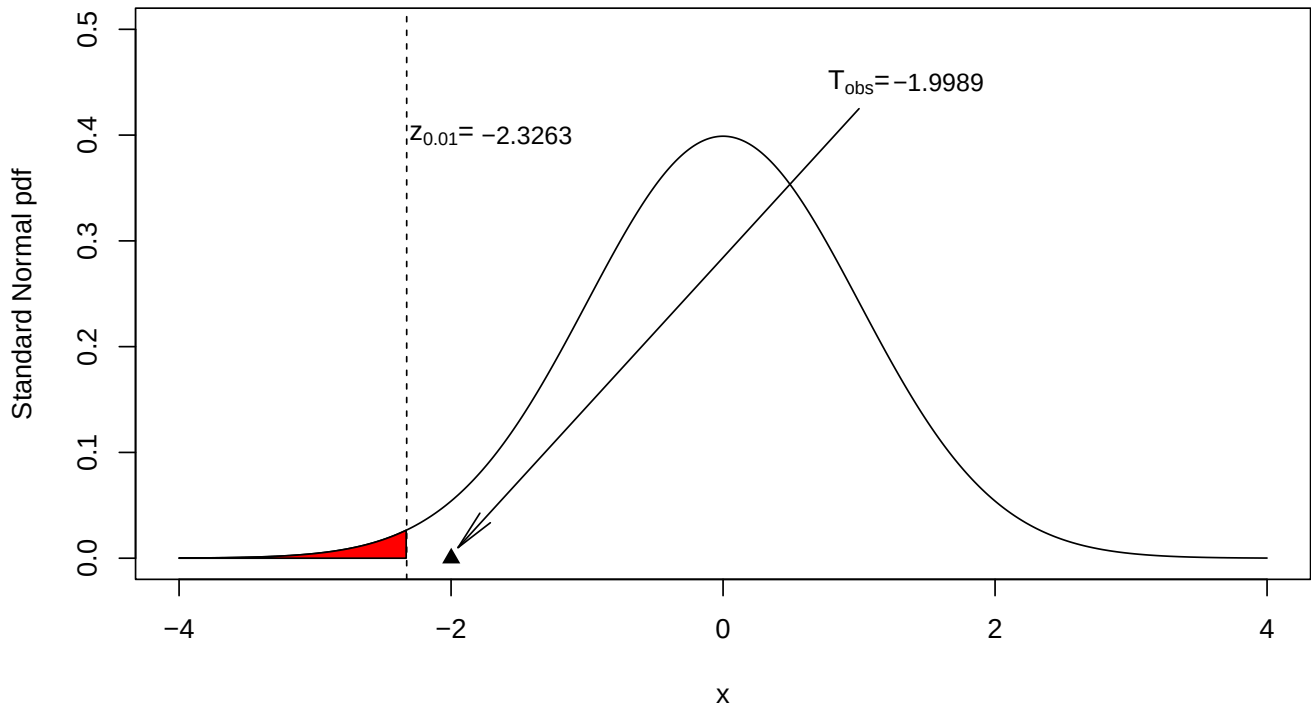
with $\mu_0 = 10.0$. Assuming that H_0 is true, we have that the null distribution is given (approximately) by the Central Limit Theorem as $T \sim Normal(0, 1)$. The upper $\alpha = 0.01$ point of the $Normal(0, 1)$ distribution is $z_{0.01} = 2.3263$. Therefore

$$\text{Rejection Region: } \{T \leq -2.3263\}.$$

We compute the observed version of the test statistic:

$$T_{obs} = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = -1.9989011.$$

We do **not** have that $T_{obs} \leq -2.3263$, T_{obs} **does not** lie in the Rejection Region, and so we **do not reject** H_0 in favour of H_a at $\alpha = 0.01$.



Finally, we can compute the p -value using the cdf function for the Normal distribution, `pnorm`, as

$$P_0(T \leq T_{obs}) = P_0(T \leq -1.9989011) = \text{pnorm}(-1.9989011) = 0.0228095.$$

If we only had access to the Normal distribution table, we could quote the approximate p -value by rounding the test statistic to two decimal places

$$P_0(T \leq -2.00) = 1 - P_0(T \leq 2.00) = 1 - 0.9772 = 0.0228.$$

Example: A random Bernoulli sample of size $n = 225$ is available, and we observe $x = 63$ successes, so that we have the point **estimate** $\hat{p} = 0.28$. We wish to test the hypotheses

$$H_0 : p = 0.25$$

$$H_a : p \neq 0.25$$

with $\alpha = 0.05$. Here we are in Case 4, and the test statistic is

$$T = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1 - p_0)}{n}}} = \frac{\hat{p} - 0.25}{\sqrt{\frac{0.25 \times 0.75}{225}}}$$

where $\hat{p}$ here is the **estimator**, $\hat{p} = X/n$, where X is the random variable recording the total number of successes. Assuming that H_0 is true, we have that the null distribution is given (approximately) by the Central Limit Theorem as $T \sim Normal(0, 1)$.

