p-adic families of theta series and arithmetic applications

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Abstract

Modular forms are functions of a complex variable which satisfy numerous symmetries. They encode deep arithmetic information such as modulo ℓ solutions of algebraic equations, representation numbers of quadratic forms, and cycles on Shimura varieties, packaged in their Fourier expansion or periods. The symmetries of modular forms lead to relations between their Fourier coefficients, providing insights into the underlying arithmetic objects. This principle is enriched if, instead of considering a single modular form, we consider families of them whose Fourier coefficients vary analytically.

In this thesis, we explore several instances of p-adic analytic families of modular forms constructed from theta series and their relations to arithmetic:

1. The first chapter presents an explicit formula relating a generating series of Heegner points on a Shimura curve to the derivative of a p-adic family of theta series of half-integral weight. This new connection, proven entirely via p-adic methods, provides an alternate proof of the Gross–Kohnen–Zagier theorem.

2. The second chapter is motivated by the search for a modular construction of Gross–Stark units for totally real fields. Considering a p-adic family obtained from pullbacks of the Eisenstein class of a torus bundle of rank n, we construct p-adic invariants associated to totally real fields of degree n where p is inert. We conjecture that such invariants are p-adic logarithms of elements in abelian extensions of totally real fields. To support the conjecture, we relate the local traces of these invariants to local traces of p-adic logarithms of Gross–Stark units.

Résumé

Les formes modulaires sont des fonctions d'une variable complexe qui satisfont de nombreuses symétries. Elles encodent plusieurs informations arithmétiques profondes, telle que les solutions d'équations algébriques modulo ℓ, les nombres de représentations de formes quadratiques, ou encore les cycles sur les variétés de Shimura, le tout encapsulé dans leur développement en série de Fourier ou dans leurs périodes. L'étude des symétries des formes modulaires permet de mieux comprendre les objets arithmétiques sous-jacents. Ce principe gagne en richesse lorsqu'on considère, non plus une seule forme modulaire, mais des familles de telles formes dont les coefficients de Fourier varient analytiquement.

Cette thèse explore plusieurs exemples de familles analytiques p-adiques de formes modulaires construites à partir de séries thêta, ainsi que leurs liens avec l'arithmétique :

1. Le premier chapitre présente une formule explicite reliant une série génératrice de points de Heegner sur une courbe de Shimura à la dérivée d'une famille p-adique de séries thêta de poids demi-entier. Cette nouvelle connexion, établie entièrement par des méthodes p-adiques, fournit une démonstration alternative du théorème de Gross–Kohnen–Zagier.

2. Le deuxième chapitre est motivé par la recherche d'une construction modulaire des unités de Gross–Stark pour les corps totalement réels. En considérant une famille p-adique obtenue comme image réciproque de la classe d'Eisenstein d'un fibré torique de rang n, nous construisons des invariants p-adiques associés à des corps totalement réels de degré n où p est inerte. Nous conjecturons que ces invariants sont les logarithmes p-adiques d'éléments appartenant à des extensions abéliennes de ces corps. Pour étayer cette conjecture, nous mettons en relation les traces locales de ces invariants avec celles des logarithmes p-adiques des unités de Gross–Stark.

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Contributions

Contribution of Authors. The main body of this thesis consists of two chapters, each stemming from a different collaboration. The first chapter, *The Gross–Kohnen–Zagier theorem via p-adic uniformization*, is joint work with Lea Beneish, Henri Darmon, and Lennart Gehrmann, and has been published in *Mathematische Annalen*, [BDGR25]. We reproduce this work in the thesis with minor reformatting. The second chapter, *The Eisenstein class of a torus bundle and a log-rigid class for* $\mathrm{SL}_n(\mathbb{Z})$, is an excerpt from an ongoing joint project with Peter Xu, which we plan to submit soon. In both collaborations, all authors contributed equally to all aspects of the work.

We obtained permission from Lea Beneish, Henri Darmon, Lennart Gehrmann, and Peter Xu to include these works in the thesis, and we are deeply grateful to them for it.

Contribution to Original Knowledge. In the first chapter of the thesis, we present a new proof of the Gross–Kohnen–Zagier theorem via p-adic methods. In doing so, we exhibit a connection between a p-adic family of theta series of half-integral weight and a generating series of Heegner points. In the second chapter of the thesis, we give a new construction of an Eisenstein group cohomology class. From there, we define a log-rigid class for $\mathrm{SL}_n(\mathbb{Z})$ and invariants associated with certain totally real fields embedded in $\mathrm{M}_n(\mathbb{Q})$. We conjecture that these invariants lie in abelian extensions of those totally real fields.

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General Introduction

Modular forms are functions of a complex variable which satisfy numerous symmetries and admit a Fourier expansion. Their Fourier coefficients encode deep arithmetic information, such as the modulo ℓ point count of algebraic equations, representation numbers of quadratic forms, and cycles on Shimura varieties. The symmetries of modular forms lead to relations between their Fourier coefficients, providing insights into the underlying arithmetic objects. The so-called *Kudla program* addresses the study of a central aspect of these phenomena.

An illustrative example of this principle is the Jacobi theta series and its relation to the problem of writing numbers as a sum of 2 squares. Let $r(n) := \#\{(x,y) \in \mathbb{Z}^2 \mid x^2+y^2 = n\}$ and consider the function on the upper half-plane $\mathscr{H}$

$$\theta(\tau) := \sum_{n \geq 0} r(n) q^n, \; q = e^{2\pi i \tau}.$$

Since θ is defined in terms of the variable q, it satisfies $\theta(\tau+1) = \theta(\tau)$. Moreover, it is a consequence of the Poisson summation formula that $\theta(\tau/(4\tau+1)) = (4\tau+1)\theta(\tau)$. These transformation properties make θ an instance of a modular form. The study of the space of holomorphic functions satisfying these symmetries using complex analysis leads to the conclusion that it is one-dimensional over $\mathbb{C}$ and generated by the function

$$E(\tau) := 1 + 4 \sum_{n \geq 1} \left(\sum_{d|n} \chi_4(d) \right) q^n, \text{ where } \chi_4 \colon (\mathbb{Z}/4\mathbb{Z})^\times \xrightarrow{\sim} \{\pm 1\}.$$

Hence $\theta(\tau) = E(\tau)$, which yields the explicit formula $r(n) = 4 \sum_{d|n} \chi_4(d)$. A similar strategy can be used to prove that every nonnegative integer can be written as a sum of 4 squares, recovering a theorem by Lagrange.

This principle is enriched if, instead of considering a modular form, we consider families of them whose Fourier coefficients vary analytically. Gross–Zagier and Gross–Kohnen–Zagier ([GZ86], [GKZ87]) followed these ideas to relate Heegner points, solutions of polynomial equations given by elliptic curves, to real-analytic families of modular forms: for F_s a family of modular forms (obtained as the product of a theta series and a real analytic family of Eisenstein series), F_s' its derivative with respect to s, and $\mathrm{pr}_E\left(e_{\mathrm{hol}}F_0'\right)$ a suitable component of the holomorphic part of F_0', a key identity of [GZ86] is

$$\sum_{n\geq 1}\langle P, T_n P\rangle q^n = \mathrm{pr}_E\left(e_{\mathrm{hol}}F_0'\right). \tag{1}$$

The coefficients on the left-hand side denote the height pairing between two Heegner points (P and its translate by a Hecke operator), which measures the complexity of these solutions. In particular, (1) determines if Heegner points are non-torsion solutions in terms of an analytic quantity.

This connection led to dramatic progress on the *Birch and Swinnerton-Dyer (BSD) Conjecture* for elliptic curves of analytic rank 1. This conjecture predicts a description of the rank of the group of solutions of an elliptic curve in terms of the order of vanishing of an analytic function at $s = 1$.

Although we focused our attention on the Fourier coefficients of modular forms and their families, modular forms also encode deep invariants not directly visible in their Fourier expansions. For instance, integrating modular forms provides an approach to constructing the previously mentioned Heegner points.

There are also p-adic (analytic) families of modular forms. These are families of modular forms such that their Fourier coefficients are analytic functions with respect to a p-adic metric. A prototypical example is the family of Eisenstein series

$$E_k(q) = (1-p^{k-1})\frac{\zeta(1-k)}{2} + \sum_{n\geq 1}\left(\sum_{p\nmid d\mid n} d^{k-1}\right)q^n, \tag{2}$$

indexed by the variable k. Indeed, as a consequence of Fermat's little theorem, the Fourier coefficients of $E_k(q)$ vary continuously in k with respect to the p-adic metric in $(\mathbb{Z}/(p-1))\times\mathbb{Z}_p$. In fact, it can be verified that the Fourier coefficients define analytic functions on k. As in the case of real-analytic families, there are deep connections between p-adic families of modular forms and arithmetic — both in the study of their Fourier coefficients and their periods. Some remarkable features of these connections are:

1. Due to the combinatorial nature inherent in the p-adic numbers, p-adic analysis is sometimes more tractable than its real counterpart, leading to simpler proofs of relations analogous to (1).

2. p-adic families of modular forms provide new connections to arithmetic which seem absent for real-analytic families.

This thesis focuses on the study of a particular type of p-adic families of modular forms, those arising from *theta series*:

1. In Chapter I, we study the Fourier coefficients of such a family to derive a formula similar to (1), or more precisely to those appearing in [GKZ87] involving modular forms of weight 3/2. Concretely, let E be an elliptic curve defined over $\mathbb{Q}$ that appears in the Jacobian of a Shimura curve. Similar to above, E is equipped with a collection of rational solutions $\{P_D\}_D$, called Heegner points, where the indexing set is over discriminants satisfying the so-called Heegner hypothesis. For p a rational prime dividing the discriminant of the Shimura curve, we construct a p-adic family of theta series Θ_k of weight $3/2 + k$. The family satisfies that $\Theta_0 = 0$ and, if we denote by Θ_0' the value of the derivative of Θ_k with respect to k at $k = 0$, we obtain a formula similar to (1)

$$\sum_{D \geq 1} \log_p(P_D) q^D = \mathrm{pr}_E \left(e_{\mathrm{ord}} \Theta_0' \right). \tag{3}$$

   In this expression, the height pairing of (1) is replaced by the so-called p-adic logarithm $\log_p \colon E(\mathbb{Q}_p) \to \mathbb{Q}_p$, and the holomorphic projector by the p-ordinary projector e_{ord}. The formula yields a new proof of the *Gross–Kohnen–Zagier theorem for Shimura curves* — originally established by Borcherds in [Bor99] — using p-adic methods.

2. In Chapter II, we consider a higher-dimensional generalization of the family of Eisenstein series in (2) and of their integrals along real quadratic geodesics in $\mathscr{H}$. For this, we use the *Eisenstein class of a torus bundle*, studied by Bergeron, Charollois, and García in [BCG20]. The p-adic properties of this family imply that their periods satisfy a p-adic coherence. This leads to consider a p-adic limit of such values (and refinements of them), $J_{\mathrm{Eis}}[\tau]$, attached to tori $T_F \subset \mathrm{SL}_n(\mathbb{Q})$ of norm 1 elements of a totally real field F where p is inert. We conjecture that $J_{\mathrm{Eis}}[\tau]$ is the p-adic logarithm of an element in an abelian extension of F and provide partial conceptual evidence for it. Notably, the quantities $J_{\mathrm{Eis}}[\tau]$ appear as the values of a modular-like object J_{Eis} at a special point τ, in line with the theory of *rigid cocycles* developed by Darmon and Vonk [DV21].

   The quantities $J_{\mathrm{Eis}}[\tau]$ were already considered by Darmon, Pozzi, and Vonk in [DPV24] for the case of real quadratic fields. Moreover, invariants of a similar flavour were constructed by Dasgupta in [Das08] and Dasgupta and Kakde proved that they serve to generate the maximal abelian extension of a totally real field in [DK23], solving *Hilbert's twelfth problem* via p-adic methods in this setting.

Thus, each of the two main chapters of the thesis illustrates a distinct feature of p-adic families discussed above. The thesis concludes with Chapter III. This shorter chapter discusses, in a more informal tone, future research directions. We place special emphasis on the following question:

Do the p-adic invariants $J_{\mathrm{Eis}}[\tau]$ *appear naturally as Fourier coefficients of derivatives of p-adic families of modular forms?*

Informally, this question asks for a formula similar to (15.2), where the Fourier coefficients of the left-hand side, given by logarithms of Heegner points in (15.2), are replaced by the values $J_{\mathrm{Eis}}[\tau]$ and generalizations of them. A positive answer to this question could shed light on the arithmetic significance of the invariants $J_{\mathrm{Eis}}[\tau]$. For instance, Darmon, Pozzi, and Vonk answered the question positively for the case of real quadratic fields in [DPV24], and used the connection between p-adic families of modular forms and deformations of Galois representations to prove that the invariant $J_{\mathrm{Eis}}[\tau]$ belongs to an abelian extension of a real quadratic field.

Chapter I

The Gross–Kohnen–Zagier theorem via p-adic uniformization

Section 1

Introduction

It is conjectured that generating series of special cycles on orthogonal Shimura varieties are modular forms. (See [Kud04] for example.) One of the first instances of this phenomenon was discovered by Gross, Kohnen and Zagier, who proved in [GKZ87] that generating series of Heegner points on the Jacobians of modular curves are modular forms of weight 3/2. The purpose of this chapter is to study the analogous statement for generating series of Heegner points on Shimura curves. We will present a new proof of the modularity of these generating series — originally established by Borcherds in [Bor99] — using p-adic methods.

1.1 Statement of the Gross–Kohnen–Zagier theorem

We proceed to explain the theorem in more detail, in a framework that encompasses both modular and Shimura curves. Let S be a finite set of places of $\mathbb{Q}$ of odd cardinality containing ∞ and let N^+ be a square-free positive integer which is not divisible by any finite place in S. This datum gives rise to a *modular* or *Shimura curve* X defined over $\mathbb{Q}$, which is an instance of an orthogonal Shimura variety. Its set $X(\mathbb{C})$ of complex points can be described in terms of an Eichler $\mathbb{Z}$-order $\mathscr{R}$ of level N^+ in a quaternion algebra $\mathscr{B}$ over $\mathbb{Q}$ ramified exactly at $S - \{\infty\}$. Namely, the set $\mathscr{V}$ of trace zero elements in $\mathscr{B}$ equipped with the quadratic form $\mathscr{Q}$ induced by the reduced norm is a quadratic space of signature $(1, 2)$, and is anisotropic at all the places $v \in S - \{\infty\}$. The action of $\mathscr{B}^\times$ on $\mathscr{V}$ via conjugation identifies $\mathscr{B}^\times$ with the group $\mathrm{GSpin}(\mathscr{V})$ of spinor similitudes of $\mathscr{V}$. It naturally acts on the conic $C_{\mathscr{V}} \subseteq \mathbb{P}(\mathscr{V})$ whose rational points over a field E of characteristic 0 are given by

$$C_{\mathscr{V}}(E) := \{\ell \in \mathbb{P}(\mathscr{V}_E) \mid \mathscr{Q}(\ell) = \{0\}\}. \tag{1.1}$$

Here and from now on, if M is an abelian group, and A is a commutative ring, write M_A for the A-module $M \otimes_{\mathbb{Z}} A$. The group Γ of units of $\mathscr{R}$ modulo $\{\pm 1\}$ acts discretely on the symmetric space

$$\mathscr{K} := C_{\mathscr{V}}(\mathbb{C}) - C_{\mathscr{V}}(\mathbb{R})$$

associated to the orthogonal group of $\mathscr{V}$. The set $X(\mathbb{C})$ of complex points of X is identified with the quotient $\Gamma \backslash \mathscr{K}$. A comparison with a more classical description of Shimura curves can be found in the Appendix of [Kud04].

Let $v \in \mathscr{V}$ be a vector for which $\mathscr{Q}(v) > 0$. Its orthogonal complement is thus a two-dimensional negative definite space, whose base change to $\mathbb{C}$ is a hyperbolic plane. Hence, there are exactly two points in $\mathscr{K}$ represented by a vector orthogonal to v. Let $\Delta(v) \subset \mathscr{K}$ be the divisor consisting of these two points. Each positive integer D in

$$\mathcal{D}_S := \{D \in \mathbb{Z}_{>0} \mid \exists v \in \mathscr{V} \text{ such that } \mathscr{Q}(v) = D\}$$

gives rise to a zero-cycle on X by setting

$$\Delta(D) := \sum_{\substack{v \in \Gamma \backslash \mathscr{R}_0, \\ \mathscr{Q}(v) = D}} \frac{1}{\#\mathrm{Stab}_{\Gamma}(v)} \Delta(v) \in \mathrm{Div}(X(\mathbb{C}))_{\mathbb{Q}}, \tag{1.2}$$

where $\mathscr{R}_0 \subseteq \mathscr{V}$ is the $\mathbb{Z}$-lattice $\mathscr{R} \cap \mathscr{V}$. The divisor $\Delta(D)$, which is supported on a finite set of CM points on X, is a simple instance of a *Heegner divisor* on this Shimura curve.

The Gross–Kohnen–Zagier theorem asserts that the classes of $\Delta(D)$ in the Jacobian of X can be packaged into a modular generating series of weight $3/2$. Namely, let $\mathcal{L}$ be the tautological line bundle of isotropic vectors whose spans are points of $\mathscr{K}$. This bundle is $\mathscr{B}^{\times}$-equivariant and, therefore, descends to a line bundle on $X(\mathbb{C})$, which is identified with the cotangent bundle of X. In particular, it has a model over $\mathbb{Q}$. Denote by $[\Delta]$ (resp. $[\mathcal{L}^{\vee}]$) the class in $\mathrm{Pic}(X)(\mathbb{Q})$ of a divisor Δ (resp. of the dual $\mathcal{L}^{\vee}$ of the line bundle $\mathcal{L}$) on X. Then, the formal generating series

$$G(q) := [\mathcal{L}^{\vee}] + \sum_{D \in \mathcal{D}_S} [\Delta(D)] q^D \in \mathrm{Pic}(X)(\mathbb{Q})_{\mathbb{Q}}[[q]], \tag{1.3}$$

is a modular form of weight $3/2$ and level $\Gamma_0(4N)$, where N is the product of N^+ with all finite places in S. Remember that, given an abelian group A, a formal q-series $f \in A[[q]]$ with coefficients in A is called a modular form of weight $3/2$ and level $\Gamma_0(4N)$ if for every homomorphism $\varphi\colon A \to \mathbb{C}$ the generating series $\varphi(f) \in \mathbb{C}[[q]]$, obtained by applying φ to each of the coefficients of f, is the q-expansion of a modular form of weight $3/2$ and level $\Gamma_0(4N)$.

The Gross–Kohnen–Zagier theorem was first proved in [GKZ87] in the case of modular curves (i.e., where $S = \{\infty\}$) by calculating the Arakelov intersection pairings of the divisors $\Delta(D)$ with a fixed CM divisor. It was extended by Borcherds [Bor99] to the setting of orthogonal groups of real signature $(n, 2)$, encompassing Shimura curves as a special case where the underlying quadratic space is of signature $(1, 2)$, as a consequence of his theory of singular theta lifts. The work of Yuan, Zhang, and Zhang [YZZ09] proves Theorem 1.1.1 in much greater generality, for certain orthogonal groups over totally real fields. It should be noted that the theorem also holds if one replaces the lattice $\mathscr{R}_0$ by a suitable weighted sum of lattice cosets or, equivalently, by a suitable Schwartz–Bruhat function. In order to keep the exposition as simple as possible we refrain from stating the most general version of the theorem in the introduction.

The goal of Chapter I is to describe a new proof of the Gross–Kohnen–Zagier theorem in the case where $S \neq \{\infty\}$, i.e., when X is not a modular curve. To simplify the exposition we will also assume that $2 \nmid N$.

Theorem 1.1.1. *The generating series* $G(q) \in \mathrm{Pic}(X)(\mathbb{Q})_{\mathbb{Q}}[[q]]$ *of* (1.3) *is a modular form of weight* $3/2$ *and level* $\Gamma_0(4N)$.

1.2 Strategy for the proof and main formula

Our approach to this theorem rests on the fact that, at a finite place $p \in S$, the curve $X(\mathbb{C}_p)$ admits a p-adic analytic uniformization. More precisely, $X(\mathbb{C}_p)$ can be described as the quotient of the p-adic upper half-plane by the discrete action of the norm one elements of an Eichler $\mathbb{Z}[1/p]$-order R of level N^+ in the (definite) quaternion algebra ramified exactly at $S - \{p\}$. Furthermore, the Heegner divisors $\Delta(D)$ can be described p-adically in terms of this uniformization. This immediately gives an expression of the generating series of degrees

$$\deg(G)(q) = \deg(\mathcal{L}^\vee) + \sum_{D \in \mathcal{D}_S} \deg(\Delta(D))q^D$$

in terms of definite ternary theta series, recovering a well-known modularity result. (See for example [HZ76, Chapter 2] and [Kud03, Theorem I].) Thus, it is enough to prove modularity of the generating series $TG(q)$ for Hecke operators of degree 0, for which $TG(q)$ takes values in the $\mathbb{Q}$-rational points of the Jacobian J of X. The existence of a basis of modular forms with rational coefficients then reduces the problem to proving modularity of the generating series

$$\log_\omega(TG)(q) := \sum_{D \in \mathcal{D}_S} \log_\omega([T\Delta(D)])q^D \in \mathbb{Q}_p[[q]] \tag{1.4}$$

for every cotangent vector ω of $J_{\mathbb{Q}_p}$ with associated p-adic formal logarithm $\log_\omega\colon J(\mathbb{Q}_p) \to \mathbb{Q}_p$. For appropriate Hecke operators T, the p-adic description of the divisors $T\Delta(D)$

leads to an expression of this series as the ordinary projection of an infinitesimal p-adic deformation of a positive definite ternary theta series attached to the data (ω, R, T). More precisely, these data give rise to a p-adic family of weighted theta series Θ_k of weight $k+3/2$, with k in the weight space $(\mathbb{Z}/(p-1)) \times \mathbb{Z}_p$, whose specialization at weight $3/2$ vanishes. (See Chapter 7.3 for its definition.) It then follows that its derivative with respect to k evaluated at $k=0$, denoted Θ_0', is a p-adic cusp form of weight $3/2$. Let e_{ord} be p-ordinary projector acting on this space. By a classicality result, $e_{\mathrm{ord}}(\Theta_0')$ is a cusp form of weight $3/2$ and level $\Gamma_0(4N)$. Let pr_1 be the projector on the space of cusp forms of weight $3/2$ and level $\Gamma_0(4N)$ to the eigenspace of the Hecke operator U_{p^2} of eigenvalue 1. The main contribution of this chapter is the following formula.

Theorem 1.2.1. *We have*

$$\log_\omega(TG) = \mathrm{pr}_1(e_{\mathrm{ord}}(\Theta_0')).$$

To summarize, the fact that TG is a modular form is a consequence of the modularity of definite theta series and classicality of ordinary p-adic modular forms of half-integral weight.

Remark 1.2.2. As the proof of Theorem 1.1.1 is a purely p-adic analytic one, it seems likely that it carries over to more general settings, e.g., to Shimura curves over totally real fields which admit a p-adic uniformization. The assumption that N is odd and square-free stems from using p-adic families of *scalar-valued* half-integral modular forms, which seem only well-behaved in that case. Concretely, we study these p-adic families using the results of [Niw77], [Koh82], and [MRV90], which have a similar condition on the level of the modular forms. Generalizing to arbitrary level likely requires a theory of families of vector-valued modular forms, which so far has only been developed in a few instances. (See [LN20].)

The strategy sketched above bypasses the global height pairings studied by Gross, Kohnen, and Zagier, or the singular theta lifts that arise in the approach of Borcherds. It can be envisaged as fitting into the broader framework of a *p-adic Kudla program*, in which p-adic families of modular forms play much the same role as analytic families of Eisenstein series in the archimedean setting. Insofar as the generating series G are among the simplest instances of the modular generating series arising in the Kudla program, it is hoped that the p-adic techniques described here will be more widely applicable, shedding light on the connection between special cycles on orthogonal and unitary Shimura varieties, p-adic Borcherds-type lifts, and p-adic families of theta series. A general framework is laid out in the article [DGL23], which introduces the notion of rigid meromorphic cocycles for orthogonal groups. In *loc.cit.* modularity statements for generating series of special divisors on arithmetic quotients on higher-dimensional p-adic symmetric spaces are formulated. A crucial input in their proof is the injectivity of the first Chern class when the arithmetic quotient has dimension 3 and higher. Theorem 1.1.1 complements the main theorem of [DGL23] by extending it to the case of curves, where the kernel of the Chern class map needs to be considered. Relating arithmetic data to Fourier coefficients of ordinary projections of p-adic modular forms is one of the prevalent themes of the p-adic Kudla program as demonstrated, for example, in [Daa23] and [DPV24].

1.3 Structure of Chapter I

The organization of this chapter is as follows. Section 2 explains the p-adic uniformization of X and states the Gross–Kohnen–Zagier theorem in terms of this uniformization. Theorem 2.2.4 below describes the main result, which is somewhat more general than Theorem 1.1.1, since the divisors $\Delta(D)$ are replaced by linear combinations of Heegner points weighted by Schwartz–Bruhat functions. Section 3 gives a short proof of the modularity of $\deg(G)$. Section 4 introduces the p-adic Abel–Jacobi map, which gives an explicit description of the Jacobian of a Mumford curve. This description is used in Section 5 to construct certain functionals on the Jacobian, whose values at Heegner points are computed in Section 6. In particular, we give an explicit expression for the quantities $\log_\omega([T\Delta(D)])$ appearing in (1.4). In Section 7, we define the p-adic family Θ_k, prove a classicality result regarding ordinary p-adic cusp forms of half-integral weight and prove the main Theorem 1.2.1, which implies the Gross–Kohnen–Zagier theorem. Finally, Section 8 illustrates the construction of Θ_k by presenting a concrete example where $S = \{7, 13, \infty\}$ and $p = 7$. In this case, the ordinary projection of Θ_0' is computed numerically modulo p.

Section 2

The Cerednik–Drinfeld theorem

This section recalls the theorem of Cerednik–Drinfeld, which gives a rigid analytic uniformization of X at a finite prime $p \in S$ that is fixed once and for all. Moreover, we describe Heegner divisors in terms of this uniformization, which leads to a reformulation of the Gross–Kohnen–Zagier theorem in this setting.

2.1 p-adic uniformization of X

The rigid analytic uniformization of X proceeds by replacing the place ∞ in the complex uniformization of the introduction by the prime $p \in S$. To describe it, we need to introduce some notation. Let B be the quaternion algebra over $\mathbb{Q}$ ramified exactly at the places in $S - \{p\}$. Let R be an Eichler $\mathbb{Z}[1/p]$-order in B of level N^+ and denote by Γ the multiplicative group of elements of reduced norm 1 in R modulo $\{\pm 1\}$. Let Q be the restriction of the reduced norm to the space

$$V = \{b \in B \mid \mathrm{Tr}(b) = 0\}$$

of elements of reduced trace zero in B. It endows V with the structure of a quadratic space of rank 3 over $\mathbb{Q}$, which is of real signature $(3, 0)$. Denote by $\langle \cdot, \cdot \rangle$ the symmetric bilinear form attached to Q, that is, $\langle v, w \rangle := Q(v + w) - Q(v) - Q(w)$. As in the case of the quadratic space $\mathscr{V}$, the action of $B^\times$ on V via conjugation identifies $B^\times$ with the group of spinor similitudes of V. The intersection $R_0 = R \cap V$ is an even $\mathbb{Z}[1/p]$-lattice in V.

A p-adic symmetric space is associated to the orthogonal group of $V_{\mathbb{Q}_p}$ as follows. Similarly to (1.1) denote by $C_V \subseteq \mathbb{P}(V)$ the conic over $\mathbb{Q}$ attached to V whose rational points over a field E of characteristic 0 are given by

$$C_V(E) = \{\ell \in \mathbb{P}(V_E) \mid Q(\ell) = \{0\}\}.$$

This conic has no rational points, but can be identified with the projective line $\mathbb{P}$ over $\mathbb{Q}_p$ as follows: choose an isomorphism of $B_{\mathbb{Q}_p}$ with the matrix ring $\mathrm{M}_2(\mathbb{Q}_p)$. The conic is then identified with the space of non-zero nilpotent 2×2-matrices up to scaling. Mapping such a matrix to its kernel yields the desired isomorphism. The action of $B_{\mathbb{Q}_p}^\times$ is identified with the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\mathbb{P}_{\mathbb{Q}_p}$ via Möbius transformations.

Definition 2.1.1. The Drinfeld p-adic upper half plane is the $\mathbb{Q}_p$-rigid analytic space $\mathscr{H}_p$, whose E-rational points for any complete extension $E/\mathbb{Q}_p$ is the set

$$\mathscr{H}_p(E) := C_V(E) - C_V(\mathbb{Q}_p) \simeq \mathbb{P}_1(E) - \mathbb{P}_1(\mathbb{Q}_p).$$

We briefly explain the rigid analytic structure on $\mathscr{H}_p$ in terms of the reduction map to the Bruhat–Tits tree. We refer the reader to [Dar04, Chapter 5] for further details. The Bruhat–Tits tree, denoted $\mathcal{T}$, is the graph whose set of vertices is the set of *unimodular* $\mathbb{Z}_p$-lattices in $V_{\mathbb{Q}_p}$, that is, lattices which are self-dual with respect to the symmetric bilinear form $\langle \cdot, \cdot \rangle$. Two unimodular $\mathbb{Z}_p$-lattices L_1 and L_2 are joined by an edge if they are *p-neighbours*, that is,

$$[L_1 : L_1 \cap L_2] = [L_2 : L_1 \cap L_2] = p.$$

A choice of a vertex L gives a smooth $\mathbb{Z}_{(p)}$-integral structure C_L to the conic C_V, where $\mathbb{Z}_{(p)}$ denotes the localization of $\mathbb{Z}$ at the prime ideal generated by p. If L' is adjacent to L, then the image of $L \cap L'$ in L/pL is a 2-dimensional non-regular subspace, hence contains a unique isotropic subspace $\ell_{L'}$. Mapping the edge (L, L') to $\ell_{L'}$ yields a bijection between the set of lattices adjacent to L and $C_L(\mathbb{F}_p) \simeq \mathbb{P}_1(\mathbb{F}_p)$. It follows that $\mathcal{T}$ is homogeneous of degree $p+1$. The set of vertices and edges of $\mathcal{T}$ are denoted by $\mathcal{T}_0$ and $\mathcal{T}_1$ respectively, and $\mathcal{T}$ shall be viewed as a disjoint union $\mathcal{T} = \mathcal{T}_0 \sqcup \mathcal{T}_1$. Identifying the quadratic space $V_{\mathbb{Q}_p}$ with the set of trace zero endomorphisms of $\mathbb{Q}_p^2$ endowed with the norm form gives the more familiar description of the tree in terms of similarity classes of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$. Indeed, the assignment $[\Lambda] \mapsto \mathrm{Hom}_0(\Lambda, \Lambda)$ is a bijection between such similarity classes and unimodular lattices in $V_{\mathbb{Q}_p}$. Moreover, two classes $[\Lambda_1]$, $[\Lambda_2]$ are joined by an edge if they admit representatives Λ_1 and Λ_2 satisfying $p\Lambda_1 \subset \Lambda_2 \subset \Lambda_1$. From this description one easily deduces that $\mathcal{T}$ is indeed a tree. The identification of the two graphs is compatible with the natural actions of $B_{\mathbb{Q}_p}^\times$ and $\mathrm{GL}_2(\mathbb{Q}_p)$. We define a notion of parity on the vertices of $\mathcal{T}$ by requiring that every edge connects an even vertex with an odd one. There are exactly two possible choices for this and we choose one of them. The action of the elements of reduced norm one in $B_{\mathbb{Q}_p}$ on $\mathcal{T}$ is parity-preserving.

We proceed by describing the well-known *reduction map*

$$\mathrm{red} \colon \mathscr{H}_p(\mathbb{C}_p) \longrightarrow \mathcal{T}$$

in the language of quadratic forms. For that let $\mathcal{O}_{\mathbb{C}_p}$ denote the ring of integers of $\mathbb{C}_p$ and $\mathfrak{m} \subseteq \mathcal{O}_{\mathbb{C}_p}$ its maximal ideal. Every unimodular $\mathbb{Z}_p$-lattice $L \subseteq V_{\mathbb{Q}_p}$ induces a reduction map

$$C_V(\mathbb{C}_p) = C_L(\mathcal{O}_{\mathbb{C}_p}) \twoheadrightarrow C_L(\overline{\mathbb{F}_p}).$$

1. Let $L \subseteq V_{\mathbb{Q}_p}$ be a unimodular $\mathbb{Z}_p$-lattice. Then $\mathrm{red}^{-1}(L)$ is the complement of the $p+1$ residue discs around the points in $C_L(\mathbb{F}_p)$.

2. Let $L, L' \subseteq V_{\mathbb{Q}_p}$ be two unimodular $\mathbb{Z}_p$-lattices that are p-neighbours and $\ell_{L'} \in C_L(\mathbb{F}_p)$ the corresponding isotropic line. The preimage of the edge (L, L') under the reduction map consists of those elements $z \in C_L(\mathcal{O}_{\mathbb{C}_p})$ that are congruent to $\ell_{L'}$ modulo $\mathfrak{m}$ but not modulo p.

One readily checks that the reduction map is $B^\times_{\mathbb{Q}_p}$-equivariant. A *finite closed subgraph* of $\mathcal{T}$ is a finite set $\mathcal{G} \subset \mathcal{T}$ satisfying

$$(v_1, v_2) \in \mathcal{G} \cap \mathcal{T}_1 \Rightarrow v_1, v_2 \in \mathcal{G} \cap \mathcal{T}_0.$$

A *standard affinoid subset* of $\mathscr{H}_p$ is a set of the form $\mathrm{red}^{-1}(\mathcal{G})$, where $\mathcal{G}$ is a finite closed subgraph of $\mathcal{T}$.

Definition 2.1.2. A function on $\mathscr{H}_p$ is said to be *rigid analytic* if its restriction to any standard affinoid subset $\mathcal{A} \subset \mathscr{H}_p$ can be written as a uniform limit of rational functions having poles outside of $\mathcal{A}$. A function on $\mathscr{H}_p$ is said to be *rigid meromorphic* if it is the quotient of two rigid analytic functions, where the denominator is non-zero.

The group Γ acts naturally on $\mathscr{H}_p$ by conjugation. This action is discrete because Γ is a p-arithmetic subgroup of an algebraic group that is compact at ∞. It follows from there that the quotient space $\Gamma\backslash\mathscr{H}_p$ has a natural structure of a rigid analytic variety over $\mathbb{Q}_p$. On the other hand, the analytification of X gives a rigid analytic space over $\mathbb{Q}_p$. The Cerednik-Drinfeld theorem states that these two spaces can be identified after base change to the unramified quadratic extension $\mathbb{Q}_{p^2}$ of $\mathbb{Q}_p$. This identification depends on choices. To make this precise, let us introduce the following notation: for a finite set Σ of places of $\mathbb{Q}$ write $\mathbb{A}^\Sigma \subset \prod_{v \notin \Sigma} \mathbb{Q}_v$ for the ring of finite adéles away from Σ. Moreover, let $\hat{\mathbb{Z}}$ (resp. $\hat{\mathbb{Z}}^{(p)}$) be the maximal order of $\mathbb{A}^\infty$ (resp. of $\mathbb{A}^{p,\infty}$). Given a finitely generated $\mathbb{Z}[1/p]$-module M, we put $\hat{M} = M \otimes \hat{\mathbb{Z}}^{(p)}$. Fix an identification

$$\mathscr{V}_{\mathbb{A}^{p,\infty}} \simeq V_{\mathbb{A}^{p,\infty}} \tag{2.1}$$

sending the $\hat{\mathbb{Z}}^{(p)}$-lattice $\hat{\mathscr{R}}_0$ to $\hat{R}_0$.

Theorem 2.1.3 (Cerednik–Drinfeld)**.** *The identification* (2.1) *induces an isomorphism*

$$X \xrightarrow{\sim} \Gamma\backslash\mathscr{H}_p. \tag{2.2}$$

of rigid analytic spaces over $\mathbb{Q}_{p^2}$*.*

Proof. See [Cer76], [Dri76] and [BC91]. □

2.2 p-adic analytic description of Heegner divisors

In analogy with the cycles defined in the introduction, every non-zero element $v \in V$ yields a cycle $\Delta(v)$ on $\mathscr{H}_p$: $\Delta(v)$ is the sum of those points in $\mathscr{H}_p$ that are orthogonal to v. This cycle has degree 0 or 2 depending on whether the orthogonal complement of v in $V_{\mathbb{Q}_p}$ represents 0 or not. Observe that the orthogonal complement of v in $V_{\mathbb{Q}_p}$, being 2-dimensional, represents 0 if and only if the negative of its discriminant is a square in $\mathbb{Q}_p$. Since the discriminant of $V_{\mathbb{Q}_p}$ is a square, the discriminant of the orthogonal complement of v is equal to the discriminant of v, which is $Q(v)$, modulo squares. Hence, we deduce $\Delta(v) \neq 0$ if and only if $\sqrt{-Q(v)} \notin \mathbb{Q}_p$. By the Hasse–Minkowski theorem the set $\mathcal{D}_S$ from the introduction is characterized locally. In particular, one gets the description

$$\mathcal{D}_S = \{D \in \mathbb{Z} \mid \exists v \in V - \{0\} \text{ such that } Q(v) = D \text{ and } \Delta(v) \neq 0\},$$

where we used that B ramifies exactly at $S - \{p\}$ while $\mathscr{B}$ ramifies exactly at $S - \{\infty\}$. Indeed, the quadratic spaces V and $\mathscr{V}$ are locally isomorphic at all places except at ∞ and p. This is why the condition $D > 0$ (or equivalently, $\sqrt{-D} \notin \mathbb{R}$) appearing in the description of $\mathcal{D}_S$ in the introduction is replaced by the condition $\Delta(v) \neq 0$ (or equivalently, $\sqrt{-D} \notin \mathbb{Q}_p$). Note that the first condition is automatic for the nonzero lengths of elements of V, as B is ramified at ∞ and the second condition is automatic for nonzero lengths of elements of $\mathscr{V}$, as $\mathscr{B}$ is ramified at p.

Lemma 2.2.1. *Let D be an element of $\mathcal{D}_S$ and $v \in V$ with $Q(v) = D$.*

1. *If $\mathrm{ord}_p(D) = 0$, there exists a unique unimodular $\mathbb{Z}_p$-lattice L in $V_{\mathbb{Q}_p}$ containing v. The support of $\Delta(v)$ is contained in $\mathrm{red}^{-1}(L)$.*
2. *If $\mathrm{ord}_p(D) = 1$, there exist exactly two unimodular $\mathbb{Z}_p$-lattices L_1, L_2 in $V_{\mathbb{Q}_p}$ containing v, which are p-neighbours. The support of $\Delta(v)$ is contained in $\mathrm{red}^{-1}((L_1, L_2))$.*

Proof. Let W be the orthogonal complement of v in $V_{\mathbb{Q}_p}$. As W is anisotropic, it contains a unique maximal $\mathbb{Z}_p$-lattice L_W, on which Q takes values in $\mathbb{Z}_p$ and it is characterized by the property that its discriminant module is an $\mathbb{F}_p$-vector space. (See for example [AN02, Corollary 11].) Remember that the discriminant module of a $\mathbb{Z}_p$-lattice $L \subseteq W$ with $Q(L) \subseteq \mathbb{Z}_p$ is the quotient $L^\sharp/L$ where $L^\sharp \subseteq W$ denotes the dual of L with respect to the bilinear pairing $\langle \cdot, \cdot \rangle$. Suppose that L is a unimodular lattice containing v. By [DGL23, Lemma 1.1], the discriminant module of $L \cap W$ is isomorphic to the discriminant module of $L \cap \mathbb{Z}_p v$. Thus, the discriminant module of $L \cap W$ is an $\mathbb{F}_p$-vector space, which implies that $L \cap W = L_W$. In particular, L must contain $\mathbb{Z}_p v \oplus L_W$.

If $\mathrm{ord}_p(D) = 0$, one easily checks that L_W is unimodular. Thus, $\mathbb{Z}_p v \oplus L_W$ is the unique unimodular $\mathbb{Z}_p$-lattice containing v. If $\mathrm{ord}_p(D) = 1$, then L_W is of index p in its dual. Thus, the discriminant module M of $\mathbb{Z}_p v \oplus L_W$ is a 2-dimensional $\mathbb{F}_p$-vector space. A quick calculation shows that Q induces a hyperbolic form on M. There exist exactly two self-dual lattices containing $\mathbb{Z}_p v \oplus L_W$ corresponding to the two isotropic lines in M.

Let σ_v be the unique simplex of $\mathcal{T}$ corresponding to v and $\mathbb{Q}(v)$ the $\mathbb{Q}$-subalgebra of B generated by v. Since $\mathbb{Q}(v)^\times$ fixes v, it follows that it also fixes σ_v. The statements about the support of $\Delta(v)$ follow from the $B_{\mathbb{Q}_p}^\times$-invariance of the reduction map. □

The space $\mathscr{S}(V_{\mathbb{A}^{p,\infty}})$ of $\mathbb{Z}$-valued Schwartz–Bruhat functions on $V_{\mathbb{A}^{p,\infty}}$ admits an action of $B_{\mathbb{A}^{p,\infty}}^\times$ induced by the conjugation action on $V_{\mathbb{A}^{p,\infty}}$. Attached to an $\hat{R}^\times$-invariant function $\Phi \in \mathscr{S}(V_{\mathbb{A}^{p,\infty}})$ and a non-zero rational number D is the zero-cycle

$$\Delta_{\Phi,\Gamma}(D) = \sum_{v \in \Gamma\backslash V,\ Q(v)=D} \frac{1}{\#\mathrm{Stab}_\Gamma(v)} \Phi(v)\Delta(v) \in \mathrm{Div}(\Gamma\backslash\mathscr{H}_p)_\mathbb{Q}, \tag{2.3}$$

on $\Gamma\backslash\mathscr{H}_p$. Note the formal similarities between (1.2) and (2.3) when Φ is the characteristic function of $\hat{R}_0$, that will simply be denoted as 1_{R_0}. We proceed to make them precise. By the theory of complex multiplication, the Heegner points appearing in the divisors $\Delta(D)$ of the introduction are defined over $\overline{\mathbb{Q}}$. Hence, after fixing an embedding $\overline{\mathbb{Q}} \subset \mathbb{C}_p$, these divisors can be viewed as elements in $\mathrm{Div}(X(\mathbb{C}_p))$.

Proposition 2.2.2. *Let $D \in \mathcal{D}_S$. The cycle $\Delta_{1_{R_0},\Gamma}(D)$ of (2.3) viewed as an element of* $\mathrm{Div}(X(\mathbb{C}_p))$ *via the Cerednik–Drinfeld uniformization theorem is equal to the cycle $\Delta(D)$ of (1.2).*

Proof. Since R_0 is a $\mathbb{Z}[1/p]$-lattice,

$$\Delta_{1_{R_0},\Gamma}(D) = \Delta_{1_{R_0},\Gamma}(Dp^{2n})$$

for every $n \geq 0$. On the other hand, $\mathscr{B}$ is ramified at p, and, thus, the unique maximal $\mathbb{Z}_p$-order of $\mathscr{B}_{\mathbb{Q}_p}$ is given by the elements whose reduced norm has non-negative p-adic valuation. Therefore, multiplication by p gives a bijection between elements of length

D and elements of length Dp^2 in $\mathscr{R}_0$, which yields the equality $\Delta(D) = \Delta(Dp^{2n})$, for every $n \geq 0$. It is then enough to prove the identification when $D \in \mathcal{D}_S$ is such that $\mathrm{ord}_p(D) \in \{0, 1\}$. The case when $\mathrm{ord}_p(D) = 0$ is treated in Theorem 5.3 of [BD98] and the case when $\mathrm{ord}_p(D) = 1$ follows from [LP18, Section 3.3] and [Mol12, Proposition 5.12]. □

Let $\psi\colon \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$ be the standard character, that is, its local component $\psi_\ell\colon \mathbb{Q}_\ell \to \mathbb{C}^\times$ at a prime ℓ is given by

$$\psi_\ell(x) = e^{-2\pi i q} \quad \text{for } x \in q + \mathbb{Z}_\ell,\ q \in \mathbb{Q}.$$

The Weil oscillator representation attached to V and ψ induces an action of the metaplectic group $\widetilde{\mathrm{SL}}_2(\mathbb{A}^{p,\infty})$ on $\mathscr{S}(V_{\mathbb{A}^{p,\infty}})_\mathbb{C}$ that commutes with the $B^\times_{\mathbb{A}^{p,\infty}}$-action. (See, for example, Section 2.3 of [Gel76] for its construction.) For $M \geq 1$, let $K_0(M)^{(p)}$ be the subgroup of $\mathrm{SL}_2(\mathbb{A}^{p,\infty})$ consisting of matrices in $\mathrm{SL}_2(\hat{\mathbb{Z}}^{(p)})$ with left lower entry divisible by M. Since $K_0(4M)^{(p)}$ splits the exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \widetilde{\mathrm{SL}}_2(\mathbb{A}^{p,\infty}) \longrightarrow \mathrm{SL}_2(\mathbb{A}^{p,\infty}) \longrightarrow 1$$

defining the metaplectic group, it can be regarded as a subgroup of $\widetilde{\mathrm{SL}}_2(\mathbb{A}^{p,\infty})$. (See [Gel76], Proposition 2.14.) We similarly define $K_0(4M)$ and view it as a subgroup of $\mathrm{SL}_2(\mathbb{A}^\infty)$ and of $\widetilde{\mathrm{SL}}_2(\mathbb{A}^\infty)$.

Definition 2.2.3. A Schwartz–Bruhat function $\Phi \in \mathscr{S}(V_{\mathbb{A}^{p,\infty}})$ is called special if:

1. Φ is $\hat{R}^\times$-invariant,
2. Φ is $K_0(4N)^{(p)}$-invariant, and
3. $\Phi(pv) = \Phi(v)$ for all $v \in V_{\mathbb{A}^{p,\infty}}$.

The characteristic function 1_{R_0} is the prime example of a special Schwartz–Bruhat function. Let Φ be a special Schwartz–Bruhat function. Property (2) implies that for $D \in \mathbb{Z}_{(p)} - \{0\}$ we have $\Delta_{\Phi,\Gamma}(D) = 0$ unless $D \in \mathcal{D}_S$: indeed, by [Gel76], Theorem 2.22, the matrix

$$u_\ell = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_\ell)$$

for ℓ a prime different from p acts via the Weil representation by

$$(u_\ell\Phi)(v) = \psi_\ell(Q(v_\ell)) \cdot \Phi(v) \quad \forall \Phi \in \mathscr{S}(V_{\mathbb{A}^{p,\infty}})_\mathbb{C},\ v = (v_\ell)_{\ell \neq p} \in V_{\mathbb{A}^{p,\infty}}.$$

Thus, if Φ is $K_0(4N)^{(p)}$-invariant, it follows that every element v in the support of Φ fulfils $Q(v) \in \hat{\mathbb{Z}}^{(p)}$. Furthermore, the equality

$$\Delta_{\Phi,\Gamma}(p^2 D) = \Delta_{\Phi,\Gamma}(D) \tag{2.4}$$

holds for all $D \in \mathcal{D}_S$ by Property (3). The remainder of this work will solely be concerned in proving the following p-adic analytic version of the Gross–Kohnen–Zagier theorem, which implies Theorem 1.1.1 in view of Proposition 2.2.2 and the fact that 1_{R_0} is special.

Theorem 2.2.4. *Let Φ be a special Schwartz–Bruhat function. The generating series*

$$G_{\Phi,\Gamma}(q) := \Phi(0)[\mathcal{L}^\vee] + \sum_{D \in \mathcal{D}_S} [\Delta_{\Phi,\Gamma}(D)] q^D \in \mathrm{Pic}(\Gamma \backslash \mathscr{H}_p)_{\mathbb{Q}}[[q]],$$

is a modular form of weight $3/2$ and level $\Gamma_0(4N)$.

Remark 2.2.5. The divisors described above are compatible under pullback in the following sense. Let Φ be a Schwartz–Bruhat function on $V_{\mathbb{A}^{p,\infty}}$ invariant under $\hat{R}^\times$. Suppose that R' is an Eichler $\mathbb{Z}[1/p]$-order contained in R, denote by Γ' the group of reduced norm 1 units in R' modulo $\{\pm 1\}$ and consider the projection map $\pi \colon \Gamma' \backslash \mathscr{H}_p \to \Gamma \backslash \mathscr{H}_p$. Then it can be seen in a similar way as in the proof of [Kud97, Proposition 5.10] that, for every $D \in \mathcal{D}_S$,

$$\pi^*(\Delta_{\Phi,\Gamma}(D)) = \Delta_{\Phi,\Gamma'}(D).$$

Using that $\pi_* \circ \pi^*$ is equal to multiplication by the degree of π on $\mathrm{Div}(\Gamma \backslash \mathscr{H}_p)$ and the previous identity, we deduce that R can be replaced by R' in the proof of Theorem 2.2.4. In particular, we will assume from now on that Γ is torsion-free by choosing an appropriate level N^+. This will simplify some calculations as the group Γ will act freely on $\mathcal{H}_p$ and $\mathcal{T}$. In particular, it is a free group on finitely many generators. Moreover, under this assumption the coefficients of the divisors $\Delta_{\Phi,\Gamma}(D)$ are integral.

2.3 Hecke action on divisors

Let $\mathbb{T}^N$ be the integral Hecke algebra away from N, which is generated by the standard generators $\{T_\ell\}_{\ell \nmid N}$. (See [JL95, Section 1.2] for its definition.) We conclude the section describing the action of $\mathbb{T}^N$ on the divisors as well as on the space of $\hat{R}^\times$-invariant Schwartz–Bruhat functions. Let ℓ be a prime not dividing N and fix $\alpha \in B^\times \cap R$ an element of reduced norm ℓ. Consider the maps

$$\Gamma \backslash \mathscr{H}_p \xleftarrow{\pi_1} (\alpha^{-1}\Gamma\alpha \cap \Gamma) \backslash \mathscr{H}_p \xrightarrow{\alpha} (\Gamma \cap \alpha\Gamma\alpha^{-1}) \backslash \mathscr{H}_p \xrightarrow{\pi_2} \Gamma \backslash \mathscr{H}_p.$$

Then, define the action of the Hecke operator T_ℓ on divisors as

$$T_\ell(\Delta) := (\pi_{2,*} \circ \alpha \circ \pi_1^*)(\Delta) \quad \text{for } \Delta \in \mathrm{Div}(\Gamma\backslash\mathscr{H}_p).$$

On the other hand, the action of T_ℓ on $\hat{R}^\times$-invariant Schwartz–Bruhat functions is determined as follows. If $\Gamma = \sqcup_j (\Gamma \cap \alpha\Gamma\alpha^{-1})\delta_j$ for $\{\delta_j\}_j \subset \Gamma$ we define

$$T_\ell(\Phi) := \sum_j \Phi \cdot (\alpha^{-1}\delta_j),$$

where if $\beta \in B^\times$, $\Phi \cdot \beta(v) := \Phi(\beta v \beta^{-1})$. Note that, since $\ell \nmid N$, $R \cap \alpha R \alpha^{-1}$ is an Eichler $\mathbb{Z}[1/p]$-order. Hence, by strong approximation ([Vig80, Ch. III, §4]), the double coset space $\left(\hat{R} \cap \alpha\hat{R}\alpha^{-1}\right)^\times \backslash \hat{B}^\times / B^\times$ has precisely one element. Using this, together with the fact that $R \cap \alpha R \alpha^{-1}$ has an element of reduced norm p ([BD96, Lemma 1.5]), we deduce: for the same $\{\delta_j\}_j \subset \Gamma$ as above $\hat{R}^\times = \sqcup_j (\hat{R} \cap \alpha\hat{R}^\times\alpha^{-1})\delta_j$. Hence, $\hat{R}^\times\alpha^{-1}\hat{R}^\times = \sqcup_j \hat{R}^\times\alpha^{-1}\delta_j$ and it follows from there that $T_\ell(\Phi)$ is $\hat{R}^\times$-invariant. It follows from this description that the Hecke action preserves the subspace of special Schwartz–Bruhat functions.

Lemma 2.3.1. *Let Φ be a Schwartz–Bruhat function on $V_{\mathbb{A}^{p,\infty}}$ invariant under $\hat{R}^\times$. The following identity of divisors holds:*

$$T_\ell(\Delta_{\Phi,\Gamma}(D)) = \Delta_{T_\ell(\Phi),\Gamma}(D).$$

Proof. Using that Γ is torsion-free, the equalities

$$\begin{aligned} T_\ell(\Delta_{\Phi,\Gamma}(D)) &= (\pi_{2,*} \circ \alpha \circ \pi_1^*)\left(\Delta_{\Phi,\Gamma}(D)\right) \\ &= (\pi_{2,*} \circ \alpha)\left(\Delta_{\Phi,\alpha^{-1}\Gamma\alpha\cap\Gamma}(D)\right) \\ &= \pi_{2,*} \circ \alpha\Delta_{\Phi\cdot\alpha^{-1},\Gamma\cap\alpha\Gamma\alpha^{-1}}(D) \\ &= \Delta_{T_\ell(\Phi),\Gamma}(D) \end{aligned}$$

can be proven in the same way as [Kud97, Prop. 5.9 and Prop. 5.10]. □

When it is clear from context that we are viewing Φ as a Γ-invariant Schwartz–Bruhat function, we simply write $\Delta_\Phi(D)$ (resp. G_Φ) to denote $\Delta_{\Phi,\Gamma}(D)$ (resp. $G_{\Phi,\Gamma}$).

Section 3

Modularity of degrees of Heegner divisors

Fix a special Schwartz–Bruhat function Φ. In this section, we prove that

$$\deg(G_\Phi)(q) = \Phi(0)\deg(\mathcal{L}^\vee) + \sum_{D \in \mathcal{D}_S} \deg(\Delta_\Phi(D))q^D$$

is a modular form by comparing $\deg(G_\Phi)$ to a genus theta series attached to V.

3.1 Ternary theta series and Siegel–Weil theorem

Fix $L_0, \ldots, L_r$ unimodular $\mathbb{Z}_p$-lattices in $V_{\mathbb{Q}_p}$ that give a set of representatives of $\Gamma\backslash\mathcal{T}_0$. For every i consider the ternary theta series attached to the Schwartz–Bruhat function $\Phi \otimes 1_{L_i}$ on $V_{\mathbb{A}^\infty}$

$$\Theta_i = \sum_{v \in V} \Phi(v) 1_{L_i}(v) q^{Q(v)}.$$

Note that theta series Θ_i only depends on the class of L_i in $R^\times\backslash\mathcal{T}_0$. Since Φ is invariant under $K_0(4N/p)^{(p)}$ and L_i is a unimodular $\mathbb{Z}_p$-lattice, $\Phi \otimes 1_{L_i}$ is invariant under $K_0(4N/p)$. It is well known that Θ_i is a modular form of weight $3/2$ and level $\Gamma_0(4N/p)$ for every i. (See for example Theorem 4.1 of [Bor98].) Define the modular form

$$E_\Phi := \sum_{i=1}^{r} \Theta_i.$$

The following lemma relates the degrees of those $\Delta_\Phi(D)$ with $\operatorname{ord}_p(D) \in \{0, 1\}$ with the corresponding Fourier coefficients of E_Φ.

Lemma 3.1.1. *Let $D \in \mathcal{D}_S$, then:*

1. *if $\operatorname{ord}_p(D) = 0$, then $2a_D(E_\Phi) = \deg(\Delta_\Phi(D))$, and*

2. *if* $\mathrm{ord}_p(D) = 1$ *then,* $a_D(E_\Phi) = \deg(\Delta_\Phi(D))$.

Proof. Since Γ is torsion-free, it does not stabilize any vertex of $\mathcal{T}_0$. Thus, Lemma 2.2.1 implies that

$$\bigsqcup_{i=1}^{r} \{v \in V \cap L_i \mid Q(v) = D\} \xrightarrow{\sim} \{v \in \Gamma\backslash V \mid Q(v) = D\}$$
$$v \longmapsto [v]$$

is bijective if $\mathrm{ord}_p(D) = 0$ and surjective and two-to-one if $\mathrm{ord}_p(D) = 1$, which implies the assertion. □

Let $M \in \mathbb{Z}_{>0}$ and k such that $2k \in \mathbb{Z}_{>0}$. When k is a half-integer, we will always assume that the level M is divisible by 4. Denote by $M_k(\Gamma_0(M))$ (resp. $S_k(\Gamma_0(M))$) the space of modular forms (resp. cusp forms) forms of weight k and level $\Gamma_0(M)$. Consider the subspaces $M_k(\Gamma_0(M), \mathbb{Z})$ (resp. $S_k(\Gamma_0(M), \mathbb{Z})$) of forms whose q-expansion has integer coefficients and, for any abelian group A, put

$$M_k(\Gamma_0(M), A) := M_k(\Gamma_0(M), \mathbb{Z}) \otimes_{\mathbb{Z}} A \quad \text{and} \quad S_k(\Gamma_0(M), A) := S_k(\Gamma_0(M), \mathbb{Z}) \otimes_{\mathbb{Z}} A.$$

We view these as subspaces of the group $A[[q]]$ of formal q-series with coefficients in A. By [SS77, Lemma 8] there exists a basis of $M_k(\Gamma_0(M))$ consisting of forms with integer coefficients. Hence, the natural homomorphisms

$$M_k(\Gamma_0(M), \mathbb{C}) \xrightarrow{\sim} M_k(\Gamma_0(M)) \quad \text{and} \quad S_k(\Gamma_0(M), \mathbb{C}) \xrightarrow{\sim} S_k(\Gamma_0(M))$$

are bijective. We now introduce several operators acting on $M_{3/2}(\Gamma_0(M), A)$. For that let

$$f = \sum_{n \geq 0} a_n q^n \in A[[q]]$$

be a formal q-series with coefficients in A. Following Shimura ([Shi73, Theorem 1.7]), define

$$T_{p^2}(f) := \sum_{n \geq 0} \left(a_{p^2 n} + \left(\frac{-n}{p} \right) a_n + p a_{n/p^2} \right) q^n,$$
$$U_{p^2}(f) := \sum_{n \geq 0} a_{p^2 n} q^n,$$

and put $V_{p^2} f := T_{p^2} f - U_{p^2} f$. Now suppose that $f \in M_{3/2}(\Gamma_0(M), R)$ is a modular form. Then $T_{p^2} f \in M_{3/2}(\Gamma_0(M), R)$ if $p \nmid M$ and, in case $p \mid M$, we have $U_{p^2} f \in M_{3/2}(\Gamma_0(M), R)$.

Proposition 3.1.2. *The modular form* E_Φ *is an Eisenstein series of level* $\Gamma_0(4N/p)$*. In particular, it satisfies* $T_{p^2}(E_\Phi) = (p+1)E_\Phi$.

Proof. Strong approximation implies that the modular form E_Φ is the genus theta function associated to the Schwartz–Bruhat function $\Phi \otimes 1_{L_i}$. Thus E_Φ is an Eisenstein series by the Siegel–Weil theorem ([Kud03, Theorem 4.1 (ii)]). A direct proof of this fact can be found in [KK22, Corollary 4.3]. Moreover, since Φ is special, Theorem 4.2 of *loc.cit.* shows that E_Φ is an eigenvector of T_{p^2} with eigenvalue $(p+1)$. □

3.2 p-stabilization of the Eisenstein series

Since Φ is special, (2.4) implies that

$$U_{p^2}(G_\Phi(q)) = G_\Phi(q).$$

We proceed to modify E_Φ so that it becomes invariant under U_{p^2} as well. For that, put $E^1_\Phi := E_\Phi - V_{p^2}(E_\Phi) \in M_{3/2}(\Gamma_0(4N))$.

Corollary 3.2.1. *We have* $U_{p^2}(E^1_\Phi) = E^1_\Phi$.

Proof. Since $(U_{p^2} \circ V_{p^2})(E_\Phi) = pE_\Phi$ (which can be verified directly from the description of U_{p^2} and V_{p^2} given above), we have

$$U_{p^2}(E^1_\Phi) = U_{p^2}(E_\Phi) - pE_\Phi.$$

Using that $U_{p^2} = T_{p^2} - V_{p^2}$ and Proposition 3.1.2 yields the desired result. □

We can finally prove the main result of this section.

Proposition 3.2.2. *The equality* $\deg(G_\Phi)(q) = E^1_\Phi$ *holds. In particular,* $\deg(G_\Phi)(q)$ *is an Eisenstein series of weight* $3/2$ *and level* $\Gamma_0(4N)$.

Proof. By Lemma 3.2.1, the equality $U_{p^2}(E^1_\Phi) = E^1_\Phi$ holds. On the other hand, since Φ is special we have $\deg(\Delta_\Phi(Dp^2)) = \deg(\Delta_\Phi(D))$ for all D. Hence, it is enough to verify that the Fourier coefficients of E^1_Φ and of $\deg(G_\Phi)$ are equal in the following cases:

- If $\mathrm{ord}_p(D) = 1$, the second point of Lemma 3.1.1 implies

$$a_D(E^1_\Phi) = a_D(E_\Phi) = \deg(\Delta_\Phi(D)).$$

- If $\mathrm{ord}_p(D) = 0$ and $\left(\frac{-D}{p}\right) = -1$, the first point of Lemma 3.1.1 gives

$$a_D(E^1_\Phi) = 2a_D(E_\Phi) = \deg(\Delta_\Phi(D)).$$

- If $\mathrm{ord}_p(D) = 0$ and $\left(\frac{-D}{p}\right) = 1$, one calculates

$$a_D(E^1_\Phi) = a_D(E_\Phi) - a_D(E_\Phi) = 0.$$

  On the other hand, we have that $\Delta_\Phi(D) = 0$, as $G_\Phi(q)$ is supported only on non-negative integers that belong to $\mathcal{D}_S$.

- If $D = 0$, we have $a_0(E^1_\Phi) = \Phi(0)(1-p)r$, where we recall that $r = \#(\Gamma\backslash\mathcal{T}_0)$. Now, since Γ is torsion-free, it follows that $\Gamma\backslash\mathcal{T}$ is a $(p+1)$-regular graph. Thus, we readily compute its first Betti number

$$g(\Gamma\backslash\mathcal{T}) = 1 - \#(\Gamma\backslash\mathcal{T}_0) + \#(\Gamma\backslash\mathcal{T}_1) = 1 - r + \frac{p+1}{2}r,$$

  which by [FvdP04, Theorem 5.4.1] equals the genus g of X. The degree of the cotangent bundle of X is equal to $2g-2$. This implies that

$$a_0(E^1_\Phi) = \Phi(0)(1-p)r = \Phi(0)(2-2g) = \Phi(0)\deg(\mathcal{L}^\vee).$$

Therefore, we obtain the desired equality $\deg(G_\Phi) = E^1_\Phi$. □

Section 4

The Abel–Jacobi map

Because the curve X is a *Mumford curve* over $\mathbb{Q}_{p^2}$, its Jacobian, denoted by J, has purely toric reduction and admits a concrete description in terms of equivalence classes of automorphy factors of rigid meromorphic functions in $\mathscr{H}_p$. In this section, we explain how the class in J of a degree zero divisor can be described explicitly in these terms. Then, we introduce the notion of divisors of strong degree 0, for which there exists a preferred choice of automorphy factor describing its class in J. Finally, we use this notion to reduce Theorem 2.2.4 to the case where all divisors appearing as coefficients of the generating series G_Φ have strong degree 0.

4.1 Definition and properties of the Abel–Jacobi map

A *formal divisor* on $\mathscr{H}_p$ is a formal, possibly infinite $\mathbb{Z}$-linear combination of points in $\mathscr{H}_p$. A formal divisor

$$\hat{\mathscr{D}} = \sum_{x \in \mathscr{H}_p} m_x(x)$$

is said to be *discrete* if the formal divisor

$$\hat{\mathscr{D}} \cap \mathcal{A} := \sum_{x \in \mathcal{A}} m_x(x)$$

is an actual divisor, i.e., involves a finite sum for all standard affinoid subsets $\mathcal{A} \subset \mathscr{H}_p$. The set of all discrete formal divisors on $\mathscr{H}_p$ is denoted by $\mathrm{Div}^\dagger(\mathscr{H}_p)$. Denote by $\mathrm{Div}(\mathscr{H}_p)$ (resp. $\mathrm{Div}^0(\mathscr{H}_p)$) the subset of finite divisors (resp. finite divisors of degree 0). The quotient map $\pi \colon \mathscr{H}_p \to \Gamma\backslash\mathscr{H}_p$ induces pushforward and pullback maps

$$\pi_* \colon \mathrm{Div}(\mathscr{H}_p) \longrightarrow \mathrm{Div}(\Gamma\backslash\mathscr{H}_p), \qquad \pi^* \colon \mathrm{Div}(\Gamma\backslash\mathscr{H}_p) \longrightarrow \mathrm{Div}^\dagger(\mathscr{H}_p),$$

since Γ acts on $\mathscr{H}_p$ with discrete orbits. Given $\Delta \in \mathrm{Div}(X(\mathbb{C}_p))$, let $\mathscr{D} \in \mathrm{Div}(\mathscr{H}_p)$ and $\hat{\mathscr{D}} \in \mathrm{Div}^\dagger(\mathscr{H}_p)$ be (formal) divisors satisfying

$$\pi_*(\mathscr{D}) = \Delta, \quad \hat{\mathscr{D}} = \pi^*(\Delta). \tag{4.1}$$

The divisor $\mathscr{D}$ is not unique, while the formal divisor $\hat{\mathscr{D}}$ is completely determined by Δ.

Given any degree zero divisor $\mathscr{D}$ on $C_V(\mathbb{C}_p) \simeq \mathbb{P}_1(\mathbb{C}_p)$, there is a rational function $f_{\mathscr{D}}$ on $C_V(\mathbb{C}_p)$ having $\mathscr{D}$ as a divisor, which is unique up to a multiplicative constant. A rational function f is extended multiplicatively to any divisor $\mathscr{D} = \sum_{x \in C_V(\mathbb{C}_p)} m_x \cdot (x)$ by setting

$$f(\mathscr{D}) := \prod_{x \in C_V(\mathbb{C}_p)} f(x)^{m_x}.$$

Definition 4.1.1. The *Weil symbol* attached to two degree zero divisors $\mathscr{D}_0$ and $\mathscr{D}_1$ on $C_V(\mathbb{C}_p)$ with disjoint supports is the quantity

$$[\mathscr{D}_0; \mathscr{D}_1] := f_{\mathscr{D}_0}(\mathscr{D}_1) \in \mathbb{C}_p.$$

The Weil symbol generalises the familiar cross-ratio which one recovers when $\mathscr{D}_0$ and $\mathscr{D}_1$ are both differences of two points, and satisfies the following familiar properties:

1. It is bilinear: for all degree zero divisors $\mathscr{D}_0$, $\mathscr{D}_1$ and $\mathscr{D}_2$

   $$[\mathscr{D}_0; \mathscr{D}_1 + \mathscr{D}_2] = [\mathscr{D}_0; \mathscr{D}_1] \times [\mathscr{D}_0, \mathscr{D}_2], \quad [\mathscr{D}_0 + \mathscr{D}_1; \mathscr{D}_2] = [\mathscr{D}_0; \mathscr{D}_2] \times [\mathscr{D}_1; \mathscr{D}_2],$$

2. It is $B_{\mathbb{C}_p}^\times$-equivariant:

   $$[\gamma\mathscr{D}_0; \gamma\mathscr{D}_1] = [\mathscr{D}_0; \mathscr{D}_1] \quad \text{for all } \gamma \in B_{\mathbb{C}_p}^\times.$$

3. It is symmetric (Weil reciprocity):

   $$[\mathscr{D}_0; \mathscr{D}_1] = [\mathscr{D}_1; \mathscr{D}_0].$$

4. Given any pair $\mathscr{D}_0$ and $\mathscr{D}_1$ of degree zero divisors on $\mathscr{H}_p$ for which the support of $\mathscr{D}_0$ is disjoint from the Γ-orbit of the support of $\mathscr{D}_1$, the infinite product

   $$[\mathscr{D}_0; \mathscr{D}_1]_\Gamma := \prod_{\gamma \in \Gamma} [\mathscr{D}_0; \gamma\mathscr{D}_1]$$

   converges absolutely in $\mathbb{C}_p^\times$. (See page 47 of [GvdP80].)

The quantity $[\mathscr{D}_0; \mathscr{D}_1]_\Gamma$ is called the *modular Weil symbol* attached to the divisors $\mathscr{D}_0$ and $\mathscr{D}_1$ on $\mathscr{H}_p$ and to the discrete p-arithmetic group Γ. It can be used to describe the Jacobian of X as follows: let L be a complete extension of $\mathbb{Q}_{p^2}$, $\mathscr{D} \in \mathrm{Div}^0(\mathscr{H}_p(L))$ a divisor of degree 0 and choose $\eta \in \mathscr{H}_p(L)$ such that (η) and $\alpha\mathscr{D}$ have disjoint support for all $\alpha \in \Gamma$. Then, define $\theta_{\mathscr{D}}$ via

$$\theta_{\mathscr{D}}(z) = [(z) - (\eta); \mathscr{D}]_\Gamma \quad \forall z \in \mathscr{H}_p(L).$$

Note that for $\gamma \in \Gamma$ one gets

$$\frac{\theta_{\mathscr{D}}(\gamma z)}{\theta_{\mathscr{D}}(z)} = [(\gamma z) - (z); \mathscr{D}]_\Gamma = [(\gamma \eta) - (\eta); \mathscr{D}]_\Gamma \in L^\times, \tag{4.2}$$

where in the second equality we used that the modular Weil symbol is invariant under the action of Γ on any of the two divisors, and therefore $[(\gamma z)-(\gamma\eta); \mathscr{D}]_\Gamma = [(z)-(\eta); \mathscr{D}]_\Gamma$, which implies the desired equality by the linearity of the Weil symbol. Thus, the automorphy factor in (4.2) is independent of z. We then denote

$$j_{\mathscr{D}}(\gamma) = [(\gamma\eta) - (\eta); \mathscr{D}]_\Gamma. \tag{4.3}$$

The function $j_{\mathscr{D}}$ defines an element in $\mathrm{Hom}(\Gamma, L^\times) = \mathrm{Hom}(\Gamma_{\mathrm{ab}}, L^\times)$, where Γ_{ab} is the abelianization of Γ. Note that the group Γ_{ab} is a finitely generated free abelian group of rank equal to the genus g of the Shimura curve X. We need to introduce one more ingredient, the so-called *p-adic period pairing.* Define

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle \colon \Gamma \times \Gamma \to \mathbb{C}_p^\times,$$

by choosing arbitrary base points $\tau_1, \tau_2 \in \mathscr{H}_p$ that are not Γ-equivalent and setting

$$\langle\!\langle \gamma_1, \gamma_2 \rangle\!\rangle := [(\gamma_1\tau_1) - (\tau_1); (\gamma_2\tau_2) - (\tau_2)]_\Gamma.$$

In a similar way as above, it can be seen that this expression does not depend on the choice of τ_1 and τ_2, and is a homomorphism in each argument. Moreover, it descends to a pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle \colon \Gamma_{\mathrm{ab}} \times \Gamma_{\mathrm{ab}} \to \mathbb{Q}_p^\times$, which gives an embedding (see VI.2 and VIII.3 of [GvdP80])

$$j \colon \Gamma_{\mathrm{ab}} \hookrightarrow \mathrm{Hom}(\Gamma_{\mathrm{ab}}, \mathbb{Q}_p^\times) \simeq (\mathbb{Q}_p^\times)^g.$$

Now, for a given $\Delta \in \mathrm{Div}^0(\Gamma\backslash\mathscr{H}_p(L))$, choose $\mathscr{D} \in \mathrm{Div}^0(\mathscr{H}_p(L))$ such that $\pi_*\mathscr{D} = \Delta$ and define

$$\mathrm{AJ} \colon \mathrm{Div}^0(\Gamma\backslash\mathscr{H}_p(L)) \longrightarrow \mathrm{Hom}(\Gamma_{\mathrm{ab}}, L^\times)/j(\Gamma_{\mathrm{ab}}), \; \Delta \longmapsto [j_{\mathscr{D}}].$$

It is a calculation to verify that the equivalence class of $j_{\mathscr{D}}$ is independent of the choice of lift of Δ, showing that the map AJ is well-defined. Remember that J denotes the Jacobian of the curve X.