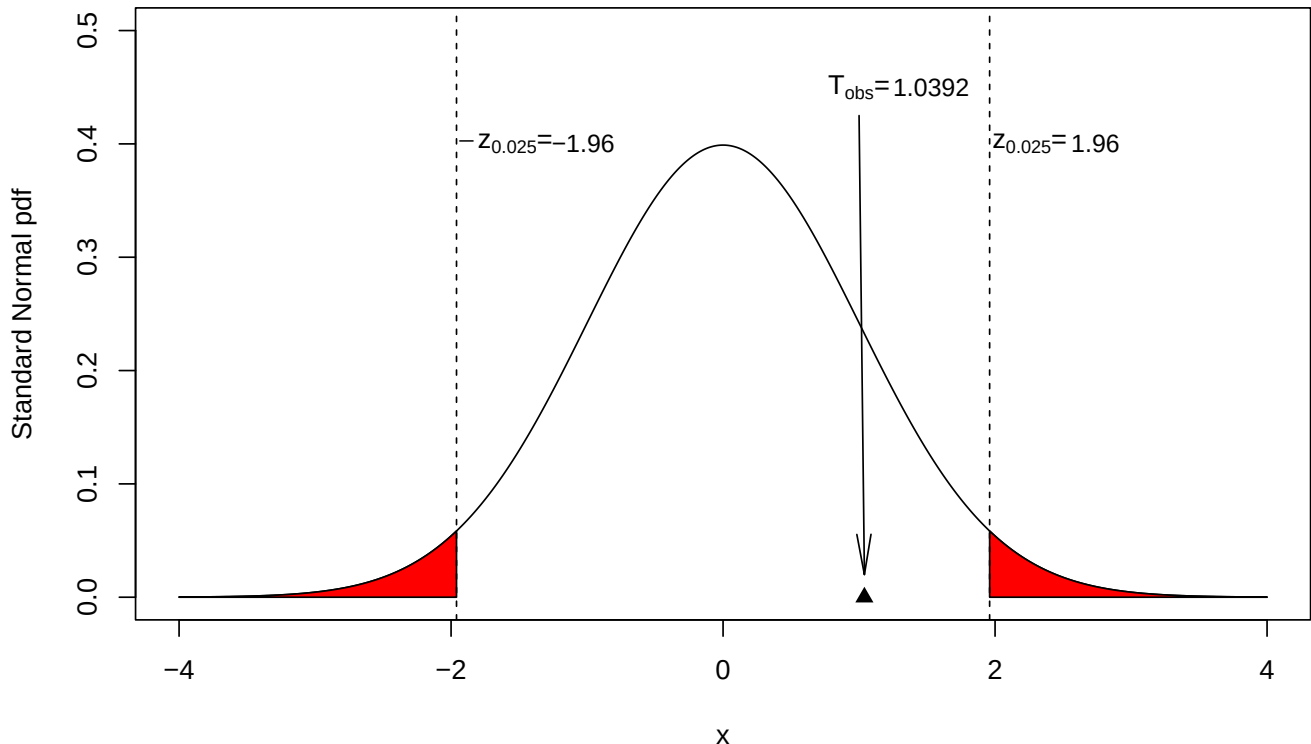
For the two-sided alternative hypothesis, we need the upper $\alpha/2 = 0.025$ point of the standard Normal distribution, which is $z_{0.05/2} = 1.9600$. Therefore

$$\text{Rejection Region: } \{|T| \geq 1.9600\}.$$

We compute the observed version of the test statistic:

$$T_{obs} = \frac{0.28 - 0.25}{\sqrt{\frac{0.25 \times 0.75}{225}}} = 1.0392305.$$

We do **not** have that $|T_{obs}| \geq 1.9600$, T_{obs} **does not** lie in the Rejection Region, and so we **do not reject** H_0 in favour of H_a at $\alpha = 0.05$.



Finally, we can compute the p -value using the cdf function for the Normal distribution, `pnorm`, as

$$\begin{aligned}2P_0(T \geq |T_{obs}|) &= 2P_0(T \geq 1.0392305) \\&= 2 \times (1 - P_0(T < 1.0392305)) \\&= 2 \times (1 - 0.8506512) \\&= 2 \times (0.1493488) \\&= 0.2986976\end{aligned}$$

If we only had access to the Normal distribution table, we could quote the approximate p -value by rounding the test statistic to two decimal places

$$\begin{aligned}2P_0(T \geq |T_{obs}|) &= 2P_0(T \geq 1.0392305) \\&= 2 \times (1 - P_0(T < 1.0392305)) \\&\approx 2 \times (1 - P_0(T < 1.04)) \\&= 2 \times (1 - 0.8508) \\&= 2 \times 0.1492 \\&= 0.2984\end{aligned}$$