Proposition 4.1.2. *The map* AJ *defined above is trivial on the group of principal divisors and, for every complete extension* L *of* $\mathbb{Q}_{p^2}$*, it induces an identification*

$$J(L) \simeq \mathrm{Hom}(\Gamma_{\mathrm{ab}}, L^\times)/j(\Gamma_{\mathrm{ab}}).$$

Moreover, if $L/\mathbb{Q}_{p^2}$ *is a Galois extension, the identification is* $\mathrm{Gal}\,(L/\mathbb{Q}_{p^2})$*-equivariant.*

Proof. See VI.2. and VIII.4 of [GvdP80]. □

In view of the previous proposition, AJ can be interpreted as a p-adic Abel–Jacobi map. We also note that, by the positive definiteness of the pairing $\mathrm{ord}_p \circ \langle\langle \cdot, \cdot \rangle\rangle$ (see VI.2 and VIII.3 of [GvdP80]), the natural homomorphism from $\mathrm{Hom}(\Gamma_{\mathrm{ab}}, \mathbb{Z}_{p^2}^\times)$ to $\mathrm{Hom}(\Gamma_{\mathrm{ab}}, \mathbb{Q}_{p^2}^\times)/j(\Gamma_{\mathrm{ab}})$ is an injection, whose image has finite index. This gives the explicit description

$$J(\mathbb{Q}_{p^2})_{\mathbb{Q}} \simeq H^1(\Gamma_{\mathrm{ab}}, \mathbb{Z}_{p^2}^\times)_{\mathbb{Q}}.$$

4.2 Divisors of strong degree 0

For any vertex $L \in \mathcal{T}_0$, consider the affinoid $\mathcal{A}_L := \mathrm{red}^{-1}(L) \subset \mathscr{H}_p$ and the wide open $\mathcal{W}_L \subset \mathscr{H}_p$ given as the preimage by red of the union of the vertex L and all the (open) edges of $\mathcal{T}$ that have L as one of its endpoints.

Definition 4.2.1. Let $\mathscr{D}$ be a finite divisor on $\mathscr{H}_p$.

1. $\mathscr{D}$ is of *strong degree* 0 *in the even sense* if $\mathscr{D} \cap \mathcal{W}_L$ is of degree 0 for every even vertex $L \in \mathcal{T}_0$, and $\mathscr{D} \cap \mathcal{A}_L$ is of degree 0 for every odd vertex $L \in \mathcal{T}_0$.

2. $\mathscr{D}$ is of *strong degree* 0 *in the odd sense* if $\mathscr{D} \cap \mathcal{W}_L$ is of degree 0 for every odd vertex $L \in \mathcal{T}_0$, and $\mathscr{D} \cap \mathcal{A}_L$ is of degree 0 for every even vertex $L \in \mathcal{T}_0$.

A divisor $\Delta \in \mathrm{Div}(\Gamma\backslash\mathscr{H}_p)$ is of *strong degree zero* if the following equivalent conditions hold:

1. There exists divisors $\mathscr{D}_e, \mathscr{D}_o \in \mathrm{Div}(\mathscr{H}_p)$ of strong degree 0 in an even and odd sense respectively such that $\pi_*(\mathscr{D}_e) = \pi_*(\mathscr{D}_o) = \Delta$.

2. The formal divisor $\hat{\mathscr{D}} = \pi^*\Delta$ satisfies that, for every $L \in \mathcal{T}_0$, the divisors $\hat{\mathscr{D}} \cap \mathcal{W}_L$ and $\hat{\mathscr{D}} \cap \mathcal{A}_L$ have degree 0.

We denote by $\mathrm{Div}_s^0(\Gamma\backslash\mathscr{H}_p)$ the group of divisors of strong degree 0 on $\Gamma\backslash\mathscr{H}_p$. We also denote $\mathrm{Div}_{s,e}^0(\mathscr{H}_p)$ (resp. $\mathrm{Div}_{s,o}^0(\mathscr{H}_p)$) the group of divisors of strong degree 0 on $\mathscr{H}_p$ in an even (resp. odd) sense. The motivation for these notions is explained in the next lemma.

Lemma 4.2.2. *Let* Δ *be an element of* $\mathrm{Div}_s^0(\Gamma\backslash\mathscr{H}_p)$*. The homomorphism* $j_{\mathscr{D}_e} \in \mathrm{Hom}(\Gamma_{\mathrm{ab}}, \mathbb{C}_p^\times)$ *does not depend on a choice of* $\mathscr{D}_e \in \mathrm{Div}_{s,e}^0(\mathscr{H}_p)$ *with* $\pi_*(\mathscr{D}_e) = \Delta$*. In particular, the morphism*

$$\mathrm{Div}_s^0(\Gamma\backslash\mathscr{H}_p) \longrightarrow \mathrm{Hom}(\Gamma_{\mathrm{ab}}, \mathbb{C}_p^\times), \ \Delta \longmapsto j_{\mathscr{D}_e},$$

is a well-defined lift of the restriction of AJ *to* $\mathrm{Div}_s^0(\Gamma\backslash\mathscr{H}_p)$*. The same is true if one replaces* e *by* o *everywhere.*

Proof. Let $\mathscr{D}$, $\mathscr{D}' \in \mathrm{Div}_{s,e}^0(\mathscr{H}_p)$ be such that $\pi_*\mathscr{D} = \pi_*\mathscr{D}' = \Delta$. By the strong degree 0 assumption there exist vertices $L_1, \ldots, L_r \in \mathcal{T}_0$ and a decomposition

$$\mathscr{D} = \mathscr{D}_1 + \cdots + \mathscr{D}_r$$

such that for $1 \leq i \leq r$ the divisor $\mathscr{D}_i$ is of degree 0 and supported on

1. $\mathcal{W}_{L_i}$ if L_i is an even vertex, or
2. $\mathcal{A}_{L_i}$ if L_i is an odd vertex.

Since $j_{\mathscr{D}_j} = j_{\gamma\mathscr{D}_j}$, for every $\gamma \in \Gamma$, we can suppose that the vertices $L_1, \ldots, L_r$ are not Γ-equivalent. Proceeding similarly for $\mathscr{D}'$, there exist lattices $L'_1, \ldots, L'_{r'} \in \mathcal{T}_0$ and degree 0 divisors $\mathscr{D}'_1, \ldots, \mathscr{D}'_{r'} \in \mathrm{Div}^0(\mathscr{H}_p)$ satisfying the same conditions as above. We have

$$\hat{\mathscr{D}} = \sum_{i=1}^{r} \sum_{\alpha\in\Gamma} \alpha\mathscr{D}_i = \sum_{i=1}^{r'} \sum_{\alpha\in\Gamma} \alpha\mathscr{D}'_i.$$

For $\alpha \in \Gamma$, the divisor $\alpha\mathscr{D}_i$ has support in $\mathcal{W}_{\alpha L_i}$ if L_i is even and has support in $\mathcal{A}_{\alpha L_i}$ if L_i is odd. Note that these supports are disjoint when i varies from 1 to r and α varies over Γ, as Γ does not stabilize any vertex because it is torsion-free. Moreover, the same holds for the divisors $\alpha\mathscr{D}'_i$. Thus, we conclude that $r = r'$ and there exist $\alpha_1, \ldots, \alpha_r \in \Gamma$ such that

$$\mathscr{D}_i = \alpha_i\mathscr{D}'_i$$

for every i (after rearranging terms, if needed). We therefore have that $j_{\mathscr{D}_i} = j_{\mathscr{D}'_i}$ for all i and the equality $j_{\mathscr{D}} = j_{\mathscr{D}'}$ follows. □

If $\Delta \in \mathrm{Div}_s^0(\Gamma\backslash\mathscr{H}_p)$ is a divisor supported on preimages of vertices by the reduction map, both lifts $\mathscr{D}_e$ and $\mathscr{D}_o$ are divisors of strong degree 0 in an even sense and in an odd sense simultaneously. We will sometimes drop the subindices e and o in this case.

4.3 Reduction of theorem to convenient functions

Recall the action of $\mathbb{T}^N$ on $\hat{R}^\times$-invariant Schwartz–Bruhat functions introduced in Section 2. We can similarly define an action of $\mathbb{T}^N$ on the space $\mathrm{Funct}(\Gamma\backslash\mathcal{T}_0,\mathbb{Z})$ of Γ-invariant integral functions on $\mathcal{T}_0$.

Definition 4.3.1. A Schwartz–Bruhat function Φ on $V_{\mathbb{A}^{p,\infty}}$ is *convenient* if it is special, $\Phi(0)=0$, and for every $D\in\mathcal{D}_S$ the divisor $\Delta_\Phi(D)$ is of strong degree 0.

Lemma 4.3.2. *Let Φ be a special Schwartz–Bruhat function and let $T\in\mathbb{T}^N$ be a Hecke operator that annihilates the space $\mathrm{Funct}(\Gamma\backslash\mathcal{T}_0,\mathbb{Z})$. Then, the Schwartz–Bruhat function $T(\Phi)$ is convenient.*

Proof. For $L\in\mathcal{T}_0$, denote by δ_L the characteristic function of L. Define the homomorphism $\deg_{\mathcal{T}_0}\colon \mathrm{Div}(\mathscr{H}_p)\to\mathrm{Funct}(\mathcal{T}_0,\mathbb{Z})$ by

$$\deg_{\mathcal{T}_0}((P))=\begin{cases}\delta_L, & \text{if } \mathrm{red}(P)=L\in\mathcal{T}_0,\\ \delta_L+\delta_{L'} & \text{if } \mathrm{red}(P)=(L,L')\in\mathcal{T}_1.\end{cases}$$

This morphism is $B^\times$-equivariant, hence induces a $\mathbb{T}^N$-equivariant morphism

$$\deg_{\mathcal{T}_0}\colon \mathrm{Div}(\Gamma\backslash\mathscr{H}_p)\longrightarrow\mathrm{Funct}(\Gamma\backslash\mathcal{T}_0,\mathbb{Z}).$$

We proceed to verify that $\Delta_{T(\Phi)}(D)$ is of strong degree 0 for a fixed $D\in\mathcal{D}_S$. From the Hecke equivariance of $\deg_{\mathcal{T}_0}$, we have

$$\deg_{\mathcal{T}_0}(\Delta_{T(\Phi)}(D))=\deg_{\mathcal{T}_0}(T(\Delta_\Phi(D)))=T(\deg_{\mathcal{T}_0}(\Delta_\Phi(D)))=0,$$

where we used Lemma 2.3.1 in the first equality. Since $\Delta_{T(\Phi)}(D)$ is supported on preimages of vertices (resp. edges) if $\mathrm{ord}_p(D)$ is even (resp. odd), the fact that $\deg_{\mathcal{T}_0}(\Delta_{T(\Phi)}(D))=0$ implies that $\Delta_{T(\Phi)}(D)$ is of strong degree 0. Finally, from the fact that T sends the constant functions on $\mathrm{Funct}(\Gamma\backslash\mathcal{T}_0,\mathbb{Z})$ to 0, it follows that $(T(\Phi))(0)=0$. □

Let $G_\Phi(q)\in\mathrm{Pic}(\Gamma\backslash\mathscr{H}_p)[[q]]$ be the generating series introduced in Theorem 2.2.4. We now use the Jacquet–Langlands correspondence to justify that to prove Theorem 2.2.4 it is enough to prove it for the particular case where Φ is convenient.

Proposition 4.3.3. *The following statements are equivalent:*

1. *The generating series $G_\Phi(q)$ is a modular form of weight $3/2$ and level $\Gamma_0(4N)$ for every special Schwartz–Bruhat function Φ.*

2. *The generating series $G_\Phi(q)$ is a cusp form of weight $3/2$ and level $\Gamma_0(4N)$ for every convenient Schwartz–Bruhat function Φ.*

Proof. Clearly (1) implies (2). We justify the reverse implication. By Jacquet–Langlands, we have:

- The action of $\mathbb{T}^N$ on $\mathrm{Funct}(\Gamma\backslash\mathcal{T}_0, \mathbb{Z})$ factors through the action of the Hecke algebra (away from N) on $M_2(\Gamma_0(N/p), \mathbb{Q})$.
- The action of $\mathbb{T}^N$ on $J(\mathbb{C}_p)_\mathbb{Q}$ factors through the action of the Hecke algebra (away from S) on the space of forms in $S_2(\Gamma_0(N), \mathbb{Q})$ that are new at p.

Let ℓ be a prime such that $\ell \notin S$ and denote by $T_\ell \in \mathbb{T}^N$ the corresponding Hecke operator. From the second point and the fact that the map $T_\ell - \ell - 1$ is an isomorphism on $S_2(\Gamma_0(N), \mathbb{Q})$, as the eigenvalues of T_ℓ acting on $S_2(\Gamma_0(N), \mathbb{Q})$ have absolute value less than or equal to $2\sqrt{\ell}$ (see [Eic54], [Shi58], and [Igu59]), we deduce that we have an isomorphism

$$\mathrm{Pic}(X)(\mathbb{C}_p)_\mathbb{Q} \xrightarrow{\sim} J(\mathbb{C}_p)_\mathbb{Q} \oplus \mathbb{Q}, \ [\Delta] \longmapsto ((T_\ell - \ell - 1)\Delta, \deg(\Delta)) .$$

Let Φ be a special Schwartz–Bruhat function. Since we proved that $\deg(G_\Phi)$ is a modular form in Proposition 3.2.2, after replacing Φ by $(T_\ell - \ell - 1)(\Phi)$ (and by Lemma 2.3.1) we may suppose that $G_\Phi(q) \in J(\mathbb{C}_p)_\mathbb{Q}[[q]]$. Now, choose $T \in \mathbb{T}^N$ satisfying

- T annihilates $\mathrm{Funct}(\Gamma\backslash\mathcal{T}_0, \mathbb{Z})$ and
- $T\colon J(\mathbb{C}_p)_\mathbb{Q} \to J(\mathbb{C}_p)_\mathbb{Q}$ is a bijection.

Such a $T \in \mathbb{T}^N$ exists because there are Hecke operators acting on $M_2(\Gamma_0(N), \mathbb{Q})$ which are 0 on $M_2(\Gamma_0(N/p), \mathbb{Q})$ and are isomorphisms on the subspace of cusp forms which are new at p. By the first property and Lemma 4.3.2, $T(\Phi)$ is convenient and therefore $G_{T(\Phi)}(q) = T(G_\Phi(q))$ is a modular form by hypothesis. Here $T(G_\Phi(q))$ denotes the q-expansion obtained by applying T to each of the coefficients of $G_\Phi(q)$. The fact that $G_\Phi(q) \in J(\mathbb{C}_p)_\mathbb{Q}[[q]]$ is a modular form follows from the bijectivity of T on the Jacobian. □

Let Φ be a convenient Schwartz–Bruhat function. For every $D \in \mathcal{D}_S$, fix $\mathscr{D}_\Phi(D)_e \in \mathrm{Div}^0_{s,e}(\mathscr{H}_p)$ and $\mathscr{D}_\Phi(D)_o \in \mathrm{Div}^0_{s,o}(\mathscr{H}_p)$ such that $\pi_*\mathscr{D}_\Phi(D)_e = \pi_*\mathscr{D}_\Phi(D)_o = \Delta_\Phi(D)$. Note that, since the cycles $\Delta(v)$ for $v \in V$ introduced in Section 2.2 are invariant under the action of $\mathrm{Aut}(\mathbb{C}_p/\mathbb{Q}_p)$, these lifts of $\Delta_\Phi(D)$ can be chosen such that they are invariant under the action of $\mathrm{Aut}(\mathbb{C}_p/\mathbb{Q}_p)$. It follows from there that the homomorphisms $j_{\mathscr{D}_\Phi(D)_e}$ and $j_{\mathscr{D}_\Phi(D)_o}$ take values in $\mathbb{Q}_p^\times$. Consider the generating series

$$G_\Phi^+(q) = \sum_{D \in \mathcal{D}_S} j_{\mathscr{D}_\Phi(D)_e} \cdot j_{\mathscr{D}_\Phi(D)_o} q^D \in \mathrm{Hom}(\Gamma_{\mathrm{ab}}, \mathbb{Q}_p^\times)[[q]].$$

Note that $a_D(G_\Phi^+(q)) = a_{Dp^{2n}}(G_\Phi^+(q))$ for all $n \geq 0$ since Φ is special.

Observe that by Lemma 4.2.2, if we apply the quotient map

$$\mathrm{Hom}(\Gamma_{\mathrm{ab}}, \mathbb{Q}_p^\times) \longrightarrow\!\!\!\!\rightarrow \mathrm{Hom}(\Gamma_{\mathrm{ab}}, \mathbb{Q}_p^\times)/j(\Gamma)$$

to each of the coefficients of $G_\Phi^+(q)$ we obtain $2\mathrm{AJ}(G_\Phi)(q)$. Here $2\mathrm{AJ}(G_\Phi)(q)$ is the generating series obtained by applying 2AJ to each of the coefficients of $G_\Phi(q)$. Hence, by Proposition 4.1.2 the modularity of $G_\Phi^+(q)$ implies the modularity of $G_\Phi(q)$. The remainder of this section is dedicated to proving modularity of $G_\Phi^+(q)$.

The advantage of working with $G_\Phi^+(q)$ over $\mathrm{AJ}(G_\Phi)(q)$ is that the group of continuous homomorphisms from $\mathrm{Hom}(\Gamma_{\mathrm{ab}}, \mathbb{Q}_p^\times)$ to $\mathbb{Q}_p$ can be described explicitly. Indeed, generators of this space are given by homomorphisms of the form $j \mapsto \log_p(j(\gamma))$ respectively $j \mapsto \mathrm{ord}_p(j(\gamma))$ with $\gamma \in \Gamma$, where $\log_p$ is the branch of the p-adic logarithm for which $\log_p(p) = 0$.

Section 5

Values of p-adic theta functions

The goal of this section is to give an explicit expression for the quantity $j_{\mathscr{D}}(\gamma)$ when $\gamma \in \Gamma$ is hyperbolic at p and $\mathscr{D}$ is a divisor on $\Gamma \backslash \mathscr{H}_p$ of strong degree 0. The formulas we will present have a similar flavor to the ones for toric values of lifting obstructions of rigid meromorphic cocycles given in [DV22, Section 5.3]. There, the orthogonal group of signature $(3,0)$ is replaced by an orthogonal group of signature $(1,2)$.

Fix an element $\gamma \in \Gamma$ that is hyperbolic at p. It has two distinct fixed points

$$\xi^+, \xi^- \in C_V(\mathbb{Q}_p)$$

on the boundary of $\mathscr{H}_p$. We order them in such a way that ξ^+ and ξ^- are the attractive and repulsive fixed points of γ, i.e.,

$$\lim_{M \to +\infty} \gamma^M \tau = \xi^+, \quad \lim_{M \to -\infty} \gamma^M \tau = \xi^-,$$

for all $\tau \in \mathscr{H}_p$.

Lemma 5.0.1. *For every $\mathscr{D} \in \mathrm{Div}^0(\mathscr{H}_p)$ the following equality holds:*

$$j_{\mathscr{D}}(\gamma) = \prod_{\alpha \in \gamma^{\mathbb{Z}} \backslash \Gamma} \left[(\xi^+) - (\xi^-); \alpha \mathscr{D} \right].$$

Proof. Recall first from (4.3) that

$$j_{\mathscr{D}}(\gamma) = \prod_{\alpha \in \Gamma} \left[(\alpha \gamma \tau) - (\alpha \tau); \mathscr{D} \right],$$

where τ is an arbitrary base point in $\mathscr{H}_p$. Since this infinite product converges absolutely, it can be rearranged by grouping together the factors that belong to the same coset for $\gamma^{\mathbb{Z}}$ in Γ

$$j_{\mathscr{D}}(\gamma) = \prod_{\alpha \in \Gamma / \gamma^{\mathbb{Z}}} \left(\prod_{i=-\infty}^{\infty} \left[(\alpha \gamma^{i+1} \tau) - (\alpha \gamma^i \tau); \mathscr{D} \right] \right).$$

The innermost product on the right hand side is equal to

$$\lim_{M\to\infty}\prod_{i=-M}^{M}\left[(\alpha\gamma^{i+1}\tau)-(\alpha\gamma^{i}\tau);\mathscr{D}\right]=\lim_{M\to\infty}\left[(\alpha\gamma^{M+1}\tau)-(\alpha\gamma^{-M}\tau);\mathscr{D}\right]$$
$$=\left[(\alpha\xi^{+})-(\alpha\xi^{-});\mathscr{D}\right].$$

It follows that

$$j_{\mathscr{D}}(\gamma)=\prod_{\alpha\in\Gamma/\gamma^{\mathbb{Z}}}\left[(\alpha\xi^{+})-(\alpha\xi^{-});\mathscr{D}\right]=\prod_{\alpha\in\gamma^{\mathbb{Z}}\backslash\Gamma}\left[(\xi^{+})-(\xi^{-});\alpha\mathscr{D}\right],$$

where the last equation was obtained by substituting α for α^{-1} and exploiting the fact that the Weil Symbol is $B^{\times}_{\mathbb{Q}_p}$-equivariant. □

5.1 The quotient $\gamma^{\mathbb{Z}}\backslash\mathcal{T}$

We will rewrite the infinite product of Lemma 5.0.1 by making an explicit choice of coset representatives for $\gamma^{\mathbb{Z}}$ in Γ, well adapted to the calculation at hand. To make this choice, we will exploit the action of $\gamma^{\mathbb{Z}}$ on the Bruhat-Tits tree $\mathcal{T}$. We explain some of the properties of such action.

Since the element $\gamma\in\Gamma$ is hyperbolic at p, and therefore its image in $\mathrm{SL}_2(\mathbb{Q}_p)$ diagonalizes by the fixed isomorphism $B_{\mathbb{Q}_p}\simeq\mathrm{M}_2(\mathbb{Q}_p)$, we deduce that it acts by conjugation on $V_{\mathbb{Q}_p}$ with three distinct eigenvalues $\varpi, 1$, and ϖ^{-1}, where ϖ is a global p-unit of norm 1 in the quadratic imaginary field that splits the characteristic polynomial of γ (relative to an embedding of this quadratic imaginary field into $\mathbb{Q}_p$). The valuation $\mathrm{ord}_p(\varpi)=2t>0$ is an even integer. Letting $V[\lambda]$ denote the eigenspace in V on which γ acts as multiplication by λ, one obtains the decomposition

$$V_{\mathbb{Q}_p}=V[\varpi]\oplus V[\varpi^{-1}]\oplus V[1].$$

The first two eigenspaces are isotropic and together generate a hyperbolic plane in $V_{\mathbb{Q}_p}$ whose orthogonal complement is $V[1]$. The fixed points ξ^{+} and ξ^{-} of γ in $C_V(\mathbb{Q}_p)$ correspond to the isotropic lines $V[\varpi]$ and $V[\varpi^{-1}]$ respectively. Let w^{+} be a generator of $V[\varpi]$, let w^{-} be a generator of $V[\varpi^{-1}]$, and e a generator of $V[1]$. It follows from this discussion that we can scale these vectors so that

$$L_0:=\langle w^{+},w^{-},e\rangle$$

is a unimodular $\mathbb{Z}_p$-lattice of $V_{\mathbb{Q}_p}$. Then, L_0 admits an eigenspace decomposition under the conjugation action of γ as a module over $\mathbb{Z}_p$. The same is true for the lattices

$$L_j:=\langle w_j^{+}:=p^{j}w^{+},w_j^{-}:=p^{-j}w^{-},e\rangle,\qquad j\in\mathbb{Z}.$$

The unimodular lattices L_j and L_{j+1} are p-neighbours, and the element γ sends L_j to L_{j+2t}. The sequence of successive p-neighbours

$$g_\gamma = \Big\{ \ldots, L_{-2}, L_{-1}, L_0, L_1, L_2, L_3, \ldots \Big\}$$

determines an infinite geodesic on $\mathcal{T}$ which is globally preserved by γ. We suppose that the scaling of the eigenvectors w^+, w^- and e are chosen so that the lattice L_0 is an even vertex of $\mathcal{T}$.

Definition 5.1.1. Let $L \subseteq V_{\mathbb{Q}_p}$ be a unimodular $\mathbb{Z}_p$-lattice in $V_{\mathbb{Q}_p}$. The lattice $L_i \in g_\gamma$ that is closest to L is called the *parent* of L. The distance from L to its parent L_i is called the *depth* of L with respect to γ.

A fundamental region for $\gamma^{\mathbb{Z}}\backslash\mathcal{T}_0$ can therefore be defined by setting

$$\mathcal{T}_{0,\gamma} := \Big\{L \in \mathcal{T}_0 \text{ with Parent}(L) \in \{L_0, L_1, L_2, \ldots L_{2t-1}\}\Big\}.$$

The subset $\mathcal{T}_{0,\gamma} \subset \mathcal{T}$ can be written as an increasing union of finite subsets

$$\mathcal{T}_{0,\gamma} = \bigcup_{n\geq 0} \mathcal{T}_{0,\gamma}^{\leq n}, \qquad \mathcal{T}_{0,\gamma}^{\leq n} := \{L \in \mathcal{T}_{0,\gamma} \text{ with depth}(L) \leq n\}.$$

Let $\mathcal{A}_\gamma$ respectively $\mathcal{A}_\gamma^{\leq n}$ be the subsets of $\mathscr{H}_p$ given as the preimages of $\mathcal{T}_{0,\gamma}$ respectively $\mathcal{T}_{0,\gamma}^{\leq n}$ under the reduction map. The set $\mathcal{A}_\gamma$ can thus be expressed as an increasing union of affinoid subsets,

$$\mathcal{A}_\gamma = \bigcup_{n\geq 0} \mathcal{A}_\gamma^{\leq n}. \tag{5.1}$$

Using $\mathcal{T}_{0,\gamma}$, we proceed to give several fundamental regions for $\gamma^{\mathbb{Z}}\backslash\mathcal{T}_1$. Define $\mathcal{T}_{1,\gamma,e}$ to be the set of edges in $\mathcal{T}_1$ such that its even vertex is in $\mathcal{T}_{0,\gamma}$. For a given vertex $L \in \mathcal{T}_0$, let $W_L \subset \mathcal{T}_1$ be the set of open edges that have L as one of its endpoints. We then have,

$$\mathcal{T}_{1,\gamma,e} = \bigcup_{n\geq 0} \mathcal{T}_{1,\gamma,e}^{\leq n}, \qquad \mathcal{T}_{1,\gamma,e}^{\leq n} := \bigcup_{\substack{L \text{ even} \\ L\in\mathcal{T}_{0,\gamma}^{\leq n}}} W_L.$$

Let $\mathcal{W}_{\gamma,e}$ and $\mathcal{W}_{\gamma,e}^{\leq n}$ be the preimage by red of $\mathcal{T}_{1,\gamma,e}$ and $\mathcal{T}_{1,\gamma,e}^{\leq n}$, respectively. We then have

$$\mathcal{W}_{\gamma,e} = \bigcup_{n\geq 0} \mathcal{W}_{\gamma,e}^{\leq n}. \tag{5.2}$$

Observe that for every n the set $\mathcal{W}_{\gamma,e}^{\leq n}$ can be written as the disjoint union of sets of the form $\mathcal{W}_L - \mathcal{A}_L$, where L runs over even vertices in $\mathcal{T}_{0,\gamma}^{\leq n}$. Similarly, define $\mathcal{T}_{1,\gamma,o}$, $\mathcal{T}_{1,\gamma,o}^{\leq n}$, $\mathcal{W}_{\gamma,o}$ and $\mathcal{W}_{\gamma,o}^{\leq n}$ by replacing even by odd everywhere.

5.2 Formula for $j_{\mathscr{D}}(\gamma)$ for divisors of strong degree 0

With the notations given in the previous section in place, we can prove the following formulas.

Proposition 5.2.1. *Let $\mathscr{D}$ be a divisor on $\mathscr{H}_p$ of strong degree zero supported on preimages of vertices of $\mathcal{T}$ under the reduction map. Let $\hat{\mathscr{D}} = \sum_{\alpha\in\Gamma} \alpha\mathscr{D} \in \mathrm{Div}^\dagger(\mathscr{H}_p)$. Then,*

$$j_{\mathscr{D}}(\gamma) = \lim_{n\to\infty} \left[(\xi^+) - (\xi^-); \hat{\mathscr{D}} \cap \mathcal{A}_\gamma^{\leq n}\right].$$

Proof. Since $\mathscr{D}$ is of strong degree 0 we can write $\mathscr{D} = \sum_{i=1}^r \mathscr{D}_{L_i}$, where $\mathscr{D}_{L_i}$ is a degree 0 divisor supported on $\mathcal{A}_{L_i}$ and the L_i are vertices in $\mathcal{T}_0$. By Lemma 5.0.1, we have

$$j_{\mathscr{D}}(\gamma) = \prod_{\alpha\in\gamma^{\mathbb{Z}}\backslash\Gamma} \left[(\xi^+) - (\xi^-); \alpha\mathscr{D}\right] = \prod_{i=1}^{r} \prod_{\alpha\in\gamma^{\mathbb{Z}}\backslash\Gamma} \left[(\xi^+) - (\xi^-); \alpha\mathscr{D}_{L_i}\right]. \tag{5.3}$$

Now, it follows from the definition of $\mathcal{T}_{0,\gamma}$, that for every $i \in \{1, \ldots, r\}$ and for every class $[\alpha] \in \gamma^{\mathbb{Z}}\backslash\Gamma$, there is precisely one representative $\gamma^{k_{i,\alpha}}\alpha$ such that $\gamma^{k_{i,\alpha}}\alpha\mathscr{D}_{L_i}$ is supported on $\mathcal{A}_\gamma$. This implies that if we write

$$\hat{\mathscr{D}} = \sum_{\alpha\in\Gamma} \alpha\mathscr{D} = \sum_{i=1}^{r} \sum_{\alpha\in\gamma^{\mathbb{Z}}\backslash\Gamma} \sum_{k=-\infty}^{+\infty} \gamma^k\alpha\mathscr{D}_{L_i},$$

we have

$$\hat{\mathscr{D}} \cap \mathcal{A}_\gamma = \sum_{i=1}^{r} \sum_{\alpha\in\gamma^{\mathbb{Z}}\backslash\Gamma} \gamma^{k_{i,\alpha}}\alpha\mathscr{D}_{L_i}.$$

On the other hand, using (5.3) and that ξ^+ and ξ^- are fixed by γ we can write

$$j_{\mathscr{D}}(\gamma) = \prod_{i=1}^{r} \prod_{\alpha\in\gamma^{\mathbb{Z}}\backslash\Gamma} \left[(\xi^+) - (\xi^-); \gamma^{k_{i,\alpha}}\alpha\mathscr{D}_{L_i}\right].$$

Combining the last two equalities, and specifying the order of multiplication on the last expression for $j_{\mathscr{D}}(\gamma)$ given by the increasing union of (5.1), we obtain the desired result. □

We can obtain similar expressions for $j_{\mathscr{D}}(\gamma)$ when $\mathscr{D}$ is a divisor of strong degree 0 supported on preimages of edges by the reduction map.

Proposition 5.2.2. *Let $\mathscr{D}$ be a divisor on $\mathscr{H}_p$ supported on preimages of edges of $\mathcal{T}$ by the reduction map. Let $\hat{\mathscr{D}} = \sum_{\alpha\in\Gamma} \alpha\mathscr{D} \in \mathrm{Div}^\dagger(\mathscr{H}_p)$. We have:*

1. *If $\mathscr{D} = \mathscr{D}_e$ is of strong degree 0 in an even sense, then*

$$j_{\mathscr{D}_e}(\gamma) = \lim_{n\to+\infty} \left[(\xi^+) - (\xi^-); \hat{\mathscr{D}} \cap \mathcal{W}_{\gamma,e}^{\leq n}\right].$$

2. If $\mathscr{D} = \mathscr{D}_o$ is of strong degree 0 *in an odd sense, then*

$$j_{\mathscr{D}_o}(\gamma) = \lim_{n\to+\infty} \left[(\xi^+) - (\xi^-); \hat{\mathscr{D}} \cap \mathcal{W}_{\gamma,o}^{\leq n}\right].$$

Proof. We only give the proof for the first case, the second being similar. Write $\mathscr{D} = \sum_{i=1}^{r} \mathscr{D}_{L_i}$, where $\mathscr{D}_{L_i}$ is a degree 0 divisor supported on $\mathcal{W}_{L_i} - \mathcal{A}_{L_i}$ and the set $\{L_1, \dots, L_r\}$ consists of even vertices of $\mathcal{T}_0$. Hence, we have

$$j_{\mathscr{D}}(\gamma) = \prod_{\alpha\in\gamma^{\mathbb{Z}}\backslash\Gamma} \left[(\xi^+) - (\xi^-); \alpha\mathscr{D}\right] = \prod_{i=1}^{r} \prod_{\alpha\in\gamma^{\mathbb{Z}}\backslash\Gamma} \left[(\xi^+) - (\xi^-); \alpha\mathscr{D}_{L_i}\right].$$

Now, for every class $[\alpha] \in \gamma^{\mathbb{Z}}\backslash\Gamma$ and L_i even vertex as above, there exists precisely one representative $\gamma^{k_{i,\alpha}}\alpha$ such that $\gamma^{k_{i,\alpha}}\alpha L_i \in \mathcal{T}_{0,\gamma}$. It follows from there that the divisor $\gamma^{k_{i,\alpha}}\alpha\mathscr{D}_{L_i}$ is supported on $\gamma^{k_{i,\alpha}}\alpha\mathcal{W}_{L_i} - \gamma^{k_{i,\alpha}}\alpha\mathcal{A}_{L_i} = \mathcal{W}_{\gamma^{k_{i,\alpha}}\alpha L_i} - \mathcal{A}_{\gamma^{k_{i,\alpha}}\alpha L_i} \subset \mathcal{W}_{\gamma,e}$. This implies that if

$$\hat{\mathscr{D}} = \sum_{i=1}^{r} \sum_{\alpha\in\gamma^{\mathbb{Z}}\backslash\Gamma} \sum_{k=-\infty}^{+\infty} \gamma^k\alpha\mathscr{D}_{L_i},$$

we have

$$\hat{\mathscr{D}} \cap \mathcal{W}_{\gamma,e} = \sum_{i=1}^{r} \sum_{\alpha\in\gamma^{\mathbb{Z}}\backslash\Gamma} \gamma^{k_{i,\alpha}}\alpha\mathscr{D}_{L_i}.$$

On the other hand,

$$j_{\mathscr{D}}(\gamma) = \prod_{\alpha\in\gamma^{\mathbb{Z}}\backslash\Gamma} \left[(\xi^+) - (\xi^-); \gamma^{k_{i,\alpha}}\alpha\mathscr{D}_{L_i}\right]. \tag{5.4}$$

Note that $\mathcal{W}_{\gamma,e}^{\leq n}$ can be written as a union of sets of the form $\mathcal{W}_{L_i} - \mathcal{A}_{L_i}$, where the union is over even vertices in $\mathcal{T}_{0,\gamma}^{\leq n}$. Hence, $\hat{\mathscr{D}} \cap \mathcal{W}_{\gamma,e}^{\leq n}$ is a degree 0 divisor (because $\mathscr{D}$ is of strong degree 0). Moreover, the increasing union over n of the sets $\mathcal{W}_{\gamma,e}^{\leq n}$ covers $\mathcal{W}_{\gamma,e}$, as we deduced in (5.2). This implies that we can use these sets to specify an order of multiplication on (5.4) to obtain the desired expression. □

Section 6

Abel–Jacobi images of Heegner divisors

We use the results of Section 5 to compute Abel–Jacobi images of the Heegner divisors introduced in Section 2. More precisely, let Φ be a convenient Schwartz–Bruhat function and fix $D \in \mathcal{D}_S$. Choose $\mathscr{D}_\Phi(D)_e, \mathscr{D}_\Phi(D)_o$ divisors on $\mathscr{H}_p$ of strong degree 0 in an even and odd sense respectively that lift $\Delta_\Phi(D)$ and let $\hat{\mathscr{D}}_\Phi(D) = \pi^*\Delta_\Phi(D)$. If $\Delta_\Phi(D)$ is supported on preimages of vertices under the reduction map, we suppose that $\mathscr{D}_\Phi(D)_e = \mathscr{D}_\Phi(D)_o$ and we will drop the subindices e and o. At last, let $\gamma \in \Gamma$ be an element hyperbolic at p. We will compute $j_{\mathscr{D}_\Phi(D)_e} \cdot j_{\mathscr{D}_\Phi(D)_o}(\gamma)$.

6.1 Values of theta functions of Heegner divisors

Since Φ is invariant under multiplication by p, we have $\Delta_\Phi(D) = \Delta_\Phi(Dp^{2n})$ for every $n \geq 0$. Therefore, we will assume here and for the rest of the section that D is an element of $\mathcal{D}_S$ with $\mathrm{ord}_p(D) \in \{0, 1\}$. In view of the notion of depth of a lattice with respect to γ, which was introduced in Definition 5.1.1, the following definition will be relevant.

Definition 6.1.1. Let $v \in V$ be a vector such that $Q(v) = D$. The *depth* of v with respect to γ is

$$\mathrm{depth}(v) := \min_{L \ni v}\{\mathrm{depth}(L)\},$$

where the minimum is taken over all unimodular $\mathbb{Z}_p$-lattices in $V_{\mathbb{Q}_p}$ such that $v \in L$.

Note that by Lemma 2.2.1 there exist at most two unimodular $\mathbb{Z}_p$-lattices containing v. If there is a unique unimodular lattice containing v, denote it by $L_v \in \mathcal{T}_0$. If there are two unimodular lattices containing v, denote by $e_v \in \mathcal{T}_1$ the edge connecting them.

We now present the computation of $j_{\mathscr{D}_\Phi(D)_e} \cdot j_{\mathscr{D}_\Phi(D)_o}(\gamma)$, which is slightly different according to the p-adic valuation of D. Let $n \geq 1$. If $\mathrm{ord}_p(D) = 0$, consider

$$\hat{\mathscr{D}}_\Phi(D) \cap \mathcal{A}_\gamma^{\leq n} = \sum_{\substack{Q(v)=D,\\ L_v \in \mathcal{T}_{0,\gamma}^{\leq n}}} \Phi(v)\Delta(v),$$

where the sum is over the vectors $v \in V$. Since this divisor is of degree zero, as Φ is convenient, the function on C_V given by

$$\xi \longmapsto \prod_{\substack{Q(v)=D,\\ L_v \in \mathcal{T}_{0,\gamma}^{\leq n}}} \langle \tilde{\xi}, v \rangle^{\Phi(v)},$$

where $\tilde{\xi}$ is any vector in the isotropic line generated by ξ in $V_{\mathbb{C}_p}$, is well-defined and has divisor equal to $\hat{\mathscr{D}}_\Phi(D) \cap \mathcal{A}_\gamma^{\leq n}$. Therefore, if $\tilde{\xi}^+$ and $\tilde{\xi}^-$ are vectors in $V_{\mathbb{C}_p}$ generating the $\mathbb{C}_p$-lines ξ^+ and ξ^- introduced in Section 5, Proposition 5.2.1 implies

$$j_{\mathscr{D}_\Phi(D)}(\gamma) = \lim_{n\to\infty} \prod_{\substack{Q(v)=D,\\ L_v \in \mathcal{T}_{0,\gamma}^{\leq n}}} \left(\frac{\langle \tilde{\xi}^+, v\rangle}{\langle \tilde{\xi}^-, v\rangle} \right)^{\Phi(v)} = \lim_{n\to\infty} \prod_{\substack{v \in \gamma^{\mathbb{Z}} \backslash V\\ Q(v)=D,\\ \mathrm{depth}(v) \leq n}} \left(\frac{\langle \tilde{\xi}^+, v\rangle}{\langle \tilde{\xi}^-, v\rangle} \right)^{\Phi(v)}. \tag{6.1}$$

Here, the second equality follows from the fact that the terms appearing in the expression of the middle do not change if we replace v by γv. If $\mathrm{ord}_p(D) = 1$, we can proceed similarly. In that case, the function on $C_V(\mathbb{C}_p)$ given by

$$\xi \longmapsto \prod_{\substack{Q(v)=D\\ e_v \in \mathcal{T}_{1,\gamma,e}^{\leq n}}} \langle \xi, v \rangle^{\Phi(v)} \prod_{\substack{Q(v)=D\\ e_v \in \mathcal{T}_{1,\gamma,o}^{\leq n}}} \langle \xi, v \rangle^{\Phi(v)}$$

has divisor equal to

$$\hat{\mathscr{D}}_\Phi(D) \cap \mathcal{W}_{\gamma,e}^{\leq n} + \hat{\mathscr{D}}_\Phi(D) \cap \mathcal{W}_{\gamma,o}^{\leq n}.$$

It then follows from Proposition 5.2.2 that

$$\begin{aligned} j_{\mathscr{D}_\Phi(D)_e} \cdot j_{\mathscr{D}_\Phi(D)_o}(\gamma) &= \lim_{n\to+\infty} \prod_{\substack{Q(v)=D\\ e_v \in \mathcal{T}_{1,\gamma,e}^{\leq n}}} \left(\frac{\langle \tilde{\xi}^+, v\rangle}{\langle \tilde{\xi}^-, v\rangle} \right)^{\Phi(v)} \prod_{\substack{Q(v)=D\\ e_v \in \mathcal{T}_{1,\gamma,o}^{\leq n}}} \left(\frac{\langle \tilde{\xi}^+, v\rangle}{\langle \tilde{\xi}^-, v\rangle} \right)^{\Phi(v)} \\ &= \lim_{n\to+\infty} \prod_{\substack{v \in \gamma^{\mathbb{Z}} \backslash V\\ Q(v)=D\\ \mathrm{depth}(v) \leq n-1}} \left(\frac{\langle \tilde{\xi}^+, v\rangle}{\langle \tilde{\xi}^-, v\rangle} \right)^{\Phi(v)} \prod_{\substack{v \in \gamma^{\mathbb{Z}} \backslash V\\ Q(v)=D\\ \mathrm{depth}(v) \leq n}} \left(\frac{\langle \tilde{\xi}^+, v\rangle}{\langle \tilde{\xi}^-, v\rangle} \right)^{\Phi(v)}, \end{aligned} \tag{6.2}$$

where the second equality is obtained by rearranging the terms of the products in the following way. Recall that $\mathcal{T}_{1,\gamma,e}^{\leq n}$ is the union of edges that have an endpoint in $\mathcal{T}_{0,\gamma}$ which is even and at distance less than or equal to n from the geodesic g_γ. Similarly, $\mathcal{T}_{1,\gamma,o}^{\leq n}$ is given by replacing even by odd in the definition above. It follows from this description that, up to $\gamma^{\mathbb{Z}}$-equivariance, the double product in the first equality considers vectors $v \in V$ such that e_v is at distance less than or equal than $n-1$ twice (as e_v, or a $\gamma^{\mathbb{Z}}$-translate of it, belongs to both $\mathcal{T}_{1,\gamma,e}^{\leq n}$ and $\mathcal{T}_{1,\gamma,o}^{\leq n}$). On the other hand, the double product of the first equality considers $v \in V$ such that e_v is at distance equal to n exactly once (as e_v belongs to either $\mathcal{T}_{1,\gamma,e}^{\leq n}$ and $\mathcal{T}_{1,\gamma,o}^{\leq n}$). The equality can be deduced from there and from the fact that the expression $\langle \tilde{\xi}^+, v\rangle^{\Phi(v)}/\langle \tilde{\xi}^-, v\rangle^{\Phi(v)}$ does not change if v is replaced by γv.

6.2 Vectors of length D in $\gamma^{\mathbb{Z}}\backslash V$

Recall that $D \in \mathcal{D}_S$ is such that $\mathrm{ord}_p(D) \in \{0,1\}$. We give a concrete choice of representatives of the quotient

$$\left\{ v \in \gamma^{\mathbb{Z}}\backslash V \;\middle|\; Q(v) = D,\ \mathrm{depth}(v) \leq n \right\}. \tag{6.3}$$

This will lead to a relation between $j_{\mathscr{D}_\Phi(D)}(\gamma)$ and Fourier coefficients of theta series in the next section. We begin by recalling some of the notation introduced in Section 5.1. Recall that the conjugation action of γ on $V_{\mathbb{Q}_p}$ diagonalizes, with eigenvalues $\{\varpi, \varpi^{-1}, 1\}$, and that $\mathrm{ord}_p(\varpi) = 2t \in 2\mathbb{Z}$. Let w^+, w^-, e be the corresponding eigenvectors, scaled so that $L_0 = \langle w^+, w^-, e\rangle$ is a unimodular $\mathbb{Z}_p$-lattice of $V_{\mathbb{Q}_p}$. For $j \in \mathbb{Z}$, let

$$w_j^+ := p^j w^+,\ w_j^- := p^{-j} w^-$$

and consider the $\mathbb{Z}_p$-lattice $L_j = \langle w_j^+, w_j^-, e\rangle$. Then, as discussed in Section 5, $L_0, L_1, \ldots, L_{2t-1}$ form a set of representatives modulo $\gamma^{\mathbb{Z}}$ of the vertices in the geodesic of $\mathcal{T}$ stabilized by γ.

Define

$$L_j^+[n] := \left\{ v \in L_j \cap V \;\middle|\; Q(v) = Dp^{2n} \text{ and } \langle v, w_j^+\rangle \in \mathbb{Z}_p^\times \right\}$$

and define $L_j^-[n]$ in a similar way as above but replacing the symbol $+$ by the symbol $-$ everywhere. The motivation for the definition of $L_j^+[n]$ and $L_j^-[n]$ is the following. Let $\mathcal{T}_j^+[n]$ be the subset of $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$ of elements $x \in \mathcal{T}$ that are at distance equal to n from L_j and satisfy that:

1. If $x = L$ is a vertex and $\mathrm{Parent}(L) = L_k$, then $k \geq j$.

2. If x is an edge, for any of its endpoints L we have that if $\mathrm{Parent}(L) = L_k$, then $k \geq j$.

Define $\mathcal{T}_j^-[n]$ in a similar way but replacing the symbol $\geq$ by the symbol $\leq$ everywhere. It then follows from the description of the action of $\gamma^{\mathbb{Z}}$ in $\mathcal{T}$ that the disjoint union

$$\bigcup_{j=0}^{2t-1} \left\{ v \in V \;\middle|\; Q(v) = D, \mathrm{red}(\Delta(v)) \subset \mathcal{T}_j^+[n] \right\}$$

gives a set of representatives of (6.3). Similarly, the same holds if we replace the symbol $+$ by the symbol $-$.

Lemma 6.2.1. *The map*

$$L_j^+[n] \xrightarrow{\sim} \left\{ v \in V \;\middle|\; Q(v) = D, \mathrm{red}(\Delta(v)) \subset \mathcal{T}_j^+[n] \right\}, \; u \longmapsto u/p^n$$

is bijective. The same result holds if we replace the symbol $+$ by the symbol $-$ everywhere.

Proof. We start proving that the map is well-defined. Let $u \in L_j^+[n]$ and let $v = u/p^n$. Since u is primitive, L_v (resp. e_v) is at distance n from L_j if $\mathrm{ord}_p(D) = 0$ (resp. $\mathrm{ord}_p(D) = 1$). Moreover, the condition $\langle u, w_j^+ \rangle \in \mathbb{Z}_p^\times$ implies that if for any unimodular lattice L containing v we denote $\mathrm{Parent}(L) = L_k$, we have $k \geq j$. Hence, $\mathrm{red}(\Delta(v)) \subset \mathcal{T}_j^+[n]$.

The injectivity of the map is clear, so we are left to prove surjectivity. For that, let $v \in V$ be such that $Q(v) = D$ and $\mathrm{red}(\Delta(v)) \subset \mathcal{T}_j^+[n]$. Since there is a unimodular $\mathbb{Z}_p$-lattice in $V_{\mathbb{Q}_p}$ containing v at distance n from L_j, we have that $p^n v \in L_j$. Note that $\langle p^n v, w_j^+ \rangle \neq 0$. Indeed, for the sake of contradiction suppose that $\langle p^n v, w_j^+ \rangle = 0$. This implies that

$$p^n v = a w_j^+ + b e,$$

for $a, b \in \mathbb{Z}_p$. Then,

$$\gamma \cdot (p^n v) = a \varpi w_j^+ + b e.$$

Subtracting these two equations, we get that $\gamma v - v \in V$ is either 0 or it is an eigenvector for the $\mathbb{Q}$-linear action of γ on V of eigenvalue ϖ. Since $\gamma v - v \in V$ and $\varpi \notin \mathbb{Q}$, the only possibility is that $\gamma v - v = 0$. This implies that $v \in \langle e \rangle$, giving a contradiction with the fact that $\sqrt{-D} \notin \mathbb{Q}_p$. We can therefore choose $i \leq j$ such that $p^n v \in L_i^+[n]$. Now, the fact that the map is well-defined applied to the index i, together with the observation that the sets $\mathcal{T}_i^+[n]$ and $\mathcal{T}_j^+[n]$ are disjoint if $i \neq j$ proves that $i = j$ and we are done. $\square$

We can combine the information of Lemma 6.2.1 for $j = 0, \dots, 2t-1$ to obtain the following result.

Proposition 6.2.2. *Let $n \geq 0$, we have a bijection*

$$L_0^+[n] \cup \cdots \cup L_{2t-1}^+[n] \xrightarrow{\sim} \left\{ v \in \gamma^{\mathbb{Z}}\backslash V \;\middle|\; Q(v) = D, \; \mathrm{depth}(v) \leq n \right\}$$

given by $v \mapsto [p^{-n} v]$, where $[p^{-n} v]$ denotes the class of $p^{-n} v \in V$ modulo $\gamma^{\mathbb{Z}}$. Moreover, the same result is true if we replace the symbol $+$ by the symbol $-$.

Proof. By Lemma 6.2.1 we have that the map

$$L_0^+[n] \cup \cdots \cup L_{2t-1}^+[n] \xrightarrow{\sim} \bigcup_{j=0}^{2t-1} \left\{ v \in V \;\middle|\; Q(v) = D, \mathrm{red}(\Delta(v)) \subset \mathcal{T}_j^+[n] \right\}, \; u \longmapsto u/p^n$$

is bijective. We conclude the proof by recalling that the right hand side gives a set of representatives of

$$\left\{ v \in \gamma^{\mathbb{Z}} \backslash V \;\middle|\; Q(v) = D, \; \mathrm{depth}(v) \leq n \right\}.$$

□

As a consequence, we obtain the following expression for $j_{\mathscr{D}_\Phi(D)_e} \cdot j_{\mathscr{D}_\Phi(D)_o}(\gamma)$.

Theorem 6.2.3. *Consider the same notation as above.*

1. *If* $\mathrm{ord}_p(D) = 0$*, we have*

$$j_{\mathscr{D}_\Phi(D)}(\gamma) = \lim_{n \to +\infty} \prod_{j=0}^{2t-1} \frac{\prod_{v \in L_j^+[n]} \langle w_0^+, v \rangle^{\Phi(v)}}{\prod_{v \in L_j^-[n]} \langle w_0^-, v \rangle^{\Phi(v)}}.$$

2. *If* $\mathrm{ord}_p(D) = 1$*, we have*

$$j_{\mathscr{D}_\Phi(D)_e} \cdot j_{\mathscr{D}_\Phi(D)_o}(\gamma) = \lim_{n \to +\infty} \prod_{j=0}^{2t-1} \frac{\prod_{v \in L_j^+[n] \cup L_j^+[n+1]} \langle w_0^+, v \rangle^{\Phi(v)}}{\prod_{v \in L_j^-[n] \cup L_j^-[n+1]} \langle w_0^-, v \rangle^{\Phi(v)}}.$$

Proof. Suppose that $\mathrm{ord}_p(D) = 0$. By (6.1) and Proposition 6.2.2, we have

$$j_{\mathscr{D}_\Phi(D)}(\gamma) = \lim_{n \to +\infty} \prod_{j=0}^{2t-1} \frac{\prod_{v \in L_j^+[n]} \langle w_0^+, vp^{-n} \rangle^{\Phi(v)}}{\prod_{v \in L_j^-[n]} \langle w_0^-, vp^{-n} \rangle^{\Phi(v)}}.$$

Here we used that w_0^+ (resp. w_0^-) generates the line ξ^+ (resp. ξ^-) and that $\Phi(pv) = \Phi(v)$ for every $v \in V$. Since the divisor $\mathscr{D}_\Phi(D)$ is of degree 0, the product of the factors $p^{-n\Phi(v)}$ is equal to 1, leading to the desired expression. The case when $\mathrm{ord}_p(D) = 1$ is proven in an analogous way, but using (6.2), instead of (6.1). □

6.3 Computation of p-adic valuations

We end the section by using the previous calculations to compute the p-adic valuation of $j_{\mathscr{D}_\Phi(D)_e} \cdot j_{\mathscr{D}_\Phi(D)_o}(\gamma)$.

Proposition 6.3.1. *Let Φ be a convenient Schwartz–Bruhat function. Then*

$$\mathrm{ord}_p(j_{\mathscr{D}_\Phi(D)_e} \cdot j_{\mathscr{D}_\Phi(D)_o}(\gamma)) = 0$$

for all $\gamma \in \Gamma$. In particular, we have

$$G_\Phi^+(q) \in \mathrm{Hom}(\Gamma, \mathbb{Z}_p^\times)[[q]].$$

Proof. Using Theorem 6.2.3, together with the fact that $\mathrm{ord}_p(\langle w_0^+, v\rangle) = -j$ if $v \in L_j^+[n]$, and $\mathrm{ord}_p(\langle w_0^-, v\rangle) = j$ if $v \in L_j^-[n]$, we deduce that it is enough to show that, for every $j \in \{0, \dots, 2t-1\}$ and for every $n \geq 0$, we have

$$\sum_{v \in L_j^+[n]} \Phi(v) + \sum_{v \in L_j^+[n+1]} \Phi(v) = 0$$

and that the same statement replacing the symbol $+$ with $-$ everywhere holds (which is proven analogously). Note that the quantity on the left hand side can be interpreted as follows. Denote by $\mathcal{A}_j^+[n]$ the preimage of $\mathcal{T}_j^+[n]$ under the reduction map. Then,

$$\hat{\mathscr{D}}_\Phi(D) \cap \left(\mathcal{A}_j^+[n] \cup \mathcal{A}_j^+[n+1]\right) = \sum_{\substack{v \in V \\ Q(v)=D \\ \mathrm{red}(\Delta(v)) \subset \mathcal{T}_j^+[n]}} \Phi(v)\Delta(v) + \sum_{\substack{v \in V \\ Q(v)=D \\ \mathrm{red}(\Delta(v)) \subset \mathcal{T}_j^+[n+1]}} \Phi(v)\Delta(v),$$

and

$$\begin{aligned} \deg\left(\hat{\mathscr{D}}_\Phi(D) \cap \left(\mathcal{A}_j^+[n] \cup \mathcal{A}_j^+[n+1]\right)\right) &= \sum_{\substack{v \in V \\ Q(v)=D \\ \mathrm{red}(\Delta(v)) \subset \mathcal{T}_j^+[n]}} 2\Phi(v) + \sum_{\substack{v \in V \\ Q(v)=D \\ \mathrm{red}(\Delta(v)) \subset \mathcal{T}_j^+[n+1]}} 2\Phi(v) \\ &= \sum_{v \in L_j^+[n]} 2\Phi(v) + \sum_{v \in L_j^+[n+1]} 2\Phi(v), \end{aligned}$$

where in the last equality we used Lemma 6.2.1 together with the fact that Φ is invariant under multiplication by p. Recall that for a given lattice L we defined the wide open $\mathcal{W}_L \subset \mathscr{H}_p$ in Section 4.2. Since we have the disjoint union

$$\mathcal{A}_j^+[n] \cup \mathcal{A}_j^+[n+1] = \bigcup_{L \in \mathcal{T}_j^+[n+1] \cap \mathcal{T}_0} \mathcal{W}_L \cup \bigcup_{L \in \mathcal{T}_j^+[n] \cap \mathcal{T}_0} \mathcal{A}_L$$

and $\Delta_\Phi(D)$ is of strong degree 0, the result follows. □

Section 7

First order p-adic deformations of ternary theta series

Fix a convenient Schwartz–Bruhat function Φ. By Proposition 6.3.1 above we know that $G_\Phi^+(q)$ belongs to $\mathrm{Hom}(\Gamma_{\mathrm{ab}}, \mathbb{Z}_p^\times)[[q]]$. In order to prove modularity of $G_\Phi^+(q)$ it is therefore enough to prove that

$$\log_\gamma(G_\Phi^+)(q) := \sum_{D \in \mathcal{D}_S} \log_p(j_{\mathscr{D}_\Phi(D)_e} \cdot j_{\mathscr{D}_\Phi(D)_o}(\gamma))q^D \in \mathbb{Q}_p[[q]]$$

is a modular form for every $\gamma \in \Gamma$ hyperbolic at p, which we fix from now on. Here $\log_p$ denotes the branch of the p-adic logarithm such that $\log_p(p) = 0$. In this section, we use γ and Φ to construct a p-adic family of theta series Θ_k, of weight $k + 3/2$ and level $\Gamma_0(4N)$, satisfying the following two properties. First, $\Theta_0 = 0$. Second, if we denote by Θ_0' the derivative with respect to the p-adic variable k evaluated at $k = 0$, and e_{ord} the so-called p-ordinary projector, then $e_{\mathrm{ord}}(\Theta_0') \in S_{3/2}(\Gamma_0(4N), \mathbb{Q}_p)$. Furthermore, the generating series $\log_\gamma(G_\Phi^+)(q)$ is the projection to the $U_{p^2} = 1$ eigenspace of $2e_{\mathrm{ord}}\Theta_0'$. In particular, it is a cusp form of weight $3/2$ and level $\Gamma_0(4N)$, which proves Theorem 2.2.4.

7.1 Ordinary subspaces

Let k be a non-negative integer. For $\ell \nmid 2N$ denote by T_{ℓ^2} the associated Hecke operator acting on $S_{k+3/2}(\Gamma_0(4N), \mathbb{Z})$ and if $\ell \mid N$ denote by U_{ℓ^2} the associated Hecke operator acting on $S_{k+3/2}(\Gamma_0(4N), \mathbb{Z})$. We similarly consider the Hecke operators T_ℓ if $\ell \nmid N$, and U_ℓ if $\ell \mid N$ acting on $S_{2k+2}(\Gamma_0(2N), \mathbb{Z})$. The following key theorem relates these spaces of modular forms.

Theorem 7.1.1. *For $k \geq 0$, we have an isomorphism*

$$\mathscr{S} : S_{k+3/2}(\Gamma_0(4N), \mathbb{Q}) \xrightarrow{\sim} S_{2k+2}(\Gamma_0(2N), \mathbb{Q})$$

which is Hecke-equivariant, i.e.

$$\mathscr{S} \circ T_{\ell^2} = T_\ell \circ \mathscr{S} \quad \forall \ell \nmid 2N,$$
$$\mathscr{S} \circ U_{\ell^2} = U_\ell \circ \mathscr{S} \quad \forall \ell \mid N.$$

Proof. The result follows from the work of Niwa [Niw77, §1 and Remark 4], the fact that the $\mathbb{C}$-span of $S_{k+3/2}(\Gamma_0(4N), \mathbb{Q})$ is equal to the space of cusp forms of weight $k+3/2$ and level $\Gamma_0(4N)$ by the theorem of Serre and Stark in [SS77], together with the fact that Hecke operators preserve $S_{k+3/2}(\Gamma_0(4N), \mathbb{Q})$. Although the calculations in [Niw77] are done for $k \geq 1$, they are also valid for $k = 0$ as pointed out by Kohnen on page 59 of [Koh82]. □

Consider the space $\mathbb{Z}_p[[q]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ equipped with the norm

$$\left| \sum_{n \geq 0} a_n q^n \right| = \max_n \{|a_n|\}.$$

Since the eigenvalues of U_{p^2} acting on $S_{k+3/2}(\Gamma_0(4N), \mathbb{Q})$ are algebraic integers, the operator

$$e_{\mathrm{ord}} \colon S_{k+3/2}(\Gamma_0(4N), \mathbb{Z}_p) \longrightarrow S_{k+3/2}(\Gamma_0(4N), \mathbb{Z}_p), \;\; f \longmapsto \lim_{m \to +\infty} U_{p^2}^{m!}(f)$$

is well-defined. Denote by $S^{\mathrm{ord}}_{k+3/2}(\Gamma_0(4N), \mathbb{Z}_p)$ the image of this map, and similarly define $S^{\mathrm{ord}}_{k+3/2}(\Gamma_0(4N), \mathbb{Q}_p)$. We also consider the analogous definition for integral weight cusp forms and use similar notation.

Proposition 7.1.2. *The rank of the finitely generated modules $S^{\mathrm{ord}}_{k+3/2}(\Gamma_0(4N), \mathbb{Z}_p)$ is constant as long as k varies over non-negative integers such that $k \equiv 0 \bmod (p-1)/2$.*

Proof. It is enough to prove that $\dim_{\mathbb{Q}_p} S^{\mathrm{ord}}_{k+3/2}(\Gamma_0(4N), \mathbb{Q}_p)$ is constant as long as $k \in \mathbb{Z}_{\geq 0}$ and $k \equiv 0 \bmod (p-1)/2$. It follows from Theorem 7.1.1 that we have an isomorphism

$$S^{\mathrm{ord}}_{k+3/2}(\Gamma_0(4N), \mathbb{Q}_p) \xrightarrow{\sim} S^{\mathrm{ord}}_{2k+2}(\Gamma_0(2N), \mathbb{Q}_p).$$

But the dimensions of the right hand side are constant as long as $k \equiv 0 \bmod (p-1)$. (See the proof of Theorem 3 in Section 7.2 of [Hid93].) The result follows. □

7.2 Λ-adic forms of half-integral weight

We study the space of Λ-adic modular forms of half-integral weight and prove a classicality result in this setting. We follow [Hid93] and [Hid95].

Let $\Lambda := \mathbb{Z}_p[[T]]$ denote the Iwasawa algebra over $\mathbb{Z}_p$ and put $u = 1 + p \in 1 + p\mathbb{Z}_p$. A Λ-*adic cusp form of half-integral weight* is a formal power series

$$F = \sum_{n\geq 1} A_n q^n \in \Lambda[[q]]$$

such that there exists k_0 (dependent on F) satisfying that for all $k \geq k_0$ and $k \equiv 0 \bmod (p-1)$, the so-called weight k specialization

$$F_k := F(u^k - 1) := \sum_{n\geq 1} A_n(u^k - 1)q^n \in \mathbb{Z}_p[[q]],$$

belongs to $S_{k+3/2}(\Gamma_0(4N), \mathbb{Z}_p)$. We denote the space of such forms by $\mathbb{P}$. We define *ordinary* Λ-adic cusp forms of half-integral weight in the same way as above but replacing $S_{k+3/2}(\Gamma_0(4N), \mathbb{Z}_p)$ by $S^{\mathrm{ord}}_{k+3/2}(\Gamma_0(4N), \mathbb{Z}_p)$, and we denote this space by $\mathbb{P}^{\mathrm{ord}}$.

A key input to study the space $\mathbb{P}^{\mathrm{ord}}$ is the fact that $r^{\mathrm{ord}} = \mathrm{rank}_{\mathbb{Z}_p} S^{\mathrm{ord}}_{k+3/2}(\Gamma_0(4N), \mathbb{Z}_p)$ is constant as long as $k \geq 0$ and $k \equiv 0 \bmod (p-1)$, proven in Proposition 7.1.2.

Theorem 7.2.1. *$\mathbb{P}^{\mathrm{ord}}$ is free of finite rank over Λ. In particular,* $\mathrm{rank}_\Lambda(\mathbb{P}^{\mathrm{ord}}) \leq r^{\mathrm{ord}}$.

Proof. A proof of this statement can be found in Proposition 4 of [Hid95]. There, Hida considers different level structures than the ones considered here, but the same reasoning works in this case. □

For every $k \geq 0$, we can define a map

$$\varphi_k \colon \mathbb{P}^{\mathrm{ord}}/P_k\mathbb{P}^{\mathrm{ord}} \longrightarrow \mathbb{Z}_p[[q]], \;\; F \longmapsto F_k,$$

where $P_k = T - (u^k - 1) \in \mathbb{Z}_p[[T]]$, which is injective. The image of this map is a submodule of $\mathbb{Z}_p[[q]]$. We can also view $S^{\mathrm{ord}}_{k+3/2}(\Gamma_0(4N), \mathbb{Z}_p)$ as a submodule of $\mathbb{Z}_p[[q]]$. The relation between these two submodules is the so-called control theorem, which is again a consequence of Proposition 7.1.2.

Theorem 7.2.2. *Let $k \geq 0$ such that $k \equiv 0 \bmod (p-1)$. Then, the map φ_k induces an isomorphism*

$$\varphi_k \colon \mathbb{P}^{\mathrm{ord}}/P_k\mathbb{P}^{\mathrm{ord}} \xrightarrow{\sim} S^{\mathrm{ord}}_{k+3/2}(\Gamma_0(4N), \mathbb{Z}_p).$$

Proof. The analogous statement for ordinary cuspidal Λ-adic forms of integral weight is known. A proof can be found in Theorem 3, Section 7.3 of [Hid93]. The same proof given there works for the case of half-integral weight forms once we have Proposition 7.1.2.

Indeed, it can be proven that every element $f \in S^{\mathrm{ord}}_{k+3/2}(\Gamma_0(4N), \mathbb{Z}_p)$ is in the image φ_k as in the case of integral weight forms. For example, this is done in Proposition 5 of

[Hid95]. Since $\mathbb{P}^{\mathrm{ord}}$ is free of finite rank, this already implies the result for k large enough. To obtain the result for all k note that

$$S^{\mathrm{ord}}_{k+3/2}(\Gamma_0(4N), \mathbb{Z}_p) \subset \mathrm{Im}(\varphi_k) \subset \mathbb{Z}_p[[q]]. \tag{7.1}$$

This implies

$$r^{\mathrm{ord}} \leq \mathrm{rank}_{\mathbb{Z}_p}(\mathrm{Im}(\varphi_k)).$$

Since $\mathrm{rank}_{\mathbb{Z}_p}(\mathrm{Im}(\varphi_k)) \leq \mathrm{rank}_{\Lambda}(\mathbb{P}^{\mathrm{ord}})$, the previous inequality and Proposition 7.2.1 imply $r^{\mathrm{ord}} = \mathrm{rank}_{\mathbb{Z}_p}(\mathrm{Im}(\varphi_k))$. Hence, it follows from (7.1) that $\mathrm{Im}(\varphi_k) = S^{\mathrm{ord}}_{k+3/2}(\Gamma_0(4N), \mathbb{Z}_p)$ and we are done. □

Fix a Λ-basis $\{B_1, \dots, B_r\}$ of $\mathbb{P}^{\mathrm{ord}}$ and write

$$B_i = \sum_{n \geq 1} A_{i,n} q^n \in \Lambda[[q]].$$

By Theorem 7.2.2, the set $\{B_1(0), \dots, B_r(0)\}$ forms a $\mathbb{Z}_p$-basis of $S^{\mathrm{ord}}_{3/2}(\Gamma_0(4N), \mathbb{Z}_p)$. Thus, there exist $n_1, \dots, n_r$ such that

$$\det\left((A_{i,n_j}(0))_{1 \leq i,j \leq r}\right) \neq 0.$$

Since $\det((A_{i,n_j}(T))_{1 \leq i,j \leq r}) \in \Lambda$, it follows by continuity that there exists k_0 such that if $k \geq k_0$ and $k \equiv 0 \bmod (p-1)$, then

$$\det\left((A_{i,n_j}(u^k - 1))_{1 \leq i,j \leq r}\right) \neq 0. \tag{7.2}$$

Now define

$$\underline{b_i} = \sum_{n \geq 1} a_{i,n} q^n,$$

where $a_{i,n} \colon \mathbb{Z}_p \to \mathbb{Z}_p$ is the analytic function determined by $a_{i,n}(k) = A_{i,n}(u^k - 1)$ for every $k \geq 0$ such that $k \equiv 0 \bmod (p-1)$.

We will now prove that certain first order derivatives of Λ-adic modular forms of half-integral weight are modular forms themselves. Let F be a Λ-adic modular form of half-integral weight such that $F_0 = 0$. Let

$$F' := \frac{d}{dk} F_k|_{k=0} = \lim_{k \to 0} \frac{F_k}{k} \in \mathbb{Z}_p[[q]]$$

be the first derivative of F with respect to k evaluated at $k = 0$. It is a weight $3/2$ analogue of a p-adic modular form in the sense of Serre. Here the limit is taken in $\mathbb{Z}_p[[q]] \otimes \mathbb{Q}_p$ with respect to the norm introduced above. Recall that U_{p^2} has the following expression at the level of q-expansions:

$$\sum_{n \geq 0} a_n q^n \longmapsto \sum_{n \geq 1} a_{np^2} q^n.$$

Since $|U_p^2 f| \leq |f|$ for any $f \in \mathbb{Z}_p[[q]] \otimes \mathbb{Q}_p$ and U_{p^2} is linear, it follows that we can define the p-adic modular form of weight $3/2$

$$e_{\mathrm{ord}}(F') := \lim_{k\to 0} e_{\mathrm{ord}}\left(\frac{F_k}{k}\right).$$

Moreover, it is a calculation to verify that the limit

$$\lim_{m\to+\infty} U_{p^2}^{m!}(F')$$

exists in $\mathbb{Z}_p[[q]] \otimes \mathbb{Q}_p$ and is equal to $e_{\mathrm{ord}}(F')$.

Corollary 7.2.3. *For every $F \in \mathbb{P}$ with $F_0 = 0$ the p-adic modular form $e_{\mathrm{ord}}(F_0')$ is classical. More precisely, it belongs to $S_{3/2}^{\mathrm{ord}}(\Gamma_0(4N), \mathbb{Q}_p)$.*

Proof. By definition,

$$e_{\mathrm{ord}}(F') = \lim_{k\to 0} e_{\mathrm{ord}}\left(\frac{F_k}{k}\right).$$

Now, Theorem 7.2.2 implies that, for every $k > 0$ and $k \equiv 0 \bmod (p-1)$, we can write

$$e_{\mathrm{ord}}\left(\frac{F_k}{k}\right) = \sum_{i=1}^{r} x_i(k)\underline{b_i}(k),$$

where $x_i(k) \in \mathbb{Q}_p$ for every i. Let $n_1, \ldots, n_r$ be as above, and note that $(x_i(k))_i$ is the solution of the linear system of equations

$$\left(a_{i,n_j}(k)\right)_{j,i} (x_i(k))_i = \left(a_{n_j}\left(\frac{F_k}{k}\right)\right)_j.$$

Moreover, since the determinant of the matrix defining this system is an analytic function, which is non-zero if $k \geq k_0$ and $k \equiv 0 \bmod (p-1)$ by (7.2) and the discussion above it, we deduce that for every i the limit $\lim_{k\to 0} x_i(k)$ exists in $\mathbb{Q}_p$. Denote it by $x_i(0)$. Then,

$$e_{\mathrm{ord}}(F') = \lim_{k\to+\infty} e_{\mathrm{ord}}\left(\frac{F}{k}\right) = \sum_{i=1}^{r} x_i(0)\underline{b_i}(0)$$

and it follows from Theorem 7.2.2 in the particular case that $k = 0$ that the right hand side belongs to $S_{k+3/2}^{\mathrm{ord}}(\Gamma_0(4N), \mathbb{Q}_p)$, which concludes the proof. □

7.3 p-adic families of theta series

Recall that the element $\gamma \in \Gamma$ determines a collection of $\mathbb{Z}_p$-lattices L_j of depth zero, and let w_j^+, w_j^- and e be as in Section 5.1, so that $L_j = \langle w_j^+, w_j^-, e\rangle$. Note that w_j^+, w_j^- can be viewed both as elements of $V_{\mathbb{Q}(\gamma)}$ and $V_{\mathbb{Q}_p}$, using the embedding $\mathbb{Q}(\gamma) \hookrightarrow \mathbb{Q}_p$ satisfying that $\mathrm{ord}_p(\varpi) = 2t > 0$. These data, together with Φ, can be used to define the following Schwartz–Bruhat functions

$$\Phi_j^+ = \Phi \otimes 1_{\left\{v \in L_j \ \mid \ \langle v, w_j^+ \rangle \in \mathbb{Z}_p^\times\right\}}, \text{ and } \Phi_j^- = \Phi \otimes 1_{\left\{v \in L_j \ \mid \ \langle v, w_j^- \rangle \in \mathbb{Z}_p^\times\right\}}$$

on $V_{\mathbb{A}^\infty}$ for every $j \in \{0, \ldots, 2t-1\}$. We have that Φ is invariant under $K_0(4N/p)^{(p)}$ by assumption. Moreover,

$$1_{\left\{v \in L_j \ \mid \ \langle v, w_j^+ \rangle \in \mathbb{Z}_p^\times\right\}} = 1_{L_j} - 1_{L_j \cap L_{j-1}}, \ 1_{\left\{v \in L_j \ \mid \ \langle v, w_j^- \rangle \in \mathbb{Z}_p^\times\right\}} = 1_{L_j} - 1_{L_j \cap L_{j+1}} \tag{7.3}$$

and L_j is unimodular, while $L_j \cap L_{j-1}$ has level p for every j. It follows that $\Phi_j^\pm$ is invariant under $K_0(4N)$ for every j. Since w_j^+ and w_j^- are isotropic, the functions $v \mapsto \langle w_j^\pm, v\rangle^k$ are harmonic polynomials on $V_{\mathbb{Q}(\gamma)}$ for all integers $k \geq 0$. Hence, the q-series

$$\Theta_k := \sum_{j=0}^{2t-1} \sum_{v \in V} \Phi_j^+(v) \langle w_j^+, v\rangle^k q^{Q(v)} - \sum_{j=0}^{2t-1} \sum_{v \in V} \Phi_j^-(v) \langle w_j^-, v\rangle^k q^{Q(v)} \tag{7.4}$$

is a linear combination of classical theta-series with coefficients in the quadratic imaginary field $\mathbb{Q}(\gamma)$ of weight $k+3/2$ and level $\Gamma_0(4N)$ by [Bor98, Theorem 4.1]. Via the embedding $\mathbb{Q}(\gamma) \hookrightarrow \mathbb{Q}_p$, we can also view the Fourier coefficients of Θ_k as elements in $\mathbb{Z}_p$. Moreover, since the non-zero terms in the infinite sum defining Θ_k solely involve elements of V for which $\langle w_j^+, v\rangle$ (resp. $\langle w_j^-, v\rangle$) are p-adic units, it follows that the Fourier coefficients of Θ_k vary analytically as functions of the variable $k \in (\mathbb{Z}/(p-1)\mathbb{Z}) \times \mathbb{Z}_p$. We can therefore define Θ_k for $k \in (\mathbb{Z}/(p-1)\mathbb{Z}) \times \mathbb{Z}_p$. It gives a prototypical instance of a Λ-adic modular form of half-integral weight, in the sense that there exists a $F \in \mathbb{P}$ such that $F_k = \Theta_k$ for every $k \equiv 0 \bmod (p-1)$.

Lemma 7.3.1. *The weight $3/2$ specialization Θ_0 is identically zero.*

Proof. By (7.3), we have

$$\begin{aligned}
\Theta_0 &= \sum_{j=0}^{2t-1} \sum_{v \in V} \Phi(v) \left(1_{L_j} - 1_{L_j \cap L_{j-1}}\right)(v) q^{Q(v)} - \sum_{j=0}^{2t-1} \sum_{v \in V} \Phi(v) \left(1_{L_j} - 1_{L_j \cap L_{j+1}}\right)(v) q^{Q(v)} \\
&= -\sum_{v \in V} \Phi(v) 1_{L_0 \cap L_{-1}}(v) q^{Q(v)} + \sum_{v \in V} \Phi(v) 1_{L_{2t-1} \cap L_{2t}}(v) q^{Q(v)} = 0,
\end{aligned}$$

where in the last equality we used that $\gamma(L_0 \cap L_{-1}) = L_{2t} \cap L_{2t-1}$ and that the functions Φ and $v \mapsto Q(v)$ are invariant under the action of $\gamma^{\mathbb{Z}}$. □

Lemma 7.3.1 together with Corollary 7.2.3 immediately imply the following:

Corollary 7.3.2. *The p-adic modular form $e_{\mathrm{ord}}(\Theta_0')$ is classical. More precisely, it belongs to $S_{3/2}^{\mathrm{ord}}(\Gamma_0(4N), \mathbb{Q}_p)$.*

We now relate $e_{\mathrm{ord}}(\Theta_0')$ with the generating series $\log_\gamma(G_\Phi^+)(q)$.

Lemma 7.3.3. *For every $D \in \mathcal{D}_S$ and every $n \geq 0$ the following equality holds:*

$$a_{Dp^{2n}}\left(\log_\gamma(G_\Phi^+)(q)\right) = a_{Dp^{2n}}\left(e_{\mathrm{ord}}\left((1+U_{p^2})\Theta_0'\right)\right).$$

Note that, since $a_{Dp^{2n}}(\log_\gamma(G_\Phi^+)(q)) = a_D(\log_\gamma(G_\Phi^+)(q))$ for all $n \geq 0$, the right hand side does not depend on n.

Proof. It is enough to prove the formula when $\mathrm{ord}_p(D) \in \{0,1\}$. Using (7.4), we can compute

$$a_{Dp^{2m}}(\Theta_0') = \sum_{j=0}^{2t-1} \sum_{v \in L_j^+[m]} \Phi(v)\log_p(\langle w_j^+, v\rangle) - \sum_{j=0}^{2t-1} \sum_{v \in L_j^-[m]} \Phi(v)\log_p(\langle w_j^-, v\rangle).$$

Hence, it follows from Theorem 6.2.3 that

$$a_D\left(\log_\gamma(G_\Phi^+)(q))\right) = \lim_{m\to+\infty} a_{Dp^{2m}}(\Theta_0') + a_{Dp^{2(m+1)}}(\Theta_0'). \tag{7.5}$$

and that the limit on the right hand side exists. Indeed, the case when $\mathrm{ord}_p(D) = 1$ follows directly from the second part of Theorem 6.2.3. The case $\mathrm{ord}_p(D) = 0$ can be deduced by noting that the first part of Theorem 6.2.3 implies $\log_p(j_{\mathscr{D}_\Phi(D)}(\gamma)) = \lim_{m\to+\infty} a_{Dp^{2m}}(\Theta_0')$ (and therefore also $\log_p(j_{\mathscr{D}_\Phi(D)}(\gamma)) = \lim_{m\to+\infty} a_{Dp^{2(m+1)}}(\Theta_0')$) and taking the sum of these two equalities.

On the other hand, from the expression of the ordinary projection given above, we have that for every $n \geq 0$

$$a_{Dp^{2n}}(e_{\mathrm{ord}}(\Theta_0' + U_{p^2}(\Theta_0'))) = \lim_{m\to+\infty} a_{Dp^{2(n+m!)}}(\Theta_0') + a_{Dp^{2(n+m!+1)}}(\Theta_0').$$

Since the right hand side of the previous equation is a subsequence of the right hand side of (7.5), we deduce that

$$a_D\left(\log_\gamma(G_\Phi^+)(q))\right) = a_{Dp^{2n}}(e_{\mathrm{ord}}(\Theta_0' + U_{p^2}(\Theta_0'))),$$

which proves the desired equality. □

The action of U_{p^2} on $S_{3/2}^{\mathrm{ord}}(\Gamma_0(4N), \overline{\mathbb{Q}}_p)$ diagonalizes. This can be justified, for example, using Theorem 7.1.1 and the fact that the analogous statement for weight 2 forms of level $\Gamma_0(2N)$ is well-known. In particular, we can consider

$$\mathrm{pr}_1 \colon S_{3/2}^{\mathrm{ord}}(\Gamma_0(4N), \mathbb{Q}_p) \longrightarrow S_{3/2}^{\mathrm{ord}}(\Gamma_0(4N), \mathbb{Q}_p)$$

to be the projection to the $U_{p^2} = 1$ eigenspace. Its image consists of p-newforms. Again, this follows from Theorem 7.1.1 and the corresponding statement for weight 2 forms, which is a consequence of the Weil conjectures for abelian varieties. We prove the main identity of this work, which implies Theorem 2.2.4.

Theorem 7.3.4. *The identity*

$$\log_\gamma(G_\Phi^+)(q) = 2\mathrm{pr}_1(e_{\mathrm{ord}}(\Theta_0'))$$

of formal power series holds. In particular, $\log_\gamma(G_\Phi^+)(q)$ *is an element of* $S_{3/2}^{\mathrm{ord}}(\Gamma_0(4N), \mathbb{Q}_p)$.

Proof. Since U_{p^2} and e_{ord} commute, it is enough to prove that if

$$f = e_{\mathrm{ord}}(\Theta_0' + U_{p^2}\Theta_0') = \sum_{n\geq 1} a_n(f)q^n,$$

we have $\log_\gamma(G_\Phi^+)(q) = \mathrm{pr}_1(f)$. Note that by Theorem 7.3.3, we have that if $D \in \mathcal{D}_S$,

$$a_{Dp^{2n}}(\log_\gamma(G_\Phi^+)(q)) = a_{Dp^{2n}}(f).$$

for every $n \geq 0$. Therefore, the equality of the theorem follows from proving that, if $D \in \mathbb{Z}_{\geq 0}$ is such that $\mathrm{ord}_p(D) \in \{0, 1\}$ and $N \geq 0$, then:

1. If $\left(\frac{-D}{p}\right) = 1$, $a_{Dp^{2n}}(\mathrm{pr}_1(f)) = 0$.
2. If $\left(\frac{-D}{p}\right) \in \{0, -1\}$, $a_{Dp^{2n}}(\mathrm{pr}_1(f)) = a_{Dp^{2n}}(f)$.

We start by proving the first point. Since $\mathrm{pr}_1(f)$ is a p-newform, Theorem 1 of [MRV90] implies that the Atkin–Lehner involution at p acts by multiplication with -1 on $\mathrm{pr}_1(f)$. Then, (1) follows from the description of the eigenspaces of the Atkin–Lehner involution given in [MRV90], Remark 2. (See also [Koh82], Proposition 4.) We proceed to prove the second point. Write f as a sum of eigenvectors for U_{p^2}, namely

$$f = \sum_{i=1}^{r} f_i,$$

where $f_i \in S_{3/2}^{\mathrm{ord}}(\Gamma_0(4N), L)$ and there exists α_i such that $U_{p^2}f_i = \alpha_i f_i$ for every i. Here L is a finite extension of $\mathbb{Q}_p$ containing all the elements α_i. We can suppose without loss of generality that $\alpha_i \neq \alpha_j$ if $i \neq j$ and that $\alpha_1 = 1$. In particular, $f_1 = \mathrm{pr}_1(f)$ (which is possibly zero). Let D be such that it satisfies the conditions of (2). For every $n \geq 0$, we can consider the Dp^{2n}-th Fourier coefficient of each side of the previous equality to obtain

$$a_D(f) = a_D(f_1) + \sum_{i=2}^{r} \alpha_i^n a_D(f_i),$$

where we used that $a_D(f) = a_{Dp^{2n}}(f)$ for every $n \geq 0$, which holds by Theorem 7.3.3. Considering this equality for $n = 0, \dots, r-1$ and using that the Vandermonde matrix associated to $\{1, \alpha_2, \dots, \alpha_r\}$ is non-singular we deduce that we must have $a_D(f) = a_D(f_1)$, implying the desired equality. Once we have $\log_\gamma(G_\Phi^+)(q) = 2\mathrm{pr}_1(e_{\mathrm{ord}}(\Theta_0'))$, the fact that $\log_\gamma(G_\Phi^+)(q) \in S_{3/2}(\Gamma_0(4N), \mathbb{Q}_p)$ follows from Corollary 7.3.2. $\square$

Section 8

Numerical example

We conclude by presenting a concrete example where we numerically compute the p-adic family Θ_k and the reduction modulo p of $e_{\text{ord}}(\Theta_0'/p)$. In future work, we aim to present a complete numerical computation of $e_{\text{ord}}(\Theta_0'/p)$. However, in this section, we focus on its reduction modulo p, as these computations are technically simpler and already illustrate interesting phenomena.

Let $S = \{7, 13, \infty\}$, let $p = 7$ and consider B be the quaternion algebra over $\mathbb{Q}$ ramified exactly at $\{13, \infty\}$. It can be viewed as the algebra over $\mathbb{Q}$ generated by i, j, k where

$$i^2 = -2,\ j^2 = -13,\ ij = -ji = k.$$

Let $\tilde{R}$ be the maximal $\mathbb{Z}[1/p]$-order of B given by $\langle 1/2 + j/2 + k/2, i/4 + j/2 + k/4, j, k\rangle$, let $\alpha = 1 + i \in B^\times$, which has reduced norm $\ell = 3 \notin S$, and consider the Eichler $\mathbb{Z}[1/p]$-order $R = \tilde{R} \cap \alpha\tilde{R}\alpha^{-1}$ of level 3. Denote by Γ the group of norm one units in R modulo $\{\pm 1\}$. The quotient $\Gamma\backslash\mathscr{H}_p$ is isomorphic to the $\mathbb{C}_p$-points of the Shimura curve X.

8.1 Construction of the p-adic family Θ_k

Recall the definition of the p-adic family Θ_k given in (7.4). This family depends on a choice of a Schwartz–Bruhat function Φ, an element $\gamma \in \Gamma$ hyperbolic at p and the eigenvectors of the action of γ on $V_{\mathbb{Q}_p}$. We proceed to fix these data. Let $\tilde{R}_0$ be the subgroup of elements of $\tilde{R}$ of reduced norm zero. As before, write $1_{\tilde{R}_0}$ for the characteristic function of $\tilde{R}_0 \otimes \hat{\mathbb{Z}}^{(p)}$. Consider the $\hat{R}^\times \times K_0(4 \cdot 13)^{(p)}$-invariant Schwartz–Bruhat function

$$\Phi = 1_{\tilde{R}_0} - 1_{\tilde{R}_0} \cdot \alpha^{-1}.$$

Since we have the factorization of ideals $(7) = (7, x+3)(7, x+4)$ in the ring of integers of $\mathbb{Q}[x]/(x^2+5)$, the element $(x+3)/(-x+3) = 3x/7 + 2/7$ is a p-unit in $\mathbb{Q}[x]/(x^2+5)$.

Its image in B with respect to the embedding

$$\mathbb{Q}[x]/(x^2+5) \hookrightarrow B,\ x \longmapsto \frac{i}{4}+\frac{j}{2}+\frac{k}{4},$$

is equal to

$$\gamma = \frac{2}{7}+\frac{3i}{28}+\frac{3j}{14}+\frac{3k}{28},$$

and it can be verified that $\gamma \in \Gamma$. Let $\mathfrak{p}$ be the prime ideal spanned by 7 and $x+4$ and fix the embedding

$$\mathbb{Q}[x]/(x^2+5) \hookrightarrow \mathbb{Q}_p \tag{8.1}$$

such that $\mathrm{ord}_p((\mathfrak{p})) = 1$. Using that γ is hyperbolic at p, we deduce that its action on $V \otimes \mathbb{Q}[x]/(x^2+5)$ (and therefore on $V_{\mathbb{Q}_p}$) diagonalizes. The eigenvectors of γ are

$$w^+ = i + \left(\frac{4x}{39}-\frac{2}{39}\right)j + \left(-\frac{4x}{39}-\frac{1}{39}\right)k$$

$$e = i+2j+k$$

$$w^- = i + \left(-\frac{4x}{39}-\frac{2}{39}\right)j + \left(\frac{4x}{39}-\frac{1}{39}\right)k$$

with eigenvalues $\varpi = -12x/49 - 41/49$, 1 and ϖ^{-1} respectively. Since $v_{\mathfrak{p}}(\varpi) = 2$, we have that $t = 1$. Note that $\langle w^+, w^-\rangle \in \mathbb{Z}_p^\times$, which implies that $\{w^+, w^-\}$ generate a hyperbolic plane. Finally, consider the unimodular $\mathbb{Z}_p$-lattices

$$L_0 = \langle w^+, e, w^-\rangle = \langle i, j, k\rangle$$

$$L_1 = \langle pw^+, e, w^-/p\rangle = \left\langle i+2j+k, 14i - \frac{28j}{39} - \frac{14k}{39}, \frac{i}{7}+j+\frac{8k}{7}\right\rangle.$$

We can therefore consider the p-adic family Θ_k given in (7.4) attached to the data Φ, γ and $\{w^+, e, w^-\}$.

8.2 Calculation of Θ_0 and $e_{\mathrm{ord}}(\Theta_0')$

Consider the same notation as above. For every $M \le 421 \cdot p^2$ we can run over the following sets:

$$\left\{v \in V \;\middle|\; \langle v,v\rangle = M, 1_{\tilde{R}_0}(v)\cdot 1_{L_0}(v) = 1, \langle v, w^+\rangle \text{ or } \langle v, w^-\rangle \in \mathbb{Z}_p^\times\right\},$$

$$\left\{v \in V \;\middle|\; \langle v,v\rangle = M, 1_{\tilde{R}_0}(v)\cdot 1_{L_1}(v) = 1, \langle v, pw^+\rangle \text{ or } \langle v, p^{-1}w^-\rangle \in \mathbb{Z}_p^\times\right\},$$

$$\left\{v \in V \;\middle|\; \langle v,v\rangle = M, 1_{\alpha\cdot\tilde{R}_0}(v)\cdot 1_{L_0}(v) = 1, \langle v, w^+\rangle \text{ or } \langle v, w^-\rangle \in \mathbb{Z}_p^\times\right\},$$

$$\left\{v \in V \;\middle|\; \langle v,v\rangle = M, 1_{\alpha\cdot\tilde{R}_0}(v)\cdot 1_{L_1}(v) = 1, \langle v, pw^+\rangle \text{ or } \langle v, p^{-1}w^-\rangle \in \mathbb{Z}_p^\times\right\}.$$

From there, it is possible to compute the first $421 \cdot p^2$ Fourier coefficients of Θ_k, for $k \in \mathbb{Z}$, as well as of Θ_0'. The necessity to calculate exactly this number of Fourier coefficients comes from the Sturm bound, which is used in the proof of Proposition 8.2.1 below. In particular, define

$$\Theta^+_{\tilde{R}_0,j} := \sum_{\substack{v \in V \\ \langle v, p^{+j} w^+ \rangle \in \mathbb{Z}_p^\times}} 1_{\tilde{R}_0}(v) \cdot 1_{L_j}(v) q^{Q(v)}$$

and define $\Theta^-_{\tilde{R}_0,j}$ with the same expression but replacing the symbol $+$ by the symbol $-$ everywhere. Define also $\Theta^+_{\alpha\cdot\tilde{R}_0,j}$ and $\Theta^-_{\alpha\cdot\tilde{R}_0,j}$ analogously. Then,

$$\Theta_0 = \left(\Theta^+_{\tilde{R}_0,L_0} + \Theta^+_{\tilde{R}_0,L_1} - \Theta^-_{\tilde{R}_0,L_0} - \Theta^-_{\tilde{R}_0,L_1}\right) - \left(\Theta^+_{\alpha\cdot\tilde{R}_0,L_0} + \Theta^+_{\alpha\cdot\tilde{R}_0,L_1} - \Theta^-_{\alpha\cdot\tilde{R}_0,L_0} - \Theta^-_{\alpha\cdot\tilde{R}_0,L_1}\right)$$

and we verify that the first $421 \cdot p^2$ Fourier coefficients are 0. For example, the first 4 terms that appear in the previous expression are given below.

Theta series	q-expansion																
	2	5	6	7	8	11	13	15	18	19	20	21	24	26	28	31	32
$\Theta^+_{\tilde{R}_0,L_0}$	2	2	4	4	6	8	2	8	6	6	8	8	8	6	6	10	14
$\Theta^-_{\tilde{R}_0,L_0}$	2	0	4	2	6	8	0	8	6	4	8	8	12	6	6	8	14
$\Theta^+_{\tilde{R}_0,L_1}$	2	0	4	2	6	8	0	8	6	4	8	8	12	6	6	8	14
$\Theta^-_{\tilde{R}_0,L_1}$	2	2	4	4	6	8	2	8	6	6	8	8	8	6	6	10	14

Table 8.1 First Fourier coefficients of the theta series $\Theta^\pm_{\tilde{R}_0,L_j}$.

The coefficients of q^n for $n < 32$ that do not appear in the table are 0, as theta series attached to lattices in V have non-zero Fourier coefficients only if $-D$ is not a square modulo 13. We will follow a similar convention from now on. The forms on the previous table belong to $M_{3/2}(\Gamma_0(4 \cdot 91), \mathbb{Q})$, a space of dimension 32, and these coefficients fully determine them.

From (7.4), we see that the derivative of Θ_k with respect to k evaluated at $k = 0$ is equal to

$$\Theta_0' = \sum_{j=0}^{2t-1} \sum_{v \in V} \Phi_j^+(v) \log_p \langle p^j w^+, v \rangle q^{Q(v)} - \sum_{j=0}^{2t-1} \sum_{v \in V} \Phi_j^-(v) \log_p \langle p^{-j} w^-, v \rangle q^{Q(v)}. \qquad (8.2)$$

Note that the dot products $\langle v, w^\pm \rangle$ belong to $\mathbb{Q}[x]/(x^2+5)$ and have $\mathfrak{p}$-adic valuation 0. Using the embedding (8.1), we can view them as elements in $\mathbb{Z}_p^\times$. Therefore, the p-adic logarithm of these numbers lies in $p\mathbb{Z}_p$. We can then consider Θ_0'/p as an element in $\mathbb{Z}_p[[q]]$ and study its reduction modulo p.

Similarly as above, we can calculate the first $421 \cdot p^2$ Fourier coefficients of Θ'_0/p modulo p. The first ones are

$$\frac{\Theta'_0}{p} = 2q^2 + 3q^5 + 2q^6 + 4q^7 + 5q^8 + 4q^{11} + 3q^{13} + 3q^{15} + 2q^{18} + 3q^{20} + 6q^{21} + \cdots .$$

Since it is possible to calculate the first $421 \cdot p^2$ Fourier coefficients of Θ'_0/p mod p, we obtain the first 421 Fourier coefficients of $U_{p^2}(\Theta'_0/p)$ modulo p. The first ones are

$$\begin{aligned} U_{p^2}(\Theta'_0/p) = {} & 3q^2 + 3q^5 + 5q^6 + 2q^7 + 3q^{11} + 6q^{13} + 3q^{15} + 5q^{18} + q^{19} + 2q^{20} + 2q^{21} \\ & + 2q^{24} + 3q^{26} + q^{28} + 6q^{31} + q^{32} + q^{33} + 4q^{34} + q^{37} + 3q^{39} + q^{44} + \cdots . \end{aligned}$$

The following proposition, which is verified experimentally using the calculations mentioned above and Magma, is key for the next calculations.

Proposition 8.2.1. *There exists a cusp form in $S_{3/2}(\Gamma_0(4\cdot 91), \mathbb{Z})$ whose reduction modulo p is equal to $U_{p^2}(\Theta'_0/p)$ mod p.*

Proof. Since $\Theta_0 = 0$, we deduce from the expressions of Θ'_0 in (8.2) and of Θ_k in (7.4) that

$$\frac{\Theta'_0}{p} \equiv \frac{\Theta_{p-1}}{p(p-1)} \bmod p = 7.$$

In particular, $U_{p^2}(\Theta'_0/p)$ is the reduction mod p of an element $g_1 \in S_{3/2+6}(\Gamma_0(4 \cdot 91), \mathbb{Z})$. We can then verify experimentally using Magma that the first 421 Fourier coefficients of g_1 are congruent modulo p to the first 421 Fourier coefficients of a modular form $g_2 \in S_{3/2}(\Gamma_0(4 \cdot 91), \mathbb{Z})$.

We claim that this implies $g_1 \equiv g_2 \bmod p$. Indeed, let $\tilde{g}_2 \in S_{3/2+6}(\Gamma_0(4 \cdot 91), \mathbb{Z})$ be such that $g_2 \equiv \tilde{g}_2 \bmod p$. Then, the modular form $g_1 - \tilde{g}_2 \in S_{3/2+6}(\Gamma_0(4 \cdot 91), \mathbb{Z})$ has the first 421 Fourier coefficients equal to 0 modulo p. This implies that the first $4 \cdot 421$ Fourier coefficients of $(g_1 - \tilde{g}_2)^4 \in S_{30}(\Gamma_0(4 \cdot 91), \mathbb{Z})$ are congruent to 0 modulo p. Since

$$421 \cdot 4 > \frac{30 \cdot [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4 \cdot 91)]}{12} = 1680 = 420 \cdot 4,$$

it follows from the Sturm bound ([Stu87, Theorem 1]) that $g_1 - \tilde{g}_2 \equiv 0 \bmod p$, implying the desired result. □

Using a basis of $S_{3/2}(\Gamma_0(4 \cdot 91), \mathbb{Z})$ given by Magma, and using Proposition 8.2.1, we can then compute $U^2_{p^2}(\Theta'_0/p)$ and verify the following:

1. $\frac{1}{2}(U_{p^2} + U^2_{p^2})(\Theta'_0/p)$ mod p is an eigenvector for U_{p^2} of eigenvalue 1.

2. $\frac{1}{2}(U_{p^2} - U^2_{p^2})(\Theta'_0/p)$ mod p is an eigenvector for U_{p^2} of eigenvalue -1.

It follows from there that, modulo p,

$$e_{\mathrm{ord}}\left(\frac{\Theta_0'}{p}\right) = \lim_{n\to+\infty} U_{p^2}^{n!}\frac{\Theta_0'}{p} = \lim_{n\to+\infty} U_{p^2}^{n!-1}U_{p^2}\left(\frac{\Theta_0'}{p}\right)$$
$$= \lim_{n\to+\infty} U_{p^2}^{n!-1}\left(\frac{1}{2}(U_{p^2}+U_{p^2}^2)(\Theta_0'/p) + \frac{1}{2}(U_{p^2}-U_{p^2}^2)(\Theta_0'/p)\right)$$
$$= \frac{1}{2}(U_{p^2}+U_{p^2}^2)(\Theta_0'/p) - \frac{1}{2}(U_{p^2}-U_{p^2}^2)(\Theta_0'/p) = U_{p^2}^2\left(\Theta_0'/p\right).$$

Based on this decomposition of $e_{\mathrm{ord}}\left(\Theta_0'/p\right)$, we will write

$$\mathrm{pr}_1(e_{\mathrm{ord}}(\Theta_0'/p)) = (U_{p^2}(\Theta_0'/p) + U_{p^2}^2(\Theta_0'/p))/2,$$
$$\mathrm{pr}_{-1}(e_{\mathrm{ord}}(\Theta_0'/p)) = -(U_{p^2}(\Theta_0'/p) - U_{p^2}^2(\Theta_0'/p))/2.$$

The results of the calculation are summarized in the following table.

Modular form mod $p=7$	q-expansion														
	2	5	6	7	8	11	13	15	18	19	20	21	24	26	28
Θ_0'/p	2	3	2	4	5	4	3	3	2	0	3	6	0	3	4
$U_{p^2}(\Theta_0'/p)$	3	3	5	2	0	3	6	3	5	1	2	2	2	3	1
$U_{p^2}^2(\Theta_0'/p)$	3	4	2	4	0	3	1	3	5	6	5	6	5	4	4
$(U_{p^2}+U_{p^2}^2)(\Theta_0'/p)/2$	3	0	0	3	0	3	0	3	5	0	0	4	0	0	6
$(U_{p^2}-U_{p^2}^2)(\Theta_0'/p)/2$	0	3	5	6	0	0	1	0	0	1	2	5	2	3	2
$e_{\mathrm{ord}}\Theta_0'/p$	3	4	2	4	0	3	1	3	5	6	5	6	5	4	4

Table 8.2 Dth Fourier coefficients of linear combinations of $U_{p^2}^n(\Theta_0'/p)$ for D such that $\left(\frac{-D}{13}\right) \neq 1$. For every D, we consider the color code blue: $\left(\frac{-D}{p}\right) = -1$, grey: $\left(\frac{-D}{p}\right) = 1$, red: $\left(\frac{-D}{p}\right) = 0$.

Remark 8.2.2. In the decomposition $e_{\mathrm{ord}}(\Theta_0'/p) = \mathrm{pr}_1(e_{\mathrm{ord}}(\Theta_0'/p)) + \mathrm{pr}_{-1}(e_{\mathrm{ord}}(\Theta_0'/p))$ both summands are non-zero. The first summand is related to the Gross–Kohnen–Zagier generating series, as proved in Theorem 7.3.4. It would be interesting to find an arithmetic interpretation of the second summand, namely $\mathrm{pr}_{-1}(e_{\mathrm{ord}}(\Theta_0'/p))$.

8.3 Shimura lift and Hecke equivariance

The space $S_2^{\mathrm{new}}(\Gamma_0(7\cdot 13))$ has dimension 7, and there is a unique (up to scalars) cuspidal form such that U_7 acts by 1 and has odd analytic rank (so in particular, the Hecke operator U_{13} acts also by 1). Its Fourier expansion is given by

$$f = q - 2q^3 - 2q^4 - 3q^5 + q^7 + q^9 + 4q^{12} + q^{13} + 6q^{15} + 4q^{16} - 6q^{17} - 7q^{19} + \dots$$
$$\equiv q + 5q^3 + 5q^4 + 4q^5 + q^7 + q^9 + 4q^{12} + q^{13} + 6q^{15} + 4q^{16} + q^{17} \dots \bmod p = 7.$$

Recall the Shimura lift

$$\mathscr{S}_D := \mathscr{S}_{D,0,91} \colon S_{3/2}(\Gamma_0(4\cdot 91), \mathbb{Q}) \longrightarrow S_2(\Gamma_0(2\cdot 91), \mathbb{Q})$$

defined in Section 7.1, where D is a square-free integer such that $-D > 0$. We computed the Shimura lift of $e_{\mathrm{ord}}(\Theta_0'/p)$ for different values of D and obtained the following identities modulo p

$$\begin{aligned} \mathscr{S}_{-2}\left(\mathrm{pr}_1(e_{\mathrm{ord}}(\Theta_0'/p))\right) &\equiv 6f \bmod p, \\ \mathscr{S}_{-11}\left(\mathrm{pr}_1(e_{\mathrm{ord}}(\Theta_0'/p))\right) &\equiv 3f + U_2 f \bmod p, \\ \mathscr{S}_{-15}\left(\mathrm{pr}_1(e_{\mathrm{ord}}(\Theta_0'/p))\right) &\equiv 3f + 6(U_2 f) \bmod p. \end{aligned} \tag{8.3}$$

In particular, we see that $\mathrm{pr}_1 e_{\mathrm{ord}}(\Theta_0'/p)$ is a Hecke eigenvector (mod p) with Hecke eigenvalues congruent to those of f.

The Schwartz–Bruhat function Φ is convenient. Indeed, since Φ is the difference of characteristic functions of the trace zero elements of two maximal orders, we deduce that $\Delta_\Phi(D)$ is of degree 0 for every D. Moreover, $\deg_{\mathcal{T}_0}(\Phi)$ lands in the subspace of $\mathrm{Funct}(\Gamma\backslash\mathcal{T}_0, \mathbb{Z})$ corresponding to weight two cusp forms of level $\Gamma_0(13\cdot 3)$ that are old at 3. Since $S_2(\Gamma_0(13), \mathbb{Q}) = 0$, we deduce $\deg_{\mathcal{T}_0}(\Phi) = 0$ implying the desired claim by the proof of Lemma 4.3.2. Applying Theorem 7.3.4 to the Schwartz–Bruhat function Φ one obtains the equality

$$\log_\gamma(G_\Phi^+)(q) = 2\mathrm{pr}_1(e_{\mathrm{ord}}(\Theta_0')).$$

In particular, $(1/p)\log_p(G_\Phi^+(\gamma)) \bmod p$ is a Hecke cuspidal eigenform of weight $3/2$ with the same Hecke eigenvalues as the cusp form f of weight 2 and level $\Gamma_0(91)$.

On the other hand, consider $G_\Phi(q) \in J(\mathbb{Q}_{p^2})_{\mathbb{Q}}[[q]]$ and observe:

- The classes $[\Delta_\Phi(D)]$ are invariant under the action of $R^\times$ for every $D \in \mathcal{D}_S$.
- The projection of the class $[\Delta_\Phi(D)]$ to a Hecke eigenspace is non-zero only if the eigenspace corresponds to an eigenform of rank 1 by the Gross–Zagier formula.
- Via Jacquet–Langlands the Hecke action of $\mathbb{T}^N$ on $J(\mathbb{Q}_{p^2})$ factors through the action on $S_2^{91-\mathrm{new}}(\Gamma_0(91\cdot 3), \mathbb{Q}_{p^2})$.
- Since the divisors $\Delta_\Phi(D)$ on X are obtained via pullback from divisors of a Shimura curve $\tilde{X}$ that is p-adically uniformized by $\tilde{\Gamma}\backslash\mathscr{H}_p$, with $\tilde{\Gamma}$ the norm 1 units of $\tilde{R}$, it follows that the classes $[\Delta_\Phi(D)]$ belong to the subspace corresponding to forms that are old at 3.

Hence, the functionals $\varphi\colon J(\mathbb{Q}_{p^2}) \longrightarrow \mathbb{Q}_{p^2}$ such that $\varphi(G_\Phi(q))$ is non-zero are generated by projections to eigenspaces where $\mathbb{T}^N$ acts with the same eigenvalues as it acts on eigenforms on $S_2^{\text{new}}(\Gamma_0(91))$ which have rank 1 and $U_7 = 1$. As we discussed above, there is a unique (up to scaling) such eigenform in $S_2^{\text{new}}(\Gamma_0(91))$, which is f. Uniqueness implies that $G_\Phi(q) \in J(\mathbb{Q}_{p^2})_{\mathbb{Q}}[[q]]$ is a non-zero multiple of $\log_\gamma(G_\Phi^+)(q)$, which has the same Hecke eigenvalues of f modulo p. Hence, the calculation we presented gives an example (modulo p) of the Hecke equivariance property of the geometric theta lift provided by the Gross–Kohnen–Zagier generating series.

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Interlude

In Chapter I, we studied Heegner points of a Shimura curve X via its p-adic uniformization $\Gamma\backslash\mathscr{H}_p$. From there, we deduced that certain generating series of Heegner classes are modular forms of weight $3/2$. A key tool for the proof was the study of the p-adic Abel–Jacobi map, introduced in Section 4,

$$\mathrm{AJ}\colon \mathrm{Div}^0(\Gamma\backslash\mathscr{H}_p) \longrightarrow H^1(\Gamma, \mathbb{C}_p^\times)/\Lambda,$$

where $\Lambda = j(\Gamma)$ is a $\mathbb{Z}$-lattice of rank equal to the rank of Γ_{ab}. The map was described as an infinite product involving Weil symbols, and it also admits an interpretation via integration, analogous to the classical Abel–Jacobi map.

Recall that $H^1(\Gamma, \mathbb{C}_p^\times)/\Lambda$ can be identified with the Jacobian of $\Gamma\backslash\mathscr{H}_p$. Suppose E is an elliptic curve appearing as a quotient of this Jacobian, with $E(\mathbb{C}_p) \simeq \mathbb{C}_p^\times/q^{\mathbb{Z}}$, where $q \in p\mathbb{Z}_p$ is the Tate period of E. Following Chapter 5 of [Dar04] and Section 2.3 of [Das04], we will use integration to describe the composition

$$\Phi\colon \mathrm{Div}^0(\Gamma\backslash\mathscr{H}_p) \xrightarrow{\mathrm{AJ}} H^1(\Gamma, \mathbb{C}_p^\times)/\Lambda \twoheadrightarrow \mathbb{C}_p^\times/q^{\mathbb{Z}} \xrightarrow{\log_q} \mathbb{C}_p,$$

where $\log_q\colon \mathbb{C}_p^\times \to \mathbb{C}_p$ is the branch of the p-adic logarithm satisfying $\log_q(q) = 0$. Let f be the weight 2 eigenform on $\mathscr{H}_p$ of level Γ corresponding to E. Being a rigid analytic function on $\mathscr{H}_p$, f can be encoded in a Γ-invariant measure of total mass zero on $\mathbb{P}^1(\mathbb{Q}_p)$, the boundary of $\mathscr{H}_p$. Denote such measure by $\mu_f \in \mathbb{D}_0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Z}_p)$; it determines f via the following formula

$$f(z) = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{1}{z-x} d\mu_f.$$

Then, the map Φ is given by

$$\Phi\colon (\tau_1) - (\tau_2) \in \mathrm{Div}^0(\Gamma\backslash\mathscr{H}_p) \longmapsto \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log_q\left(\frac{\tau_1 - x}{\tau_2 - x}\right) d\mu_f.$$

Observe that the definition is similar to the classical Abel–Jacobi map via the following formal equalities:

$$\int_{\tau_1}^{\tau_2} f(z)dz = \int_{\tau_1}^{\tau_2}\int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{1}{z-x} d\mu_f dz = \int_{\mathbb{P}^1(\mathbb{Q}_p)}\int_{\tau_1}^{\tau_2} \frac{1}{z-x} dz d\mu_f = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log_q\left(\frac{\tau_1 - x}{\tau_2 - x}\right) d\mu_f.$$

In particular, if $\tau_1, \tau_2 \in \mathscr{H}_p$ are CM points, the expression

$$\int_{\mathbb{P}^1(\mathbb{Q}_p)} \log_q\left(\frac{\tau_1 - x}{\tau_2 - x}\right) d\mu_f \tag{8.4}$$

provides an explicit formula for the logarithm of a Heegner point in $E(\mathbb{C}_p)$.

There have been generalizations of the expression (8.4) to different settings. Their importance lies in the fact that they provide conjectural formulas for arithmetic objects that seem evasive via archimedean techniques. For example, in [Dar01], Darmon replaced the role of the rigid eigenform f on the Shimura curve X by a weight 2 eigenform for $\Gamma_0(p)$. This led to the definition of the so-called *Stark–Hegner points* on the elliptic curve E corresponding to f. These are local points on E conjecturally defined over abelian extensions of real quadratic fields. Later, Darmon and Dasgupta followed a similar strategy but replaced the role of the cusp form with the Eisenstein series for $\Gamma_0(p)$ in [DD06]. Their recipe yields elements in abelian extensions of real quadratic fields, more precisely *Gross–Stark units*. An informal comparison of the ingredients involved in each construction is given below.

Heegner points	Stark–Heegner points	Gross–Stark units
$B^\times$	$\mathrm{SL}_2(\mathbb{Q})$	$\mathrm{SL}_2(\mathbb{Q})$
Γ	$\Gamma^p = \mathrm{SL}_2(\mathbb{Z}[1/p])$	$\Gamma^p = \mathrm{SL}_2(\mathbb{Z}[1/p])$
f eigenform for Γ	f eigenform for $\Gamma_0(p)$	E Eis. series for $\Gamma_0(p)$
$\mu_f \in \mathbb{D}_0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Z}_p)^\Gamma$	$\mu_f \in H^1(\Gamma^p, \mathbb{D}_0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Z}_p))$	$\mu_E \in H^1(\Gamma^p, \mathbb{D}_0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Z}_p))$
Im. quad. $\tau \in \mathscr{H}_p$	Real quad. $\tau \in \mathscr{H}_p$	Real quad. $\tau \in \mathscr{H}_p$

Table i Ingredients for the p-adic constructions of Heegner points, Stark–Heegner points, and Gross-Stark units.

In Chapter II, we follow these ideas to conjecture an explicit expression for p-adic logarithms Gross–Stark units in abelian extensions of totally real fields of degree n where p is inert, partially generalizing the third column of Table i. The expressions have a similar flavor to (8.4)

$$\int_{\mathbb{X}} \log_p(\tau^t \cdot x) d\mu_{\mathrm{Eis}}(c_{U_F}). \tag{8.5}$$

An informal dictionary between the two explicit formulas is summarized below.

- Γ is replaced by $\mathrm{SL}_n(\mathbb{Z})$.
- $\mathbb{P}^1(\mathbb{Q}_p)$ is replaced by $\mathbb{X} := \mathbb{Z}_p^n - p\mathbb{Z}_p^n$.
- $\mu_f \in H^0(\Gamma, \mathbb{D}_0(\mathbb{P}_1(\mathbb{Q}_p), \mathbb{Z}_p))$ is replaced by $\mu_{\mathrm{Eis}} \in H^{n-1}(\mathrm{SL}_n(\mathbb{Z}), \mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p))$.

- The CM points τ_i in $\mathscr{H}_p$ are replaced by points in Drinfeld's symmetric domain $\mathscr{X}_p \subset \mathbb{P}^{n-1}(\mathbb{C}_p)$ represented by a vector $\tau \in \mathbb{C}_p^n$ whose components generate a fractional ideal of a totally real field of degree n where p is inert.

- c_{U_F} is a generator of $H_{n-1}(U_F, \mathbb{Z}) \simeq \mathbb{Z}$, where $U_F \subset \mathrm{SL}_n(\mathbb{Z})$ is the stabilizer of the vector τ in $\mathbb{P}^{n-1}(\mathbb{C}_p)$.

Already in the construction of Stark–Heegner points and Gross–Stark units for real quadratic fields, it is useful to lift measures from $\mathbb{P}^1(\mathbb{Q}_p)$ to $\mathbb{Z}_p^2 - p\mathbb{Z}_p^2$ to obtain finer invariants. The constructions presented in Chapter II are related to these lifts. On the other hand, there is room to explore the complete analogy between the objects appearing in Table i and the conjectural formula (8.5).

Finally, it is worth mentioning that the cases considered in Table i can be seen as particular instances of a general setting. Darmon, Gehrmann, and Lipnowski recently developed in [DGL23] the theory of rigid classes for orthogonal groups. Then, these formulas for Heegner points, Stark–Heegner points, and Gross–Stark units (for real quadratic fields) should appear as the values of rigid cocycles attached to orthogonal groups via the accidental isomorphisms between $B^\times$ and orthogonal groups of signature $(3, 0)$, and $\mathrm{SL}_2(\mathbb{Q})$ and an orthogonal group of signature $(2, 1)$. Similarly, in Chapter II we will interpret our conjectural formulas for Gross–Stark units for totally real fields as values of rigid classes for $\mathrm{SL}_n(\mathbb{Q})$.

Chapter II

The Eisenstein class of a torus bundle and a log-rigid class for $\mathrm{SL}_n(\mathbb{Z})$

Section 9

Introduction

The values of modular units at CM points, called elliptic units, have rich arithmetic significance. Notably, they generate abelian extensions of imaginary quadratic fields. In [DD06], Darmon and Dasgupta proposed a conjectural construction of elliptic units for real quadratic fields and predicted that they behave similarly to elliptic units. Their construction consists of a p-adic limiting process involving periods of logarithmic derivatives of modular units along real quadratic geodesics.

Using different methods, Dasgupta extended this construction to the case of totally real fields in [Das08] and together with Kakde proved that the recipe gives p-units in abelian extensions of totally real fields [DK23]. More precisely, they proved that their resulting objects are Gross–Stark units. Remarkably, their work provides a solution to Hilbert's twelfth problem for totally real fields via p-adic methods.

Darmon, Pozzi, and Vonk constructed analogs of modular functions, called rigid classes, which can be evaluated at real quadratic points, and expressed the original construction of [DD06] as the value of a rigid class in [DPV24]. Their work provides a modular approach to the construction of Gross–Stark units and leads to a new proof of the conjecture of [DD06] in the real quadratic setting.

In Chapter II, we construct a log-rigid analytic class for $\mathrm{SL}_n(\mathbb{Z})$ and study its values at points attached to totally real fields where p is inert. This represents a first step toward a modular construction of Gross–Stark units for totally real fields, which would generalize the results of [DD06] and [DPV24]. A key ingredient in our construction is the Eisenstein class of a torus bundle of Bergeron, Charollois and García [BCG20], that replaces the role of modular units. We provide partial evidence for our constructions by relating the local traces of the values we construct with local traces of logarithms of Gross–Stark units.

9.1 Siegel units and abelian extensions of quadratic fields

We begin by explaining the construction of Siegel units and its relation with the classical theory of complex multiplication for imaginary quadratic fields. Let E be an elliptic curve defined over a scheme S, fix a positive integer c coprime to 6, and denote by $\mathbb{N}^{(c)}$ the set of positive integers coprime to c. We have the following proposition.

Proposition 9.1.1. *Let $E[c]$ be the kernel of multiplication by c on E, and denote by $\mathcal{O}(E - E[c])^\times$ the space of meromorphic functions on E that are regular and non-vanishing outside $E[c]$. There exists a unique function ${}_c\theta_E \in \mathcal{O}(E - E[c])^\times$ satisfying:*

1. *The divisor of ${}_c\theta_E$ is $E[c] - c^2(0)$.*
2. *${}_c\theta_E$ is invariant under pushforward induced by multiplication by a for all $a \in \mathbb{N}^{(c)}$.*

Let $N \geq 3$ be a positive integer coprime to c, denote by $\Gamma(N) \subset \mathrm{SL}_2(\mathbb{Z})$ the congruence subgroup of full level N, and let $\mathscr{H}$ be the complex upper half-plane. We can then consider the universal elliptic curve

$$E := \Gamma(N)\backslash \left((\mathscr{H} \times \mathbb{C})/\mathbb{Z}^2\right) \longrightarrow Y(N) := \Gamma(N)\backslash\mathscr{H}.$$

The proposition above yields the function ${}_c\theta_E \in \mathcal{O}(E - E[c])^\times$, which can be used to construct modular units on $\Gamma(N)\backslash\mathscr{H}$ in the following way. A vector $v \in \mathbb{Q}^2/\mathbb{Z}^2 - \{0\}$ of order N induces a torsion section $v\colon Y(N) \to E - E[c]$. Then, the pullback ${}_cg_v := v^*({}_c\theta_E) \in \mathcal{O}(Y(N))^\times$ is called a *Siegel unit* and is an instance of a modular unit. It gives rise to a $\Gamma(N)$-invariant function on $\mathscr{H}$, that we will denote by the same symbol. The theory of complex multiplication implies that the values of Siegel units at special points have deep significance.

Theorem 9.1.2. *Let $\tau \in \mathscr{H}$ be a CM point attached to a quadratic imaginary field K, i.e. τ is stabilized by a subgroup of norm one elements $K^1 \subset \mathrm{SL}_2(\mathbb{Q})$ of K. Then,*

$${}_cg_v(\tau) \in K^{\mathrm{ab}} \subset \bar{\mathbb{Q}}.$$

An important question in algebraic number theory is to find an analog of this theorem for general number fields. The case of real quadratic fields has been extensively studied via different methods. We are particularly interested in the p-adic approach initiated by Darmon and Dasgupta in [DD06] and followed, among others, by Darmon, Pozzi, and Vonk in [DPV24]. We proceed to outline these works in a language suited to this chapter.

Let F be a real quadratic field and p a rational prime. Observe that $\mathscr{H}$ does not contain real quadratic points, i.e. there are no points stabilized by a torus of norm one elements $F^1 \subset \mathrm{SL}_2(\mathbb{Q})$ of F. On the other hand, $\mathscr{H}$ has geodesics stabilized by these norm one tori. Moreover, if $(z, \gamma z) \subset \mathscr{H}$ is a segment of such geodesic, where $\gamma \in F^1 \cap \Gamma(p^r)$, and $v \in \mathbb{Q}^2/\mathbb{Z}^2 - \{0\}$ is of exact order p^r, we have the so-called Meyer's Theorem

$$\frac{1}{2\pi i}\int_z^{\gamma z} \mathrm{dlog}({}_c g_v) = \zeta_c(F, [\mathfrak{b}], 0) \in \mathbb{Z}. \tag{9.1}$$

Here $\zeta_c(F, [\mathfrak{b}], 0)$ denotes the value at $s = 0$ of a c-smoothed partial zeta function attached to F and an ideal class $[\mathfrak{b}]$ in a narrow class group of conductor divisible p^r, determined by the inclusion $F^1 \subset \mathrm{SL}_2(\mathbb{Q})$ and v. In addition to encoding information about abelian extensions of totally real fields, these zeta values possess notable p-adic properties and serve for the construction of measures that yield p-adic partial zeta functions of F.

The search for a symmetric space containing real quadratic points, combined with the p-adic properties of the partial zeta values considered above, leads to replacing $\mathscr{H}$ by a p-adic symmetric space to generalize Theorem 9.1.2. More precisely, if we let $\mathscr{H}_p := \mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(\mathbb{Q}_p)$ be the p-adic upper half-plane and $\mathcal{A}$ its ring of rigid analytic functions, we have:

- $\mathscr{H}_p$ contains points stabilized by $F^1 \subset \mathrm{SL}_2(\mathbb{Q})$ if and only if p is nonsplit in F.
- There is a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism between $\mathcal{A}^\times/\mathbb{C}_p^\times$ and the space of $\mathbb{Z}$-valued measures on $\mathbb{P}^1(\mathbb{Q}_p)$ of total mass zero (see [vdP82]), suggesting that $\mathcal{A}^\times$ encodes information about p-adic zeta functions and refinements of their values.

In [DPV24], Darmon, Pozzi, and Vonk exploit the distribution relation of Siegel units attached to vectors of arbitrary p-power order to construct a cohomology class

$$\mathcal{J}_{\mathrm{DR}} \in H^1(\mathrm{SL}_2(\mathbb{Z}), \mathcal{A}^\times).$$

This class can be viewed as a generalization of a modular function. Indeed, the space of invariant functions $H^0(\mathrm{SL}_2(\mathbb{Z}), \mathcal{A}^\times) = \mathbb{C}_p^\times$ is too simple, which suggests studying the first cohomology group instead. Moreover, if $\tau \in \mathscr{H}_p$ is stabilized by $F^1 \subset \mathrm{SL}_2(\mathbb{Q})$ and $F^1 \cap \mathrm{SL}_2(\mathbb{Z}) = \langle \pm\gamma_\tau \rangle$, they define the value $\mathcal{J}_{\mathrm{DR}}[\tau] := \mathcal{J}_{\mathrm{DR}}(\gamma_\tau)(\tau) \in \mathbb{C}_p^\times$.

Theorem 9.1.3 (Darmon–Pozzi–Vonk). *Let $\tau \in \mathscr{H}_p$ be as above with stabilizer $\langle \pm\gamma_\tau \rangle \subset \mathrm{SL}_2(\mathbb{Z})$ be attached to a real quadratic field F where p is inert. Then,*

$$\log_p(\mathcal{J}_{\mathrm{DR}}[\tau]) = \log_p(u_\tau), \;\; u_\tau = \textit{Gross–Stark unit} \in F^{\mathrm{ab}} \subset \bar{\mathbb{Q}}.$$

This theorem provides a level 1 version of Theorem 9.1.2 for real quadratic fields where p is inert. Indeed, it produces nontrivial elements in abelian extensions of real quadratic fields as values of $\mathcal{J}_{\mathrm{DR}}$ at special points in $\mathscr{H}_p$.

Remark 9.1.4. The class $\mathcal{J}_{\mathrm{DR}}$ is the unique lift via the quotient map $\mathcal{A}^\times \to \mathcal{A}^\times/\mathbb{C}_p^\times$ of the restriction to $\mathrm{SL}_2(\mathbb{Z})$ of a class $J_{\mathrm{DR}} \in H^1(\mathrm{SL}_2(\mathbb{Z}[1/p]), \mathcal{A}^\times/\mathbb{C}_p^\times)$, also constructed in [DPV24]. Rigorously, it is this second class that should be regarded as a modular object, as the Hecke module $H^1(\mathrm{SL}_2(\mathbb{Z}[1/p]), \mathcal{A}^\times/\mathbb{C}_p^\times)_{\mathbb{Q}}$ is isomorphic to the sum of $H^1(\Gamma_0(p), \mathbb{Q})$ and an Eisenstein line. On the other hand, the lift $\mathcal{J}_{\mathrm{DR}}$ is important to define the values of J_{DR} and it is the object we aim to generalize in Chapter II. Here and for the rest of the chapter, the subindex $\mathbb{Q}$ denotes tensor product with $\mathbb{Q}$ over $\mathbb{Z}$.

9.2 Construction of the log-rigid class for $\mathrm{SL}_n(\mathbb{Z})$

The work of Bergeron, Charollois, and García in [BCG20] provides a generalization of logarithmic derivatives of Siegel units which is relevant for the study of totally real fields of degree n: the *Eisenstein class of a torus bundle*. Let $E \to X$ be an oriented real vector bundle of rank n over an oriented manifold X. Suppose that E contains a sub-bundle $E_{\mathbb{Z}}$ with fibers isomorphic to $\mathbb{Z}^n$. We can then construct the torus bundle $T := E/E_{\mathbb{Z}} \to X$. Consider the following class in singular cohomology with $\mathbb{Z}$-coefficients

$$T[c] - c^n\{0\} \in H^0(T[c]) \simeq H^n(T, T - T[c]),$$

where the isomorphism above is the Thom isomorphism. The long exact sequence in relative cohomology provides a map $H^{n-1}(T - T[c]) \to H^n(T, T - T[c])$. The Eisenstein class ${}_c z_T$ attached to T and c is constructed from the next theorem and is analogous to the functions ${}_c\theta_E$ determined in Proposition 9.1.1.

Theorem 9.2.1 (Sullivan, Bergeron–Charollois–García)**.** *There exists a unique class ${}_c z_T \in H^{n-1}(T - T[c], \mathbb{Z}[1/c])$ satisfying:*

1. *${}_c z_T$ is a lift of $T[c] - c^n\{0\} \in H^n(T, T - T[c], \mathbb{Z}[1/c])$.*
2. *${}_c z_T$ is invariant under pushforward induced by multiplication by a for all $a \in \mathbb{N}^{(c)}$.*

Let $\mathscr{X} := \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n$ be the symmetric space attached to $\mathrm{SL}_n(\mathbb{R})$, let $v_r \in \mathbb{Q}^n/\mathbb{Z}^n$ be the column vector $(1/p^r, 0, \dots, 0)^t$ and let Γ_r be its stabilizer in $\Gamma := \mathrm{SL}_n(\mathbb{Z})$. Finally, fix q an auxiliary integer such that the full level congruence subgroup $\Gamma(q)$ is torsion-free and $[\Gamma : \Gamma(q)]$ is prime to p, which imposes that p is sufficiently large. Then, $\Gamma_r(q) := \Gamma_r \cap \Gamma(q)$ is torsion-free. We can apply the previous theorem to the universal family of tori

$$T_r := \Gamma_r(q)\backslash(\mathscr{X} \times \mathbb{R}^n/\mathbb{Z}^n) \longrightarrow \Gamma_r(q)\backslash\mathscr{X}$$

and obtain the Eisenstein class ${}_c z_{T_r}$, that we will simply denote by z_r. The vector v_r induces a torsion section $v_r \colon \Gamma_r(q)\backslash\mathscr{X} \to T_r - T_r[c]$ and we can consider the pullback $v_r^* z_r$, which defines a Γ_r-invariant cohomology class on $\Gamma_r(q)\backslash\mathscr{X}$. This class is a higher-dimensional analog of $\mathrm{dlog}_c g_v$.

The pullbacks of Eisenstein classes by arbitrary p^r-torsion sections satisfy distribution relations parallel to those of Siegel units. In particular, $(v_r^* z_r)_r$ are compatible with respect to pushforward by the projection maps. Using these properties and Shapiro's lemma, we package the pullbacks of the Eisenstein classes by p-power torsion sections in a group cohomology class for Γ

$$\mu_0 \in H^{n-1}(\Gamma, \mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p))^{w=-1},$$

where $\mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p)$ is the space of $\mathbb{Z}_p$-valued measures on $\mathbb{X} := \mathbb{Z}_p^n - p\mathbb{Z}_p^n$ of total mass zero, and w denotes the involution given by the action of $\mathrm{GL}_n(\mathbb{Z})/\mathrm{SL}_n(\mathbb{Z})$.

There are constructions of similar cohomology classes in the literature under the name of *Eisenstein cocycles*. Notably, the work of Sczech [Scz93] together with its integral refinement by Charollois and Dasgupta [CD14, Theorem 4], and the works of Charollois, Dasgupta, Greenberg, and Spiess (see [CDG15] and [DS18]) using Shintani's method give explicit formulas for Eisenstein cocycles. These works yield cocycles for S-arithmetic groups, on the other hand, they take values in measures on $\mathbb{X}$ together with some additional data, such as a set of linear forms in n-variables (used for Q-summation), or the set of rays in $\mathbb{R}^n$ not generated by a vector in $\mathbb{Q}^n$. Even closer to our setting, Beilinson, Kings and Levin [BKL18] and Galanakis and Spiess [GS24] construct similar classes using the polylogarithm sheaf.

The class μ_0 valued in $\mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p)$ is suitable for the construction of rigid classes on Drinfeld's p-adic symmetric domain via a Poisson kernel. Let $\mathscr{X}_p := \mathbb{P}^{n-1}(\mathbb{C}_p) - \bigcup_{H\in\mathcal{H}} H$ be Drinfeld's p-adic symmetric domain, where $\mathcal{H}$ denotes the set of all $\mathbb{Q}_p$-rational hyperplanes. Denote by $\mathcal{A}_{\mathcal{L}}$ the space of log-rigid analytic functions. Informally, $\mathcal{A}_{\mathcal{L}}$ consists of the $\mathbb{C}_p$-valued functions on $\mathscr{X}_p$ such that its restriction to any affinoid is of the form

$$(\text{rigid analytic function}) + \sum_{H,H'\in H} c_{H,H'} \log_p\left(\ell_H(z)/\ell_{H'}(z)\right),$$

where $c_{H,H'} \in \mathbb{Q}_p$ are all but finitely many 0, $\ell_H(z)$ denotes the equation of the hyperplane $H \in \mathcal{H}$, and $\log_p \colon \mathbb{C}_p^\times \to \mathbb{C}_p$ is the p-adic logarithm satisfying $\log_p(p) = 0$. Integration over $\mathbb{X}$ leads to a Γ-equivariant lift

$$\mathrm{ST}\colon \mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p) \longrightarrow \mathcal{A}_{\mathcal{L}},\ \lambda \longmapsto \left(z \longmapsto \int_{\mathbb{X}} \log_p(z^t \cdot x) d\lambda\right). \tag{9.2}$$

This lift is a particular instance of the theory of p-adic Poisson kernels, which relate measures on the set of hyperplanes of $\mathbb{Q}_p^n$, or on $\mathbb{Z}_p^\times$-bundles over them, to functions on $\mathscr{X}_p$. We refer the reader to the works of Schneider–Teitelbaum [ST97], van der Put [vdP82], and Gekeler [Gek20] for more examples of this phenomenon and its generalizations.

Finally, we define our desired log-rigid analytic class as

$$J_{\mathrm{Eis}} := \mathrm{ST}(\mu_0) \in H^{n-1}(\Gamma, \mathcal{A}_{\mathcal{L}}).$$

The construction of the class J_{Eis} can be compared to that in [DPV24] when $n = 2$, leading to the relation $J_{\mathrm{Eis}} = \log_p(\mathcal{J}_{\mathrm{DR}})$. In particular, this shows that the class $\log_p(\mathcal{J}_{\mathrm{DR}})$ can be constructed solely from logarithmic derivatives of Siegel units, rather than from the full Siegel units.

Let F be a totally real field of degree n where p is inert, and let $\tau \in F^n$ be such that its coordinates give an oriented $\mathbb{Z}$-basis of $\mathfrak{a}^{-1}$, for $\mathfrak{a}$ an ideal of $\mathcal{O}_F$. Since p is inert, it follows that $\tau \in \mathscr{X}_p$. Moreover, τ is a special point in $\mathscr{X}_p$ in the sense that its stabilizer in $\mathrm{SL}_n(\mathbb{Q})$ is isomorphic to the norm 1 elements of F. In particular, its stabilizer in Γ is a group of rank $n-1$. Following a similar recipe to the case $n = 2$, we define the evaluation of $J \in H^{n-1}(\Gamma, \mathcal{A}_{\mathcal{L}})$ at $\tau \in \mathscr{X}_p$, giving $J[\tau] \in \mathbb{C}_p$. From our construction, one readily deduces $J_{\mathrm{Eis}}[\tau] \in F_p$ and the theorem below gives partial evidence regarding the arithmetic significance of this value.

Theorem 9.2.2. *For every $n \geq 2$, $\mathrm{Tr}_{F_p/\mathbb{Q}_p} J_{\mathrm{Eis}}[\tau] = \mathrm{Tr}_{F_p/\mathbb{Q}_p} \log_p(u_\tau)$, where $u_\tau \in \mathcal{O}_H[1/p]^\times \otimes \mathbb{Q}$ is a Gross–Stark unit in the narrow Hilbert class field H of F.*

The proof of this result uses that the integral of $v_r^* z_r$ along the $(n-1)$-dimensional submanifold of $\Gamma_r(q)\backslash\mathscr{X}$ determined by the inclusion $F^1 \subset \mathrm{SL}_n(\mathbb{Q})$ is a special value of a partial zeta function of F, generalizing (9.1). From there, we construct the p-adic partial zeta function of F attached to $\mathfrak{a}$ from μ_0 and express $\mathrm{Tr}_{F_p/\mathbb{Q}_p} J_{\mathrm{Eis}}[\tau]$ as its derivative at $s = 0$. Thus, Theorem 9.2.2 follows from the Gross–Stark conjecture in rank 1, proved in [DDP11] and [Ven15]. We point out that using the groundbreaking work of Dasgupta, Kakde, Silliman, and Wang [DKSW23], it should be possible to deduce that u_τ belongs to $\mathcal{O}_H[1/p]^\times$.

The previous theorem, together with Theorem 9.1.3 involving real quadratic fields suggests the following conjecture.

Conjecture 9.2.3. *We have $J_{\mathrm{Eis}}[\tau] = \log_p(u_\tau)$, where $u_\tau \in \mathcal{O}_H[1/p]^\times \otimes \mathbb{Q}$ is as above.*

Answering this conjecture affirmatively would provide a new approach to constructing Gross–Stark units and would constitute a first step towards their modular construction via the theory of rigid classes. The conjecture above can also be viewed as a refinement of the Gross–Stark conjecture, in which the quantity $J_{\mathrm{Eis}}[\tau]$ replaces the derivative of the p-adic L-function and yields a formula for the p-adic logarithm of u_τ rather than for its trace. We refer the reader to Sections 2 and 3 of [Das08] for another explicit construction of Gross–Stark units, as well as for the interpretation that these expressions refine the Gross–Stark conjecture in the rank 1 setting.

Remark 9.2.4. When n is odd, Gross–Stark units in the narrow Hilbert class field of F are trivial. It seems that to construct meaningful invariants in this setting, we would need a higher-level version of J_{Eis}, generalizing [Cha09] to totally real fields. For such construction, we would expect that the corresponding invariants belong to abelian extensions of totally real fields of larger conductor.

9.3 Structure of Chapter II

The organization of the chapter is as follows. Section 10 defines the Eisenstein class of a torus bundle and proves a distribution relation involving the pullbacks of this class by torsion sections. Section 11 explains the work of [BCG20] and introduces an explicit differential form representing the Eisenstein class for a universal family of tori over locally symmetric spaces attached to $\mathrm{SL}_n(\mathbb{R})$. We conclude by proving that the sum of the pullbacks of this form along the torsion sections of exact order p is 0. The content of these two sections are combined in Section 12 to construct the class $\mu_0 \in H^{n-1}(\Gamma, \mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p))^-$. Section 13 constructs the log-rigid class J_{Eis} from μ_0 and defines its values at points attached to totally real fields where p is inert. Finally, Section 14 proves Theorem 9.2.2, relating the local trace of these values to local traces of p-adic logarithm of Gross–Stark units.

Section 10

Eisenstein class of a torus bundle

In this section, we introduce the Eisenstein class of a torus bundle, as studied in [BCG20]. We focus specifically on the torus bundle

$$\Gamma'\backslash(\mathscr{X} \times \mathbb{R}^n/\mathbb{Z}^n) \longrightarrow \Gamma'\backslash\mathscr{X},$$

where $\mathscr{X}$ is the symmetric space attached to $\mathrm{SL}_n(\mathbb{R})$ and $\Gamma' \subset \Gamma := \mathrm{SL}_n(\mathbb{Z})$ is a congruence subgroup that is torsion-free. We then prove several properties of this class, including a distribution relation between its pullbacks by torsion sections, which parallels the distribution relations satisfied by Siegel units.

10.1 Thom and Eisenstein classes of a torus bundle

Let $\pi\colon E \to X$ be an oriented real vector bundle of rank n over an oriented manifold X. Since E is oriented, for every fiber $E_x \subset E$ over $x \in X$ we have a preferred generator

$$u_{E_x} \in H^n(E_x, E_x - \{0\}) \simeq \mathbb{Z}$$

satisfying a local compatibility condition (see [MS74, Page 96]). The Thom isomorphism theorem asserts that there is a global class which restricts to the orientation to each fiber.

Theorem 10.1.1 (Thom isomorphism theorem)**.** *There is a unique class $u_E \in H^n(E, E - \{0\})$ such that its pullback to any fiber E_x of E is equal to u_{E_x}. Moreover, for every $i \in \mathbb{Z}$, we have an isomorphism*

$$H^i(X) \xrightarrow{\sim} H^{i+n}(E, E - \{0\}), \;\; y \longmapsto \pi^* y \smile u_E.$$

Proof. See Section 10, and in particular Theorem 10.4, of [MS74]. □

Now suppose that E contains a sub-bundle $E_{\mathbb{Z}}$ with fibers isomorphic to $\mathbb{Z}^n$. We can then construct the torus bundle $T := E/E_{\mathbb{Z}} \to X$. For every $x \in X$, the orientation on E_x yields an orientation on T_x. Fix $c \in \mathbb{Z}_{\geq 1}$ and consider the class

$$u_{T_x,c} \in H^n(T_x, T_x - T_x[c]) \simeq \bigoplus_{z \in T_x[c]} H^n(T_x, T_x - \{z\}),$$

which restricts to the generator of each $H^n(T_x, T_x - \{z\})$ determined by the orientation of T_x at z, for every $z \in T_x[c]$. In a similar way than in Theorem 10.1.1, we will find a unique class in $H^n(T, T - T[c])$ restricting to $u_{T_x,c}$ for every $x \in X$.

Let $D \subset T$ be a tubular neighborhood of $T[c]$. By construction, D is diffeomorphic to an oriented vector bundle of rank n over $T[c]$. Let $\pi_D \colon D \to T[c]$ be the projection map. We can apply Theorem 10.1.1 to obtain a unique class $u_D \in H^n(D, D - T[c])$ restricting to the orientation of each fiber and an isomorphism for every $i \in \mathbb{Z}$

$$H^i(T[c]) \xrightarrow{\sim} H^{i+n}(D, D - T[c]), \ y \longmapsto \pi_D^*(y) \smile u_D.$$

Combining this map with the pullback of the inclusion $\iota \colon (D, D - T[c]) \hookrightarrow (T, T - T[c])$, which is an isomorphism by the excision theorem, we obtain the isomorphism

$$H^i(T[c]) \xrightarrow{\sim} H^{i+n}(D, D - T[c]) \xrightarrow{(\iota^*)^{-1}} H^n(T, T - T[c]). \tag{10.1}$$

Theorem 10.1.2. *There is a unique class $u_{T,c} \in H^n(T, T - T[c])$ such that its pullback to any fiber T_x of T is equal to $u_{T_x,c}$. Moreover, for every $i \in \mathbb{Z}$, the map (10.1) yields an isomorphism*

$$H^i(T[c]) \xrightarrow{\sim} H^{i+n}(T, T - T[c]).$$

Proof. By the description given above, it follows that the unique class with the desired property is the class corresponding to u_D via the isomorphism

$$\iota^* \colon H^n(T, T - T[c]) \xrightarrow{\sim} H^n(D, D - T[c]).$$

The result follows from Theorem 10.1.1 and the discussion above (10.1). □

Definition 10.1.3. The class u_E is called the Thom class of the bundle $E \to X$, and $u_{T,c}$ is the Thom class of the torus bundle $T \to X$ relative to the c-torsion.

We now outline the definition of the Eisenstein class of the torus bundle $T \to X$ relative to the c-torsion. For this, we assume that for all $i \in \mathbb{Z}$, the singular cohomology group with $\mathbb{Z}$-coefficients $H^i(X)$ is finitely generated. Consider the following class in singular cohomology

$$T[c] - c^n\{0\} \in H^0(T[c]).$$

Denote by the same symbol the image of this class in $H^n(T, T - T[c])$ via the Thom isomorphism given in Theorem 10.1.2. The long exact sequence in relative cohomology gives

$$\cdots \longrightarrow H^{n-1}(T) \longrightarrow H^{n-1}(T - T[c]) \longrightarrow H^n(T, T - T[c]) \longrightarrow H^n(T) \longrightarrow \cdots . \quad (10.2)$$

We then have the following theorem.

Theorem 10.1.4 (Sullivan, Bergeron–Charollois–García)**.** *There exists a unique class* ${}_cz_T \in H^{n-1}(T - T[c], \mathbb{Z}[1/c])$ *satisfying:*

1. *It is a lift of* $T[c] - c^n\{0\} \in H^n(T, T - T[c], \mathbb{Z}[1/c])$ *by the map in* (10.2)*.*

2. *It is invariant under pushforward induced by multiplication by* a *in* T *for all* $a \in \mathbb{N}^{(c)}$*.*

Proof. See Section 2 and Section 3 of [BCG20]. There, it is proven the existence of the class ${}_cz_T$ with coefficients in $\mathbb{Z}[1/N]$, for N divisible by c and prime to p (see the remarks below Lemma 9 and Definition 10 of [BCG20]). This is sufficient for our purposes, but we refer the reader to [Xu23, Page 14] for a proof that the coefficients can be taken to be $\mathbb{Z}[1/c]$. □

Definition 10.1.5. The class ${}_cz_T$ above is the Eisenstein class attached to T and c.

Throughout this work, we will construct invariants attached to totally real fields of degree n from periods of Eisenstein classes of torus bundles of rank n.

Remark 10.1.6. Theorem 10.1.4 has the following visual interpretation. The first point is equivalent to the fact that the image of $T[c] - c^n\{0\}$ in $H^n(T, \mathbb{Z}[1/c])$ vanishes. Informally, this means that there is a codimension $n-1$ submanifold $\Sigma \subset T - T[c]$ such that

$$\partial\Sigma = t(T[c] - c^n\{0\}), \ t \in \mathbb{Z},$$

where $\partial\Sigma$ denotes the boundary of Σ. On the other hand, the class of Σ is not unique, and the second point of the theorem provides a preferred class, ${}_cz_T$ with this property. In particular, ${}_cz_T$ allows defining linking numbers with $T[c] - c^n\{0\}$ as the intersection number with the preferred choice of Σ.

Remark 10.1.7. Let $a \in \mathbb{N}^{(c)}$. Consider inclusion

$$i\colon T - T[ac] \hookrightarrow T - T[c].$$

Multiplication by a induces a map

$$[a]\colon T - T[ac] \longrightarrow T - T[c].$$

The map pushforward induced by multiplication by $[a]$ on $H^i(T - T[c])$ appearing in Theorem 10.1.4 is defined as the composition

$$H^i(T - T[c]) \xrightarrow{i^*} H^i(T - T[ac]) \xrightarrow{[a]_*} H^i(T - T[c]).$$

We similarly define $[a]_* \colon H^i(T, T - T[c]) \to H^i(T, T - T[c])$.

10.2 Eisenstein class of universal families of tori

Let $n \geq 2$ and denote by $\mathscr{X} := \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n$ the symmetric space attached to $\mathrm{SL}_n(\mathbb{R})$. We are interested in the Eisenstein class of universal families of tori over quotients of $\mathscr{X}$ by the following congruence subgroups.

Let p be an odd prime such that $(p, c) = 1$, for $r \geq 0$ consider the column vector

$$v_r = (1/p^r, 0, \dots, 0)^t \in \mathbb{Q}^n/\mathbb{Z}^n,$$

and let Γ_r be its stabilizer in $\Gamma := \mathrm{SL}_n(\mathbb{Z})$. Fix $q \neq p$ an auxiliary prime such that the full level congruence subgroup $\Gamma(q) \subset \Gamma$ is torsion-free and has index prime to p. Observe that these conditions impose that p is sufficiently large. Finally, define $\Gamma_r(q) := \Gamma_r \cap \Gamma(q)$ and consider the torus bundle

$$T_r := \Gamma_r(q) \backslash (\mathscr{X} \times \mathbb{R}^n/\mathbb{Z}^n) \longrightarrow \Gamma_r(q) \backslash \mathscr{X}.$$

Definition 10.2.1. Denote by $z_r := {}_c z_{T_r} \in H^{n-1}(T_r - T_r[c], \mathbb{Z}[1/c])$ the Eisenstein class attached to the torus bundle T_r and c.

Remark 10.2.2. We introduced the auxiliary prime q and the congruence subgroups $\Gamma_r(q)$ to ensure that their action on $\mathscr{X}$ is free, which holds as $\Gamma_r(q)$ is torsion-free. Thus, the fibers of T_r are n-tori.

For $r \geq 1$, the vector v_r induces a section

$$v_r \colon \Gamma_r(q) \backslash \mathscr{X} \longrightarrow T_r - T_r[c], \ [g] \longmapsto [(g, v_r)].$$

We can then consider the pullback $v_r^* z_r \in H^{n-1}(\Gamma_r(q) \backslash \mathscr{X}, \mathbb{Z}[1/c])$. We proceed to study the behavior of $v_r^* z_r$ with respect to two different actions. Observe that $\Gamma_r(q)$ is a normal subgroup of Γ_r. Thus, we can define an action of Γ_r on $\Gamma_r(q) \backslash \mathscr{X}$ as follows. For $\gamma \in \Gamma_r$,

$$\gamma \colon \Gamma_r(q) \backslash \mathscr{X} \longrightarrow \Gamma_r(q) \backslash \mathscr{X}, \ [g] \longmapsto [\gamma g].$$

As a consequence, Γ_r acts on $H^{n-1}(\Gamma_r(q) \backslash \mathscr{X}, \mathbb{Z}[1/c])$ via pullback. Since Γ_r fixes v_r, we deduce that the class $v_r^* z_r$ is fixed by this action. More precisely:

Lemma 10.2.3. *Consider the same notation as above. We have*

$$v_r^* z_r \in H^{n-1}(\Gamma_r(q)\backslash\mathscr{X}, \mathbb{Z}[1/c])^{\Gamma_r}.$$

Proof. Let $\gamma \in \Gamma_r$ and define the map of torus bundles

$$\tilde{\gamma}\colon T_r \longrightarrow T_r,\ [(g,v)] \longmapsto [(\gamma g, \gamma v)].$$

We have $\tilde{\gamma}^* T_r[c] = T_r[c]$ and $\tilde{\gamma}^*\{0\} = \{0\}$ in $H^0(T_r[c])$. Moreover, since $\tilde{\gamma}^* u_{T_r,c} = u_{T_r,c}$, as $\tilde{\gamma}$ is orientation preserving, it follows that

$$\tilde{\gamma}^*(T_r[c] - c^n\{0\}) = T_r[c] - c^n\{0\} \in H^n(T_r, T_r - T_r[c]).$$

This implies that $\tilde{\gamma}^* z_r$ is a lift of $T_r[c] - c^n\{0\}$. Moreover, for every $a \in \mathbb{N}^{(c)}$, $\tilde{\gamma}^*$ commutes with $[a]_*$. Indeed, define $\widetilde{\gamma^{-1}}$ in the same way as $\tilde{\gamma}$ but replacing γ by γ^{-1}. Then, $\widetilde{\gamma^{-1}}$ is the inverse of $\tilde{\gamma}$ and therefore $\tilde{\gamma}^* = \widetilde{\gamma^{-1}}_*$. The desired commutativity follows then from taking the pushforward of the map $[a] \circ \widetilde{\gamma^{-1}} = \widetilde{\gamma^{-1}} \circ [a]$. From there, we deduce that $\tilde{\gamma}^* z_r$ is invariant under $[a]_*$. As a consequence, Theorem 10.1.4 implies $z_r = \tilde{\gamma}^* z_r$. Pulling back this equality by $v_r\colon \Gamma_r(q)\backslash\mathscr{X} \to T_r - T_r[c]$ yields the desired expression. □

Let $w = \mathrm{diag}(1, -1, 1, \ldots, 1) \in \mathrm{GL}_n(\mathbb{Z})$. Since w normalizes $\Gamma_r(q)$ and SO_n, conjugation induces the following map

$$w\colon \Gamma_r(q)\backslash\mathscr{X} \longrightarrow \Gamma_r(q)\backslash\mathscr{X},\ [g] \longmapsto [wgw^{-1}],$$

which induces an involution w on $H^{n-1}(\Gamma_r(q)\backslash\mathscr{X}, \mathbb{Z}[1/c])$ via pullback. Here and for the rest of the chapter, we will denote with a superindex $-$ the $w = -1$ eigenspace for w.

Lemma 10.2.4. *For every $r \geq 1$ we have*

$$v_r^* z_r \in H^{n-1}(\Gamma_r(q)\backslash\mathscr{X}, \mathbb{Z}[1/c])^-.$$

Proof. The proof is analog to the proof of Lemma 10.2.3. Denote by $\tilde{w}$ the morphism of torus bundles

$$\tilde{w}\colon T_r \longrightarrow T_r,\ [([g], v)] \longmapsto [(wgw^{-1}, wv)].$$

Since $\tilde{w}$ reverses the orientation on the fibers (because the determinant of the matrix defining w is -1), it follows that

$$\tilde{w}^*(T_r[c] - c^n\{0\}) = -(T_r[c] - c^n\{0\}) \in H^n(T_r, T_r - T_r[c]).$$

Indeed, $\tilde{w}^*$ fixes $T_r[c] - c^n\{0\}$ as a class in $H^0(T_r[c])$, but maps the Thom form in $H^n(T_r, T_r - T_r[c])$ to its negative. Similar as in Lemma 10.2.3, we can see $\tilde{w}^*$ commutes with $[a]_*$ for every $a \in \mathbb{N}^{(c)}$. From there, we deduce that $\tilde{w}^* z_r = -z_r$. The desired result can be deduced by pulling back this equality by v_r and observing $wv_r = v_r$. □

10.3 Distribution relations

We give some compatibility properties regarding the classes z_r and their pullbacks by torsion sections. In particular, we prove a distribution relation. We begin with the following general lemma.

Lemma 10.3.1. *Consider the commutative diagram of topological spaces, where all the maps are continuous*

$$\begin{array}{ccc} Z & \xrightarrow{f_1} & Y \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} \\ X & \xrightarrow{f_2} & S. \end{array}$$

Suppose that the following two conditions hold:

1. *h_1 and h_2 are r-sheeted covering maps, for $r \in \mathbb{Z}_{\geq 1}$.*
2. *If $x \in X$ and $\{z_j\}_{j=1}^r$ are its distinct lifts by h_1, the images $\{f_1(z_j)\}_{j=1}^r$ are distinct.*

Then, for all $i \in \mathbb{Z}_{\geq 0}$, we have

$$(h_1)_* f_1^* = f_2^* (h_2)_* \colon H^i(Y) \longrightarrow H^i(X).$$

Proof. For the proof of this lemma, we follow the same notation as in its statement. Let $\varphi \in C^i(Y, \mathbb{Z})$ be a degree i cochain and consider $\sigma : \Delta^i \to X$ a continuous map from an i-simplex Δ^i to X. Fix a vertex $u \in \Delta^i$, and let $x = \sigma(u)$.

Since h_1 is an r-sheeted covering map, there are $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_r \colon \Delta^i \to Z$ distinct lifts of σ by h_1, characterized by the property $\tilde{\sigma}_j(u) = z_j$. Then,

$$(h_1)_* f_1^* \varphi(\sigma) = \sum_j f_1^* \varphi(\tilde{\sigma}_j) = \sum_j \varphi(f_1 \circ \tilde{\sigma}_j).$$

Similarly, let $y_1, \ldots, y_r \in Y$ be the distinct lifts of $f_2(x)$, and consider $\tilde{\omega}_1, \ldots, \tilde{\omega}_r \colon \Delta^i \to Y$ the distinct lifts of $f_2 \circ \sigma$ by h_2, characterized by the property $\tilde{\omega}_j(u) = y_j$. Then,

$$f_2^* (h_2)_* \varphi(\sigma) = (h_2)_* \varphi(f_2 \circ \sigma) = \sum_j \varphi(\tilde{\omega}_j).$$

We now observe that we have the equality of sets $\{f_1 \circ \tilde{\sigma}_j\}_j = \{\tilde{\omega}_j\}_j$. Indeed, Condition (2) in the statement of the lemma implies that the simplices $\{f_1 \circ \tilde{\sigma}_j\}_j$ are all distinct, as their evaluations at u are $\{f_1 \circ \tilde{\sigma}_j(u) = f_1(z_j)\}_j$, which are all distinct. Thus, both sets have the same number of elements and to prove the equality it is enough to see the inclusion $\{f_1 \circ \tilde{\sigma}_j\} \subset \{\tilde{\omega}_j\}_j$. For that, observe that the commutativity of the diagram implies that $f_1 \circ \tilde{\sigma}_j$ is a lift of $f_2 \circ \sigma$ by h_2.

From this equality of sets and the previous two calculations, we obtain the desired equality $(h_1)_* f_1^* = f_2^* (h_2)_*$ of cochain maps, which induces the result in cohomology. $\square$

Remark 10.3.2. Condition (2) of the lemma above holds if the commutative diagram is Cartesian.

Proposition 10.3.3. *Let $r, r' \in \mathbb{Z}$ with $r \geq r' \geq 1$, consider the projection map* $\mathrm{pr}\colon T_r - T_r[c] \to T_{r'} - T_{r'}[c]$, *and denote by* pr^* *the corresponding pullback in cohomology. Then,* $\mathrm{pr}^* z_{r'} = z_r$.

Proof. The structure of the proof is analog to the proof of Lemma 10.2.3, so we only outline the key points. First, we observe

$$\mathrm{pr}^*(T_{r'}[c] - c^n\{0\}) = T_r[c] - c^n\{0\} \in H^n(T_r, T_r - T_r[c]).$$

Therefore, $\mathrm{pr}^*(z_{r'})$ is a lift of $T_r[c] - c^n\{0\}$. Second, we claim that pr^* commutes with $[a]_*$. The key to prove this statement is to apply Lemma 10.3.1 to the diagram

$$\begin{array}{ccc} T_r - T_r[ac] & \xrightarrow{\mathrm{pr}} & T_{r'} - T_{r'}[ac] \\ \downarrow{\scriptstyle [a]} & & \downarrow{\scriptstyle [a]} \\ T_r - T_r[c] & \xrightarrow{\mathrm{pr}} & T_{r'} - T_{r'}[c]. \end{array}$$

Therefore, $z_r = \mathrm{pr}^* z_{r'}$ by Theorem 10.1.4. □

From the previous proposition, we deduce that the classes $v_r^* z_r$ satisfy the following distribution relation.

Proposition 10.3.4. *Let $r \geq 1$ and consider the pushforward attached to the finite quotient map* $\mathrm{pr}\colon \Gamma_{r+1}(q)\backslash\mathscr{X} \to \Gamma_r(q)\backslash\mathscr{X}$, *namely*

$$\mathrm{pr}_*\colon H^{n-1}(\Gamma_{r+1}(q)\backslash\mathscr{X}, \mathbb{Z}[1/c]) \longrightarrow H^{n-1}(\Gamma_r(q)\backslash\mathscr{X}, \mathbb{Z}[1/c]).$$

Then, $\mathrm{pr}_*(v_{r+1}^* z_{r+1}) = v_r^* z_r$.

Proof. Consider the map

$$f_r\colon \Gamma_r(q)\backslash\mathscr{X} \longrightarrow T_r - T_r[c] \longrightarrow T_1 - T_1[c],$$

where the first arrow is induced by v_r and the second one is the quotient map. Also, observe that since $r \geq 1$ we can define

$$f_{r+1} : \Gamma_{r+1}(q)\backslash\mathscr{X} \longrightarrow T_{r+1} - T_{r+1}[pc] \longrightarrow T_1 - T_1[pc],$$

in a similar way as f_r, but where we used that v_{r+1} is of exact p^{r+1} torsion, with $p^{r+1} > p$. It is a consequence of Proposition 10.3.3 that, if $\iota\colon T_1 - T_1[pc] \to T_1 - T_1[c]$,

$$v_r^* z_r = f_r^* z_1, \quad v_{r+1}^* z_{r+1} = f_{r+1}^* \iota^* z_1.$$

We will now deduce the desired statement from the invariance of z_1 under multiplication by p. With this aim, observe that we can apply Lemma 10.3.1 to the following commutative diagram

$$\begin{array}{ccc} \Gamma_{r+1}(q)\backslash\mathscr{X} & \xhookrightarrow{f_{r+1}} & T_1 - T_1[pc] \\ \downarrow \text{pr} & & \downarrow [p] \\ \Gamma_r(q)\backslash\mathscr{X} & \xhookrightarrow{f_r} & T_1 - T_1[c]. \end{array}$$

Indeed, since $\Gamma_r(q)$ is torsion-free and the index $[\Gamma_r(q) : \Gamma_{r+1}(q)] = p^n$, both horizontal maps are p^n-sheeted covering maps, implying the Condition (1) of the lemma. Moreover, the fact that $\Gamma_r(q)$ is torsion-free implies that the maps f_r and f_{r+1} are injective, giving Condition (2) of the lemma. Therefore,

$$\text{pr}_* f_{r+1}^* = f_r^*[p]_*.$$

From there,

$$\text{pr}_* v_{r+1}^* z_{r+1} = \text{pr}_* f_{r+1}^* \iota^* z_1 = f_r^*[p]_* \iota^* z_1 = f_r^* z_1 = v_r^* z_r,$$

where we used the invariance of z_1 under multiplication by p on the second to last equality (see Theorem 10.1.4 and Remark 10.1.7). □

Section 11

Differential form representative of the Eisenstein class

In [BCG20], Bergeron, Charollois, and García construct a closed differential form on $T_r - T_r[c]$ representing the Eisenstein class z_r. Their construction departs from a transgression form obtained from the Mathai–Quillen Thom form, studied in [MQ86]. Then, inspired by the work of Bismut and Cheeger [BC92], they obtain a representative of the Eisenstein class from a regularized average of the transgression form. In this section, we outline this procedure and use the differential forms we obtain to prove some properties about pullbacks of the Eisenstein class by torsion sections. The expressions given here will also be used in the last section of this chapter to relate periods of the Eisenstein class to special values of L-functions.

11.1 Mathai–Quillen form and the transgression form

Let $S := \mathrm{GL}_n(\mathbb{R})/\mathrm{SO}_n$ and consider the real vector bundle $E := S \times \mathbb{R}^n \to S$, which is $\mathrm{GL}_n(\mathbb{R})$-equivariant for the left multiplication action on each of the components of E and on S. Mathai and Quillen construct a closed $\mathrm{GL}_n(\mathbb{R})$-equivariant differential form

$$\varphi \in \Omega^n_{\mathrm{rd}}(E)^{\mathrm{GL}_n(\mathbb{R})}$$

which has rapid decay (Gaussian shape) and integral 1 along the fibers. In particular, φ represents the Thom class of the oriented vector bundle $E \to S$ via the isomorphisms

$$H^n(\Omega^\bullet_{\mathrm{rd}}(E)) \simeq H^n(E, E - \{0\}, \mathbb{R})$$

between the cohomology of the complex of forms on E with rapid decay along the fibers $\Omega^\bullet_{\mathrm{rd}}(E)$ and relative singular cohomology (see [MQ86, Page 98 and Page 99]).

There is an explicit expression for the form φ, that we proceed to outline following [BCG20, Theorem 13] and [MQ86]. The reader is referred to these sources for further details on the construction of φ, as for our purposes it is sufficient to know the shape of its expression. Using the Iwasawa decomposition of $\mathrm{GL}_n(\mathbb{R})$, fix $h\colon S \to \mathrm{GL}_n(\mathbb{R})$ a smooth section of the quotient map $\mathrm{GL}_n(\mathbb{R}) \to S$. Then

$$\varphi = \pi^{-n/2} e^{-|h^{-1}x|^2} \sum_{\substack{I\subset\{1,\dots,n\} \\ |I| \text{ even}}} \varepsilon_{I,I'} \mathrm{Pf}(\Omega_I/2) \left(d(h^{-1}x) + \theta h^{-1}x\right)^{I'}, \tag{11.1}$$

where:

- $x \in \mathbb{R}^n$ and $|x|$ is its standard norm.
- θ is an $n\times n$ matrix of 1-forms on S, obtained as the pullback by h of the connection of the principal SO_n-bundle $\mathrm{GL}_n(\mathbb{R}) \to S$ given by $\theta_{\mathrm{GL}_n(\mathbb{R})} = (g^{-1}dg - dg^t(g^t)^{-1})/2$.
- Ω is an $n\times n$ matrix of 2-forms on S, obtained as the pullback by h of the curvature $d\theta_{\mathrm{GL}_n(\mathbb{R})} + \theta^2_{\mathrm{GL}_n(\mathbb{R})}$. Then, $\mathrm{Pf}(\Omega_I/2)$ is an $|I|$-form given as the Pfaffian of the submatrix of $\Omega/2$ of size $|I|$ involving the indices in I.
- I' denotes the complement of $I \subset \{1,\dots,n\}$, $\varepsilon_{I,I'} \in \{\pm 1\}$, and for a vector v of size n, $v^{I'} = v_{i_1}v_{i_2}\cdots v_{i_{|I'|}}$, where $I' = \{i_1,\dots,i_{|I'|}\}$.

Remark 11.1.1. We will not use the expressions for θ and Ω, aside from the fact that they are forms of degree 1 and 2 on S.

For $t \in \mathbb{R}_{>0}$, let $[t]\colon E \to E$ be multiplication by t on the fibers. An important property of φ is that for every $t \in \mathbb{R}_{>0}$, $[t]^*\varphi$ also represents the Thom class. Indeed, the Gaussian on the fibers gets dilated, but the value of the integral over the fibers is preserved and equal to 1. In particular, observe that

$$[t]^*\varphi \longrightarrow \delta_0, \quad \text{as } t \longrightarrow +\infty,$$

where δ_0 denotes the current of integration along the zero section of E, also represents the Thom class (as a current). Recall that the Eisenstein class is a lift of Thom classes of a torus bundle, by Theorem 10.1.4. The next proposition constructs a form η whose differential involves δ_0. The relevance of this form is that a (regularized) average of it will give a representative of the Eisenstein class.

Definition 11.1.2. Let $R := \sum_i x_i \frac{\partial}{\partial x_i}$ be the radial vector field on $E = S\times\mathbb{R}^n$, where $\{x_i\}$ denote the coordinates on $\mathbb{R}^n$ and define $\psi := \iota_R\varphi \in \Omega^{n-1}_{\mathrm{rd}}(E)^{\mathrm{GL}_n(\mathbb{R})}$, which can be verified to be $\mathrm{GL}_n(\mathbb{R})$-invariant.

Proposition 11.1.3. *Consider the differential form on $E - S$*

$$\eta := \int_0^{+\infty} [t]^*\psi \frac{dt}{t}. \tag{11.2}$$

Viewed as a current on E, it satisfies the transgression property

$$d\eta = \delta_0 - [0]^*\varphi.$$

Proof. The main idea for the proof of this statement lies in the following computation

$$\delta_0 - [0]^*\varphi = \int_0^{+\infty} \frac{d}{dt}[t]^*\varphi dt = \int_0^{+\infty} d[t]^*\iota_R\varphi\frac{dt}{t} = d\eta,$$

where the second equality follows from interpreting $\frac{d}{dt}[t]^*\varphi$ in terms of a Lie derivative with respect to the vector field R and using Cartan magic formula. For more details, see Section 7.2 and Section 7.3 of [BCG20] and Page 106 of [MQ86]. □

Using the explicit expression for φ given in (11.1), and following the same notation as in that equation, we obtain

$$\psi = \pi^{-n/2}e^{-|h^{-1}x|^2} \sum_{\substack{I\subsetneq\{1,\dots,n\}\\ |I| \text{ even}}} \left(\varepsilon_{I,I'}\mathrm{Pf}(\Omega_I/2)\sum_{k=1}^{|I'|}(-1)^{k+1}(h^{-1}x)_{i_k}\left(d(h^{-1}x)+\theta h^{-1}x\right)^{I'-\{i_k\}}\right),$$

$$\eta = \frac{\pi^{-n/2}}{2} \sum_{\substack{I\subsetneq\{1,\dots,n\}\\ |I| \text{ even}}} \left(\varepsilon_{I,I'}\mathrm{Pf}(\Omega_I/2)\frac{\Gamma(|I'|/2)}{|h^{-1}x|^{|I'|}}\sum_{k=1}^{|I'|}(-1)^{k+1}(h^{-1}x)_{i_k}\left(d(h^{-1}x)+\theta h^{-1}x\right)^{I'-\{i_k\}}\right).$$

Here $I' = \{i_1, \dots, i_{|I'|}\}$ is the complement of $I \subsetneq \{1, \dots, n\}$. The exact formulas will not be necessary for us. On the other hand, it will be important to note:

- φ and ψ are linear combinations of products of an exponential and a polynomial. In particular, they have rapid decay along the fibers.
- $[0]^*\psi = 0$.
- η does not have rapid decay along the fibers.

11.2 Eisenstein transgression

We proceed to consider a regularized average of the form η in (11.2) over a lattice to obtain forms on torus bundles representing the Eisenstein class. For $L \subset \mathbb{Q}^n$ a $\mathbb{Z}$-lattice and $\lambda \in L$, let

$$\mathrm{tr}_\lambda \colon E \longrightarrow E, \ (g, x) \longmapsto (g, x + \lambda).$$

Then, if $t \in \mathbb{R}_{>0}$, define

$$\theta([t]^*\psi, L) := \sum_{\lambda \in L} \mathrm{tr}_\lambda^*[t]^*\psi. \tag{11.3}$$

The sum converges as the differential form $t^*\psi$ has rapid decay on the fibers of $E \to S$.

Theorem 11.2.1. *View $\theta([t]^*\psi, L)$ as a differential form on $S \times (\mathbb{R}^n - L)$. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$, the integral*

$$E_\psi(L, s) := \int_0^{+\infty} \theta([t]^*\psi, L) t^s \frac{dt}{t}$$

converges. Furthermore, it admits a meromorphic continuation to all $\mathbb{C}$, regular at $s = 0$, and its value at every regular $s \in \mathbb{C}$ defines a differential form on $S \times (\mathbb{R}^n - L)$.

Proof. This follows from Proposition 17 and Section 8.5 of [BCG20]. In particular, the fact that the integral is regular at $s = 0$ follows from the fact that we are viewing $\theta([t]^*\psi, L)$ as a form on $S \times (\mathbb{R}^n - L)$, and $[t]^*\psi$ tends to 0 as $t \to +\infty$ on $S \times (\mathbb{R}^n - L)$. □

The previous theorem implies that $E_\psi(L, s)$ is regular at $s = 0$ and

$$E_\psi(L) := E_\psi(L, 0)$$

defines a form on $S \times (\mathbb{R}^n - L)/L$. In fact, $E_\psi(L)$ descends to a form in $\mathscr{X} \times (\mathbb{R}^n - L)/L$ by the calculation on (8.9) of [BCG20]. Moreover, if $\Gamma' \subset \mathrm{SL}_n(\mathbb{R})$ is a subgroup contained in the stabilizer of L, the form $E_\psi(L)$ is invariant under Γ'.

Remark 11.2.2. We outline how to view $E_\psi(L)$ as a regularized average of η. As we pointed out at the end of Section 11.1, the form η does not have rapid decay along the fibers. Therefore, the sum $\sum_{\lambda \in L} \mathrm{tr}_\lambda^* \eta$ does not converge. On the other hand, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$ define

$$\eta(s) := \int_0^{+\infty} [t]^*\psi t^s \frac{dt}{t}.$$

Then, $\eta(s)$ has the same expression as the one given for η at the end of Section 11.1 where the term $\Gamma(|I'|/2)/|h^{-1}x|^{|I'|}$ is replaced by $\Gamma((|I'| + s)/2)/|h^{-1}x|^{|I'|+s}$. In particular, it follows that if $\mathrm{Re}(s) \gg 0$, the sum $\sum_{\lambda \in L} \mathrm{tr}_\lambda^* \eta(s)$ is absolutely convergent and

$$E_\psi(L, s) = \int_0^{+\infty} \theta([t]^*\psi, L) t^s \frac{dt}{t} = \sum_{\lambda \in L} \mathrm{tr}_\lambda^* \eta(s),$$

where we exchanged the integral with the sum (using that for $\mathrm{Re}(s) \gg 0$, the sums are absolutely convergent). Thus, $E_\psi(L)$ is equal to the value at $s = 0$ of the meromorphic continuation of $\sum_{\lambda \in L} \mathrm{tr}_\lambda^* \eta(s)$.

Recall the torus bundle

$$T_r = \Gamma_r(q)\backslash(\mathscr{X} \times \mathbb{R}^n/\mathbb{Z}^n) \longrightarrow \Gamma_r(q)\backslash\mathscr{X}$$

introduced in Section 10.2.

Definition 11.2.3. Consider the linear combination

$${}_cE_\psi := E_\psi(c^{-1}\mathbb{Z}^n) - c^n E_\psi(\mathbb{Z}^n),$$

which we view as a differential form on $T_r - T_r[c]$ for every $r \geq 1$.

Theorem 11.2.4. *The form ${}_cE_\psi$ is closed in $T_r - T_r[c]$ and its cohomology class*

$${}_cE_\psi \in H^{n-1}_{\mathrm{dR}}(T_r - T_r[c]) \simeq H^{n-1}(T_r - T_r[c], \mathbb{R})$$

is equal to the image of the Eisenstein class z_r in $H^{n-1}(T_r - T_r[c], \mathbb{R})$.

Proof. See Theorem 19, Proposition 20, and Theorem 21 of [BCG20]. There it is explained that, since E_ψ is a regularized average of η (see Remark 11.2.2), Proposition 11.1.3 implies that

$$d({}_cE_\psi) = \delta_{T[c]} - c^n\delta_{\{0\}},$$

where $\delta_{T[c]}$ and $\delta_{\{0\}}$ denote currents of integration along $T[c]$ and $\{0\}$ (the contributions $[0]^*\varphi$ appearing in Proposition 11.1.3 vanish after the regularization). Moreover, $[a]_*E_\psi = E_\psi$ by Proposition 20 of [BCG20]. Thus, ${}_cE_\psi$ is a closed form on $T_r - T_r[c]$ satisfying the characterizing properties of the Eisenstein class z_r asserted in Theorem 10.1.4. □

11.3 Pullbacks by torsion sections

We now use the differential forms introduced above to study the pullbacks of the form ${}_cE_\psi$ by torsion sections. For $v \in \mathbb{Q}^n$, denote also by v the corresponding section $v\colon S \to E$. Then, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$, consider the differential form on S

$$\eta(v, s) := \int_0^{+\infty} v^*[t]^*\psi t^s \frac{dt}{t} = \int_0^{+\infty} (tv)^*\psi t^s \frac{dt}{t}.$$

Since $0^*\psi = 0$, which can be verified using the explicit expression given at the end of Section 11.1, we have $\eta(0, s) = 0$. From this same expression and Remark 11.2.2, we deduce that for $v \neq 0$

$$\eta(v,s) = \frac{\pi^{-n/2}}{2} \sum_{\substack{I \subsetneq \{1,\ldots,n\} \\ |I| \text{ even}}} \left(\varepsilon_{I,I'} \mathrm{Pf}(\Omega_I/2) \frac{\Gamma((|I'|+s)/2)}{|h^{-1}v|^{|I'|+s}} \sum_{k=1}^{|I'|} (-1)^{k+1} (h^{-1}v)_{i_k} \left(d(h^{-1})v + \theta h^{-1}v\right)^{I'-\{i_k\}} \right). \tag{11.4}$$

Proposition 11.3.1. *Let $L \subset \mathbb{Q}^n$ be a $\mathbb{Z}$-lattice and $v \in \mathbb{Q}^n - L$. For $s \in \mathbb{C}$ with* $\mathrm{Re}(s) \gg 0$,

$$v^* E_\psi(L, s) = \sum_{\lambda \in v+L} \eta(\lambda, s).$$

In particular, the right-hand side has a meromorphic continuation regular at $s = 0$.

Proof. This follows from Theorem 11.2.1 and Remark 11.2.2. □

Thus, if $v \in \mathbb{Q}^n - (1/c)\mathbb{Z}^n$,

$$v^*{}_c E_\psi = \lim_{s \to 0} \sum_{\lambda \in v + c^{-1}\mathbb{Z}^n} \eta(\lambda, s) - c^n \sum_{\lambda \in v + \mathbb{Z}^n} \eta(\lambda, s), \tag{11.5}$$

where here and from now on, $\lim_{s \to 0}$ denotes evaluation of the meromorphic continuation.

In fact, the right hand side of the equation appearing in Proposition 11.3.1 defines a differential form on S even if $v \in L$. More precisely,

$$\sum_{\lambda \in L} \eta(\lambda, s)$$

converges for $\mathrm{Re}(s) \gg 0$, and admits a meromorphic continuation to $\mathbb{C}$ which is regular at $s = 0$. We proceed to prove a weaker version of this statement, as this will be enough for our purposes.

Lemma 11.3.2. *Let $g \in S$, consider tangent vectors $Y_1, \ldots, Y_{n-1} \in T_g S$ and denote $Y = (Y_1, \ldots, Y_{n-1})$. Then, for $s \in \mathbb{C}$ with* $\mathrm{Re}(s) \gg 0$,

$$s \longmapsto \sum_{\lambda \in L} \eta(\lambda, s)_g(Y)$$

converges and admits a meromorphic continuation to $\mathbb{C}$ which is regular at $s = 0$.

Proof. It follows from the explicit expression of $\eta(v, s)$ given in (11.4) that the sum

$$\sum_{\lambda \in L} \eta(\lambda, s)_g(Y)$$

is absolutely convergent for $\mathrm{Re}(s) \gg 0$. From there, we deduce that if $\mathrm{Re}(s) \gg 0$, we have the equality

$$\sum_{\lambda \in L} \eta(\lambda, s)_g(Y) = \int_0^{+\infty} \sum_{\lambda \in L} \left((t\lambda)^* \psi\right)_g (Y) t^s \frac{dt}{t},$$

as we can exchange the integral with the sum. Thus, it is enough to prove that the right-hand side has a meromorphic continuation regular at $s = 0$. For that, define the function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R},\ v \longmapsto (v^* \psi)_g(Y).$$

Since ψ is a differential form which has rapid decay along the fibers, it follows that f is a Schwartz function. Hence, we need to prove that

$$\int_0^{+\infty} \sum_{\lambda \in L} f(t\lambda) t^s \frac{dt}{t} \tag{11.6}$$

has a meromorphic continuation to $s \in \mathbb{C}$ which is regular at $s = 0$. We split the integral as a sum of integrals from 1 to $+\infty$ and from 0 to 1. Observe that $f(0) = 0$, as $0^*\psi = 0$. The rapid decay of f, together with the fact that $f(0) = 0$, implies that the integral from 1 to $+\infty$ converges absolutely and defines an entire function on s. To study the integral from 0 to 1, we use Poisson summation formula

$$\int_0^1 \sum_{\lambda \in L} f(t\lambda) t^s \frac{dt}{t} = \int_0^1 \sum_{\lambda \in L^\vee} \hat{f}(\lambda/t) t^{s-n} \frac{dt}{t},$$

where $\hat{f}$ denotes the Fourier transform of f and $L^\vee$ the dual lattice of L. For $\mathrm{Re}(s) \gg n$, the previous integral can be written as

$$\frac{\hat{f}(0)}{s-n} + \int_1^{+\infty} \sum_{\lambda \in L^\vee - \{0\}} \hat{f}(\lambda u) u^{n-s} \frac{du}{u}.$$

Since $\hat{f}$ is a Schwartz function, the integral converges for all values of $s \in \mathbb{C}$ and defines an entire function. Thus, this expression gives a meromorphic continuation of the integral from 0 to 1 regular everywhere except maybe at $s = n$. The result follows from there. $\square$

Finally, we are ready to prove the following expression regarding pullbacks of the Eisenstein class by torsion sections, which will be useful for the next section.

Proposition 11.3.3. *For $v \in \mathbb{Q}^n - c^{-1}\mathbb{Z}^n$, view $v^*{}_cE_\psi$ as a differential form on $\mathscr{X}$. Then,*

$$\sum_{v \in \frac{1}{p}\mathbb{Z}^n/\mathbb{Z}^n - \{0\}} v^*{}_cE_\psi = 0.$$

Proof. By Proposition 11.3.1, and more precisely (11.5), we can write the sum of the proposition as the evaluation at $s = 0$ of the following expression

$$\sum_{v \in \frac{1}{p}\mathbb{Z}^n/\mathbb{Z}^n - \{0\}} \sum_{\lambda \in v + c^{-1}\mathbb{Z}^n} \eta(\lambda, s) - c^n \sum_{v \in \frac{1}{p}\mathbb{Z}^n/\mathbb{Z}^n - \{0\}} \sum_{\lambda \in v + \mathbb{Z}^n} \eta(\lambda, s).$$

We will verify that each of the two terms vanishes when evaluated at $s = 0$. Since the proof is analogous in the two cases, we will show that

$$\lim_{s \to 0} \sum_{v \in \frac{1}{p}\mathbb{Z}^n/\mathbb{Z}^n - \{0\}} \sum_{\lambda \in v + \mathbb{Z}^n} \eta(\lambda, s) = 0.$$

Let $g \in S$, consider tangent vectors $Y_1, \dots, Y_{n-1} \in T_g S$, and let $Y = (Y_1, \dots, Y_{n-1})$. Then, it is enough to see

$$\lim_{s \to 0} \sum_{v \in \frac{1}{p}\mathbb{Z}^n/\mathbb{Z}^n - \{0\}} \sum_{\lambda \in v + \mathbb{Z}^n} \eta(\lambda, s)_g(Y) = 0.$$

Then, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$

$$\begin{aligned} \sum_{v \in \frac{1}{p}\mathbb{Z}^n/\mathbb{Z}^n - \{0\}} \sum_{\lambda \in v + \mathbb{Z}^n} \eta(\lambda, s)_g(Y) = & \sum_{v \in \frac{1}{p}\mathbb{Z}^n - \mathbb{Z}^n} \eta(\lambda, s)_g(Y) + \sum_{\lambda \in \mathbb{Z}^n} \eta(\lambda, s)_g(Y) \\ & - \sum_{\lambda \in \mathbb{Z}^n} \eta(\lambda, s)_g(Y), \end{aligned}$$

where we added and subtracted a function which admits a meromorphic continuation to all $s \in \mathbb{C}$ and is regular at $s = 0$ by Lemma 11.3.2. Collecting the first two terms of the right hand side, we obtain that the previous expression is equal to

$$\sum_{\lambda \in \mathbb{Z}^n} \eta(\lambda/p, s)_g(Y) - \sum_{\lambda \in \mathbb{Z}^n} \eta(\lambda, s)_g(Y) = p^s \sum_{\lambda \in \mathbb{Z}^n} \eta(\lambda, s)_g(Y) - \sum_{\lambda \in \mathbb{Z}^n} \eta(\lambda, s)_g(Y).$$

Here we used that $\eta(\lambda/p, s) = p^s \eta(\lambda, s)$, which can be verified from the definition of $\eta(v, s)$. Since the meromorphic continuation of $\sum_{\lambda \in \mathbb{Z}^n} \eta(\lambda, s)_g(Y)$ is regular at $s = 0$ by Lemma 11.3.2, the evaluation of the meromorphic continuation of the expression above at $s = 0$ is zero. $\square$

Section 12

The Eisenstein group cohomology class

In this chapter, we package the pullbacks of the Eisenstein class by p-power torsion sections in a group cohomology class for $\Gamma := \mathrm{SL}_n(\mathbb{Z})$ valued in measures on $\mathbb{X} := \mathbb{Z}_p^n - p\mathbb{Z}_p^n$. Then, we discuss the process of lifting this class to a class valued in total mass zero measures of $\mathbb{X}$, which will be an important property for constructing rigid classes and p-adic invariants attached to totally real fields.

12.1 From singular to group cohomology

Let m be the least common multiple of c and $[\Gamma : \Gamma(q)]$. Observe that m is prime to p, by assumption on c and on the index $[\Gamma : \Gamma(q)]$. The group $\Gamma_r(q)$ acts freely on $\mathscr{X}$, as it is torsion-free. Thus, we have a natural isomorphism

$$H^{n-1}(\Gamma_r(q)\backslash\mathscr{X}, \mathbb{Z}[1/m]) \xrightarrow{\sim} H^{n-1}(\Gamma_r(q), \mathbb{Z}[1/m]). \tag{12.1}$$

Since $\Gamma_r(q)$ is normal in Γ_r, there are natural actions of Γ_r on both cohomology groups. Indeed, the action of Γ_r on singular cohomology is described above Lemma 10.2.3, and the action on group cohomology is induced by the conjugation action of Γ_r on $\Gamma_r(q)$. These actions are compatible with the isomorphism above leading to

$$H^{n-1}(\Gamma_r(q)\backslash\mathscr{X}, \mathbb{Z}[1/m])^{\Gamma_r} \xrightarrow{\sim} H^{n-1}(\Gamma_r(q), \mathbb{Z}[1/m])^{\Gamma_r}.$$

Finally, since $[\Gamma_r : \Gamma_r(q)]$ divides $[\Gamma : \Gamma(q)]$, and therefore it is invertible in $\mathbb{Z}[1/m]$, restriction induces an isomorphism

$$H^{n-1}(\Gamma_r, \mathbb{Z}[1/m]) \xrightarrow{\sim} H^{n-1}(\Gamma_r(q), \mathbb{Z}[1/m])^{\Gamma_r}.$$

The inverse of this map is given by the corestriction map multiplied by $[\Gamma_r : \Gamma_r(q)]^{-1}$.

For every $r \geq 1$, in Section 10.2 we constructed the classes

$$v_r^* z_r \in H^{n-1}(\Gamma_r(q)\backslash\mathscr{X}, \mathbb{Z}[1/m])^{\Gamma_r}.$$

and proved they were invariant under the action of Γ_r in Lemma 10.2.3.

Definition 12.1.1. For $r \geq 1$, let $c_r \in H^{n-1}(\Gamma_r, \mathbb{Z}[1/m])$ be the group cohomology class corresponding to $v_r^* z_r$ via the previous two isomorphisms. We will sometimes view c_r as a class with coefficients in $\mathbb{Z}_p$, $\mathbb{Z}/p^r\mathbb{Z}$ or $\mathbb{R}$ via the natural maps from $\mathbb{Z}[1/m]$ to these rings.

The trace compatibility of the singular cohomology classes $(v_r^* z_r)_r$ leads to the compatibility of the group cohomology classes $(c_r)_r$ with respect to corestriction maps.

Proposition 12.1.2. *For $r \geq 1$ let* $\mathrm{cor}\colon H^{n-1}(\Gamma_{r+1}, \mathbb{Z}[1/m]) \to H^{n-1}(\Gamma_r, \mathbb{Z}[1/m])$ *be the corestriction map. Then,* $\mathrm{cor}(c_{r+1}) = c_r$.

Proof. Denote by $c_r(q) \in H^{n-1}(\Gamma_r(q), \mathbb{Z}[1/m])$ the image of $v_r^* z_r$ via the isomorphism (12.1). Since this isomorphism is compatible with respect to pushforward and corestriction (see [Bro82, Chapter III, Section 9 (E)]), it follows from Proposition 10.3.4 that if

$$\mathrm{cor}_q : H^{n-1}(\Gamma_{r+1}(q), \mathbb{Z}[1/m]) \longrightarrow H^{n-1}(\Gamma_r(q), \mathbb{Z}[1/m]),$$

denotes corestriction in group cohomology, then $\mathrm{cor}_q(c_{r+1}(q)) = c_r(q)$. Now observe that we have a commutative diagram

$$\begin{array}{ccc} H^{n-1}(\Gamma_{r+1}(q), \mathbb{Z}[1/m]) & \longrightarrow & H^{n-1}(\Gamma_{r+1}, \mathbb{Z}[1/m]) \\ \big\downarrow \mathrm{cor}_q & & \big\downarrow \mathrm{cor} \\ H^{n-1}(\Gamma_r(q), \mathbb{Z}[1/m]) & \longrightarrow & H^{n-1}(\Gamma_r, \mathbb{Z}[1/m]), \end{array}$$

where all the maps denote corestriction maps. Since the image of $c_r(q)$ by the horizontal map is $[\Gamma_r : \Gamma_r(q)]c_r$, we deduce that

$$\mathrm{cor}([\Gamma_{r+1} : \Gamma_{r+1}(q)]c_{r+1}) = [\Gamma_r : \Gamma_r(q)]c_r.$$

Now, observing that $[\Gamma_{r+1} : \Gamma_{r+1}(q)] = [\Gamma_r : \Gamma_r(q)]$ and that this quantity is invertible in $\mathbb{Z}[1/m]$, we obtain the desired equality in $H^{n-1}(\Gamma_r, \mathbb{Z}[1/m])$. □

12.2 Cohomology class with coefficients in $\mathbb{Z}_p$-measures

For $r \geq 1$, let $\mathbb{X}_r := (\mathbb{Z}/p^r\mathbb{Z})^n - (p\mathbb{Z}/p^r\mathbb{Z})^n$ and if A is an abelian group, denote

$$\mathbb{D}(\mathbb{X}_r, A) := \mathrm{Maps}(\mathbb{X}_r, A).$$

It admits a left action of Γ given by $(g \cdot \lambda)(x) = \lambda(g^{-1}x)$, for $g \in \Gamma$, $\lambda \in \mathbb{D}(\mathbb{X}_r, A)$, and $x \in \mathbb{X}_r$. Let $x_r := (1, 0, \dots, 0)^t \in \mathbb{X}_r$. Since the stabilizer of x_r in Γ is Γ_r, we deduce that we have a Γ-equivariant isomorphism

$$\mathrm{coInd}_{\Gamma_r}^{\Gamma}(A) \xrightarrow{\sim} \mathbb{D}(\mathbb{X}_r, A), \; f \longmapsto \lambda_f,$$

where $\lambda_f(x) = f(\gamma)$ for $\gamma \in \Gamma$ such that $\gamma x_r = x$.

Definition 12.2.1. For every $r \geq 1$, define $\mu_r \in H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{Z}[1/m]))$ to be the image of c_r by the inverse of the isomorphism induced by Shapiro's lemma

$$H^{n-1}(\Gamma_r, \mathbb{Z}[1/m]) \xrightarrow{\sim} H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{Z}[1/m])).$$

Again, we will sometimes view μ_r as a class with coefficients in $\mathbb{Z}_p$, $\mathbb{Z}/p^r\mathbb{Z}$ or $\mathbb{R}$.

Consider the Γ-equivariant maps

$$u_{r+1} \colon \mathbb{D}(\mathbb{X}_{r+1}, A) \longrightarrow \mathbb{D}(\mathbb{X}_r, A), \; u_{r+1}(f)(x) = \sum_{\substack{x' \in \mathbb{X}_{r+1} \\ x' \equiv x \mod p^r}} f(x'). \tag{12.2}$$

It follows from the compatibility of the classes $(c_r)_r \in \varprojlim_r H^{n-1}(\Gamma_r, \mathbb{Z}[1/m])$ proven in Proposition 12.1.2, that we have a compatible system

$$(\mu_r)_r \in \varprojlim_r H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{Z}[1/m])),$$

where the transition maps are given by u_r for every $r \geq 2$. This statement can be proven using Chapter III, Section 9 (A) of [Bro82], which leads to describe the corestriction maps in terms of the map given by Shapiro's lemma and u_r.

Denote by $\mathbb{D}(\mathbb{X}, A)$ the space of A-valued distributions on $\mathbb{X}$. An element of $\lambda \in \mathbb{D}(\mathbb{X}, A)$ is determined by the values $\lambda(U)$ of the characteristic functions of compact open sets U. In particular, it is determined by the images of the following compact open sets. For $x \in \mathbb{X}_r$, choose any lift of it in $\mathbb{X}$, also denoted by x, and let

$$U_{x/p^r} := x + p^r\mathbb{Z}_p^n \subset \mathbb{X}. \tag{12.3}$$

Endow $\mathbb{D}(\mathbb{X}, A)$ with a left action of Γ given by $(g \cdot \lambda)(U) = \lambda(g^{-1}U)$ and define

$$\mathbb{D}(\mathbb{X}, A) \twoheadrightarrow \mathbb{D}(\mathbb{X}_r, A), \; \lambda \longmapsto \lambda_r,$$

where for $x \in \mathbb{X}_r$, $\lambda_r(x) = \lambda(U_{x/p^r})$.

Lemma 12.2.2. *Let A be an abelian group. The maps*

$$\mathbb{D}(\mathbb{X}, A) \xrightarrow{\sim} \varprojlim_r \mathbb{D}(\mathbb{X}_r, A), \ \lambda \longmapsto (\lambda_r)_r,$$

$$\mathbb{D}(\mathbb{X}, \mathbb{Z}_p) \xrightarrow{\sim} \varprojlim_r \mathbb{D}(\mathbb{X}_r, \mathbb{Z}/p^r\mathbb{Z}), \ \lambda \longmapsto (\lambda_r \mod p^r)_r,$$

are Γ-equivariant isomorphisms.

Proof. The first isomorphism follows from the fact that a measure $\lambda \in \mathbb{D}(\mathbb{X}, A)$ is determined by $\{\lambda(U_{x/p^r})\}$ for $x \in \mathbb{X}_r$ and $r \geq 1$. For the second one, observe that

$$\varprojlim_r \mathbb{D}(\mathbb{X}_r, \mathbb{Z}/p^r\mathbb{Z}) \longrightarrow \mathbb{D}(\mathbb{X}, \mathbb{Z}_p), \ (\lambda_r) \longmapsto \lambda,$$

where λ is defined by

$$\lambda(U_{x/p^r}) = \Bigg(\sum_{\substack{x' \in \mathbb{X}_m \\ x' \equiv x \mod p^r}} \lambda_r(x') \Bigg)_{m \geq r} \in \varprojlim \mathbb{Z}/p^m\mathbb{Z} = \mathbb{Z}_p$$

provides an inverse to the second morphism. □

We now aim to combine the compatible system of classes $(\mu_r)_r$ to a group cohomology class valued on $\mathbb{D}(\mathbb{X}, \mathbb{Z}_p)$. First, we need to recall the following fact regarding the cohomology of congruence subgroups of Γ.

Lemma 12.2.3. *For every $r \geq 1$ and $i \geq 0$, the cohomology group $H^i(\Gamma_r, \mathbb{Z}/p^r\mathbb{Z})$ is finite.*

Proof. By [BS73, Theorem 11.4], the group Γ_r is of type (WFL). In particular, it is of type (VFL). By the Remark in Page 101 of Section 1.8 of [Ser71], it follows that $H^i(\Gamma_r, \mathbb{Z}/p^r\mathbb{Z})$ is finitely generated over $\mathbb{Z}$. Since it is also a torsion group, we deduce that it is finite, as desired. We refer the reader to Section 1.2 and Section 1.8 of [Ser71] for the definitions and properties of groups of type (FL), (VFL) and (WFL). □

Proposition 12.2.4. *For every $i \in \mathbb{Z}_{\geq 0}$, the map $\lambda \mapsto (\lambda_r)_r$ of Lemma 12.2.2 induces an isomorphism*

$$H^i(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}_p)) \xrightarrow{\sim} \varprojlim_r H^i(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{Z}_p)).$$

Proof. We begin proving that the second map of Lemma 12.2.2 induces an isomorphism

$$H^i(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}_p)) \xrightarrow{\sim} \varprojlim_r H^i(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{Z}/p^r\mathbb{Z})). \tag{12.4}$$

To simplify the notation, denote $\mathbb{D} := \mathbb{D}(\mathbb{X}, \mathbb{Z}_p)$ and $\mathbb{D}_r := \mathbb{D}(\mathbb{X}_r, \mathbb{Z}/p^r\mathbb{Z})$. For a group G, a G-module M, and $j \in \mathbb{Z}_{\geq 0}$, let

$$C^j(G, M) := \mathrm{Hom}_G(\mathbb{Z}[G^{j+1}], M),$$

where the action of G in G^{j+1} is diagonal. The complex $C^\bullet(G, M)$ with the usual coboundary maps computes the group cohomology of G with coefficients in M.

The surjective morphisms $u_r \mod p^{r-1} \colon \mathbb{D}_r \to \mathbb{D}_{r-1}$ obtained by taking the maps in (12.2) modulo p^{r-1} induce a map

$$u = (u_r) \colon \prod_{r\geq 1} C^i(\Gamma, \mathbb{D}_r) \twoheadrightarrow \prod_{r \geq 1} C^i(\Gamma, \mathbb{D}_r).$$

Since u_r is surjective for every r, u is surjective. It can be deduced from there and the expression of u, that $1-u$ is also surjective, where 1 denotes the identity. In particular, we have a short exact sequence of complexes

$$0 \longrightarrow C^\bullet(\Gamma, \mathbb{D}) \longrightarrow \prod_{r\geq 1} C^\bullet(\Gamma, \mathbb{D}_r) \xrightarrow{1-u} \prod_{r\geq 1} C^\bullet(\Gamma, \mathbb{D}_r) \longrightarrow 0.$$

Note that to justify exactness in the middle, we used that $\mathbb{D} = \varprojlim_r \mathbb{D}_r$ by Lemma 12.2.2. The corresponding long exact sequence in cohomology yields to the short exact sequence

$$0 \longrightarrow R^1 \varprojlim_r H^{i-1}(\Gamma, \mathbb{D}_r) \longrightarrow H^i(\Gamma, \mathbb{D}) \longrightarrow \varprojlim_r H^i(\Gamma, \mathbb{D}_r) \longrightarrow 0, \tag{12.5}$$

where we used that

$$R^1 \varprojlim_r H^{i-1}(\Gamma, \mathbb{D}_r) = \operatorname{coker}\left(\prod_{r\geq 1} H^{i-1}(\Gamma, \mathbb{D}_r) \xrightarrow{1-u} \prod_{r\geq 1} H^{i-1}(\Gamma, \mathbb{D}_r) \right).$$

Finally, since $H^{i-1}(\Gamma, \mathbb{D}_r) \simeq H^{i-1}(\Gamma_r, \mathbb{Z}/p^r\mathbb{Z})$ is finite for every i by Lemma 12.2.3, it follows that $(H^{i-1}(\Gamma, \mathbb{D}_r))_r$ satisfies the Mittag–Leffler condition. Thus, $R^1 \varprojlim_r H^{i-1}(\Gamma, \mathbb{D}_r) = 0$ for every i proving the isomorphism (12.4).

We conclude the proof using (12.4). For that, observe that we have the commutative diagram

$$\begin{array}{ccccc} R^1 \varprojlim H^{i-1}(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}_p)) & \hookrightarrow & H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}_p)) & \twoheadrightarrow & \varprojlim H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{Z}_p)) \\ \downarrow & & \downarrow \scriptstyle{\mathrm{Id}} & & \downarrow \\ 0 & \hookrightarrow & H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}_p)) & \twoheadrightarrow & \varprojlim H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{Z}/p^r\mathbb{Z})), \end{array}$$

where the first row is obtained in the exact same way as above. The diagram implies that $R^1 \varprojlim H^{i-1}(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}_p)) = 0$ and we are done. □

Definition 12.2.5. Define

$$\mu \in H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}_p))$$

to be the class corresponding to $(\mu_r)_r$ via the isomorphism of Proposition 12.2.4.

12.3 Cocycle with coefficients in $\mathbb{R}$-distributions

Using the differential form ${}_cE_\psi$ introduced in Section 11, which represents the Eisenstein class, we give an explicit representative of the image of $\mu_r \in H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{Z}[1/m]))$ in $H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{R}))$. This will be used to lift μ to a class valued in measures of total mass zero and to compare our p-adic constructions to special values of L-functions.

Lemma 12.3.1. *Let $r \geq 1$ and let $z \in \mathscr{X}$ be an arbitrary point. The map*

$$c_{v_r^* {}_cE_\psi} : \Gamma_r^n \longrightarrow \mathbb{R}, \ (\gamma_0, \ldots, \gamma_{n-1}) \longmapsto \int_{\Delta(\gamma_0 z, \ldots, \gamma_{n-1} z)} v_r^* {}_cE_\psi,$$

where $\Delta(\gamma_0 z, \ldots, \gamma_{n-1} z)$ denotes the geodesic simplex in $\mathscr{X}$ with vertices $\{\gamma_i z\}_i$, defines a group cocycle and represents the class $c_r \in H^{n-1}(\Gamma_r, \mathbb{R})$.

Proof. The form $v_r^* {}_cE_\psi$ on $\mathscr{X}$ is closed and invariant under the action of Γ_r. It follows from there that $c_{v_r^* {}_cE_\psi}$ is a group cocycle and its cohomology class is independent of the choice of point $z \in \mathscr{X}$.

We proceed to see that the class of $c_{v_r^* {}_cE_\psi}$ is c_r. For this, note that Theorem 11.2.4 implies that $v_r^* {}_cE_\psi$ descends to a closed differential form on $\Gamma_r(q)\backslash\mathscr{X}$ representing $v_r^* z_r \in H^{n-1}(\Gamma_r(q)\backslash\mathscr{X}, \mathbb{R})$. Thus the image of $v_r^* z_r$ by the isomorphism (12.1) (with coefficients in $\mathbb{R}$) is represented by the restriction of $c_{v_r^* {}_cE_\psi}$ to $\Gamma_r(q)^n$. In particular, it follows from the definition of c_r that $[c_{v_r^* {}_cE_\psi}] = c_r$. □

Fix $z \in \mathscr{X}$ an arbitrary point. Define a cocycle

$$\mu_{v_r^* {}_cE_\psi} : \Gamma^n \longrightarrow \mathbb{D}(\mathbb{X}_r, \mathbb{R}), \ (\gamma_0, \ldots, \gamma_{n-1}) \longmapsto \left(\bar{x} \longmapsto \int_{\Delta(\gamma_0 z, \ldots, \gamma_{n-1} z)} (x/p^r)^* {}_cE_\psi \right),$$

where $x \in \mathbb{Z}^n$ is a lift of $\bar{x} \in \mathbb{X}_r$, $z \in \mathscr{X}$ denotes a fixed arbitrary base point, and $\Delta(\gamma_0 z, \ldots, \gamma_{n-1} z)$ is defined as in the lemma above.

Proposition 12.3.2. *We have $[\mu_{v_r^* {}_cE_\psi}] = \mu_r$ when viewed as classes in $H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{R}))$.*

Proof. First observe that $\mu_{v_r^* {}_cE_\psi}$ is a group cocycle. This follows from the fact that ${}_cE_\psi$ is closed and invariant under Γ. Now, Shapiro's lemma gives an isomorphism

$$H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{R})) \xrightarrow{\sim} H^{n-1}(\Gamma_r, \mathbb{R}), \ [\varphi] \longmapsto [c_\varphi]$$

defined as follows. If $\gamma_0, \ldots, \gamma_{n-1} \in \Gamma_r$, and $x_r = (1, 0, \ldots, 0)^t \in \mathbb{X}_r$,

$$c_\varphi(\gamma_0, \ldots, \gamma_{n-1}) = \varphi(\gamma_0, \ldots, \gamma_{n-1})(x_r).$$

Since the image of $\mu_{v_r^* {}_cE_\psi}$ by this map is $c_{v_r^* {}_cE_\psi}$, it follows from Lemma 12.3.1 and the definition of μ_r (see Definition 12.2.1) that $[\mu_{v_r^* {}_cE_\psi}] = \mu_r$ in $H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{R}))$. □

Consider the Γ-equivariant morphism given by taking the total mass of a distribution

$$\mathbb{D}(\mathbb{X}_1, \mathbb{R}) \longrightarrow \mathbb{R}, \; \lambda \longmapsto \sum_{x \in \mathbb{X}_1} \lambda(x).$$

Corollary 12.3.3. *The corestriction map $H^{n-1}(\Gamma_1, \mathbb{R}) \to H^{n-1}(\Gamma, \mathbb{R})$ maps c_1 to 0. In particular, the morphism induced by taking the total mass of a measure*

$$H^{n-1}(\Gamma_1, \mathbb{D}(\mathbb{X}_1, \mathbb{R})) \longrightarrow H^{n-1}(\Gamma, \mathbb{R})$$

maps μ_1 to 0.

Proof. The corestriction map can be written as

$$H^{n-1}(\Gamma_1, \mathbb{R}) \xrightarrow{\sim} H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_1, \mathbb{R})) \longrightarrow H^{n-1}(\Gamma, \mathbb{R}),$$

where the first map is given by the inverse of the map given by Shapiro's lemma, and the second one is the map induced by taking the total mass of a measure (see [Bro82, Chapter III, Section 9 (A)]). In view of this observation and of Proposition 12.3.2, it is enough to prove that the image of $[\mu_{v_1^* {}_c E_\psi}]$ by the second map is trivial. For that, observe that such image is represented by the cocycle

$$(\gamma_0, \dots, \gamma_{n-1}) \longmapsto \int_{\Delta(\gamma_0 z, \dots, \gamma_{n-1} z)} \sum_{x \in \mathbb{X}_1} (x/p)^* {}_c E_\psi.$$

It follows from Proposition 11.3.3 that the sum of differential forms in the integral is equal to zero, giving the desired result. □

12.4 Lifting to measures of total mass zero

To construct rigid classes, it is useful to lift the class μ to a class with coefficients in measures of total mass zero. To define such a lift, we first study the action of the involution induced by $w = \mathrm{diag}(1, -1, 1, \dots, 1) \in \mathrm{GL}_n(\mathbb{Z})$ in group cohomology. Conjugation by w induces the automorphism $\alpha \colon \Gamma \to \Gamma$, $\alpha(\gamma) = w\gamma w$. Then, for every $\mathrm{GL}_n(\mathbb{Z})$-module M, we can consider the morphism of complexes

$$C^\bullet(\Gamma, M) \longrightarrow C^\bullet(\Gamma, M), \; c \longmapsto w \circ c \circ \alpha^r,$$

which induces an involution w on $H^i(\Gamma, M)$. We will denote by $H^i(\Gamma, M)^-$ the (-1)-eigenspace for w. Observe that $\mathrm{GL}_n(\mathbb{Z})$ acts on $\mathbb{D}(\mathbb{X}, A)$ on the left via $(g \cdot \mu)(U) = \mu(g^{-1}U)$. In particular, we can consider the involution w on $H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}_p))$.

Proposition 12.4.1. *Let $\mu \in H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}_p))$ be the class constructed in Definition 12.2.5. Then, $w\mu = -\mu$ in $H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}_p))$.*

Proof. Since w normalizes Γ_r for every r, it acts on the cohomology groups $H^{n-1}(\Gamma_r, \mathbb{Z}_p)$. From there, it follows that it also acts in the inverse limit $\varprojlim_r H^{n-1}(\Gamma_r, \mathbb{Z}_p)$. Moreover, the isomorphism from Proposition 12.2.4

$$H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}_p)) \xrightarrow{\sim} \varprojlim_r H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{Z}_p)) \xrightarrow{\sim} \varprojlim_r H^{n-1}(\Gamma_r, \mathbb{Z}_p)$$

is equivariant with respect to the involution w. Therefore, the result follows from Lemma 10.2.4. □

Let $\mathbb{D}_0 := \mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p)$ be the sub-module of $\mathbb{D} := \mathbb{D}(\mathbb{X}, \mathbb{Z}_p)$ consisting of measures $\lambda \in \mathbb{D}(\mathbb{X}, \mathbb{Z}_p)$ such that $\lambda(\mathbb{X}) = 0$. Consider the short exact sequence

$$0 \longrightarrow \mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p) \longrightarrow \mathbb{D}(\mathbb{X}, \mathbb{Z}_p) \longrightarrow \mathbb{Z}_p \longrightarrow 0. \tag{12.6}$$

Proposition 12.4.2. *The image of μ by the map $H^{n-1}(\Gamma, \mathbb{D}) \to H^{n-1}(\Gamma, \mathbb{Z}_p)$ is torsion.*

Proof. The result follows from Proposition 12.3.3. □

As we explained above, $w = \mathrm{diag}(1, -1, 1, \ldots, 1) \in \mathrm{GL}_n(\mathbb{Z})$ acts on the cohomology groups $H^i(\Gamma, \mathbb{Z}_p)$, $H^i(\Gamma, \mathbb{D}_0)$, and $H^i(\Gamma, \mathbb{D})$. Moreover, (12.6) yields a long exact sequence

$$H^{n-2}(\Gamma, \mathbb{Q}_p)^- \longrightarrow H^{n-1}(\Gamma, \mathbb{D}_0)^-_{\mathbb{Q}} \longrightarrow H^{n-1}(\Gamma, \mathbb{D})^-_{\mathbb{Q}} \longrightarrow H^{n-1}(\Gamma, \mathbb{Q}_p)^-,$$

where the subindex denotes taking the tensor product with $\mathbb{Q}$ over $\mathbb{Z}$. By the previous proposition, μ lifts to a class in $H^{n-1}(\Gamma, \mathbb{D}_0)^-_{\mathbb{Q}}$. The lift is well-defined up to $H^{n-2}(\Gamma, \mathbb{Q}_p)^-$, but the following theorem asserts that this group is zero.

Theorem 12.4.3. *The cohomology group $H^{n-2}(\Gamma, \mathbb{Q}_p)^-$ is trivial.*

Proof. This follows from the work of Li–Sun, [LS19]. More precisely, Example 1.10 of [LS19] states

$$H^{n-2}(\mathrm{GL}_n(\mathbb{Z}), \mathbb{R}(\det)) = 0.$$

Using Shapiro's lemma, this is equivalent to $H^{n-2}(\Gamma, \mathbb{R})^- = 0$. From there, the universal coefficient theorem implies that $H^{n-2}(\Gamma, \mathbb{Z})^-$ is torsion. Moreover, $H^i(\Gamma, \mathbb{Z})$ is finitely generated for every i, and therefore $H^{n-2}(\Gamma, \mathbb{Z})^-$ is finite. Another application of the universal coefficient theorem implies that $H^{n-2}(\Gamma, \mathbb{Q}_p)^- = 0$. □

Here and for the rest of the paper, we will fix

$$\mu_0 \in H^{n-1}(\Gamma, \mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p))^-_{\mathbb{Q}} \tag{12.7}$$

a lift of μ, which by the previous theorem, is unique up to a torsion subgroup.

Section 13

Drinfeld's symmetric domain and log-rigid classes

In this section, we introduce Drinfeld's p-adic symmetric domain $\mathscr{X}_p$. Then, we define a lift from measures on $\mathbb{X} = \mathbb{Z}_p^n - p\mathbb{Z}_p^n$ of total mass zero to log-rigid analytic functions on $\mathscr{X}_p$. This leads to construct a log-rigid class J_{Eis} as the image of the class $\mu_0 \in H^{n-1}(\Gamma, \mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p))$ of the previous section by such lift. We conclude by defining the evaluation of J_{Eis} at points $\tau \in \mathscr{X}_p$ attached to totally real fields where p is inert.

13.1 Drinfeld's domain and rigid functions

Drinfeld's p-adic symmetric domain is defined as $\mathscr{X}_p := \mathbb{P}^{n-1}(\mathbb{C}_p) - \bigcup_{H\in\mathcal{H}} H$, where $\mathcal{H}$ is the set of all $\mathbb{Q}_p$-rational hyperplanes. It has a the structure of a rigid analytic space, which we proceed to describe following [SS91].

For a given $H \in \mathcal{H}$, let ℓ_H be an equation of H whose coefficients form a unimodular vector in $\mathbb{C}_p^n$; that is, all coefficients lie in $\mathcal{O}_{\mathbb{C}_p}$, and at least one of them has p-adic norm equal to 1. Also, if $z \in \mathbb{P}^{n-1}(\mathbb{C}_p)$, we will always assume $z = [(z_0, \dots, z_{n-1})]$ is represented by a unimodular vector. For $m \geq 1$, define

$$\mathscr{X}_p^{\leq m} := \{z \in \mathbb{P}^{n-1}(\mathbb{C}_p) \mid \mathrm{ord}_p(\ell_H(z)) \leq m, \text{ for all } H \in \mathcal{H}\}.$$

The family $\{\mathscr{X}_p^{\leq m}\}_m$ forms an admissible covering of $\mathscr{X}_p$ by open affinoid subdomains.

The ring of rigid functions on $\mathscr{X}_p^{\leq m}$ can be described as follows. Let $\mathcal{H}_m$ be the set of equivalence classes of $\mathcal{H}$ modulo p^m. Also, fix $\bar{\mathcal{H}}_m$ a set of representatives for the equivalence classes in $\mathcal{H}_{m+1}$ containing the coordinate hyperplanes $H_i = \{z_i = 0\}$ for every $i = 0, \dots, n-1$. For $H, H' \in \mathcal{H}$, define the function $f_{H,H'} \colon \mathscr{X}_p \to \mathbb{C}_p$

$$f_{H,H'}(z) := \frac{\ell_H(z)}{\ell_{H'}(z)}.$$

Then, observe that we can describe

$$\mathscr{X}_p^{\leq m} = \{z \in \mathscr{X}_p \mid \mathrm{ord}_p(f_{H,H'}(z)) \geq -m \text{ for all } H, H' \in \bar{\mathcal{H}}_m\}.$$

Let A_m be the affinoid $\mathbb{Q}_p$-algebra obtained as the quotient of the free Tate algebra over $\mathbb{Q}_p$ in the indeterminates $\{T_{H,H'}\}_{H,H' \in \bar{\mathcal{H}}_m}$ modulo the closed ideal generated by

$$\begin{aligned} &T_{H,H} - p^m, \text{ for } H \in \bar{\mathcal{H}}_m \\ &T_{H,H'}T_{H',H''} - p^m T_{H,H''}, \text{ for } H, H', H'' \in \bar{\mathcal{H}}_m, \\ &T_{H,H_j} - \sum_{i=0}^{r-1} \lambda_i T_{H_i,H_j}, \text{ if } \ell_H(z) = \sum_{i=0}^{n-1} \lambda_i z_i \text{ for } H \in \bar{\mathcal{H}}_m \text{ and } 0 \leq j \leq n-1. \end{aligned}$$

The previous descriptions of $\mathscr{X}_p^{\leq m}$ and A_m lead to the following result.

Proposition 13.1.1. *Denote by $\mathcal{A}^{\leq m}$ the ring of rigid analytic functions on $\mathscr{X}_p^{\leq m}$. Then, we have an isomorphism of $\mathbb{Q}_p$-algebras*

$$A_m \xrightarrow{\sim} \mathcal{A}^{\leq m}, \ T_{H,H'} \longmapsto p^m f_{H,H'}.$$

In particular, it induces an isomorphism of rigid spaces $\mathscr{X}_p^{\leq m} \xrightarrow{\sim} \mathrm{Sp}(A_m)$.

Proof. See proof of Proposition 4 of [SS91]. □

In particular, $\mathcal{A}^{\leq m}$ is a Banach algebra with respect to the supremum norm.

Definition 13.1.2. The ring of rigid analytic functions on $\mathscr{X}_p$, denoted by $\mathcal{A}$, is the space of functions $f\colon \mathscr{X}_p \to \mathbb{C}_p$ such that for every m, their restriction to $\mathscr{X}_p^{\leq m}$ belongs to $\mathcal{A}^{\leq m}$.

We will also consider a larger space of functions on $\mathscr{X}_p$, called log-rigid analytic functions. Let $\log_p\colon \mathbb{C}_p^\times \to \mathbb{C}_p$ be the branch of the p-adic logarithm satisfying $\log_p(p) = 0$. A function $f\colon \mathscr{X}_p^{\leq m} \to \mathbb{C}_p$ is log-rigid analytic on $\mathscr{X}_p^{\leq m}$ if it can be written as

$$f = f_0 + \sum_{H,H' \in \mathcal{H}} c_{H,H'} \log_p(f_{H,H'}(z)),$$

where $f_0 \in \mathcal{A}^{\leq m}$ and $c_{H,H'} \in \mathbb{Q}_p$ are all but finitely many equal to 0. Denote the space of log-rigid analytic functions on $\mathscr{X}_p^{\leq m}$ by $\mathcal{A}_{\mathcal{L}}^{\leq m}$.

Definition 13.1.3. The space of log-rigid analytic functions on $\mathscr{X}_p$, denoted by $\mathcal{A}_{\mathcal{L}}$, is the space of functions $f\colon \mathscr{X}_p \to \mathbb{C}_p$ such that for every m, their restriction to $\mathscr{X}_p^{\leq m}$ belongs to $\mathcal{A}_{\mathcal{L}}^{\leq m}$.

The following lemma will be useful to study log-rigid functions in the next sections.

Lemma 13.1.4. *Let $m \geq 1$, and let $H, H' \in \mathcal{H}$ be hyperplanes with equations ℓ_H and $\ell_{H'}$ which are congruent modulo p^{m+1}. Then, the function*

$$f\colon \mathscr{X}_p^{\leq m} \longrightarrow \mathbb{C}_p,\ z \longmapsto \log_p\left(f_{H,H'}(z)\right)$$

is rigid analytic on $\mathscr{X}_p^{\leq m}$.

Proof. Observe that we can write

$$f(z) = \log_p\left(1 - \frac{\ell_{H'}(z) - \ell_H(z)}{\ell_{H'}(z)}\right).$$

Moreover, since $\ell_H \equiv \ell_{H'} \mod p^{m+1}$ and $z \in \mathscr{X}_p^{\leq m}$, we have

$$\mathrm{ord}_p\left(\frac{\ell_{H'}(z) - \ell_H(z)}{\ell_{H'}(z)}\right) \geq 1.$$

Therefore,

$$f(z) = \sum_{k \geq 1} \frac{1}{k}\left(\frac{\ell_{H'}(z) - \ell_H(z)}{\ell_{H'}(z)}\right)^k,$$

which is rigid analytic on $\mathscr{X}_p^{\leq m}$. □

Observe that matrix multiplication induces a right action of $\mathrm{SL}_n(\mathbb{Q}_p)$ on $\mathscr{X}_p$ given as follows. For $g \in \mathrm{SL}_n(\mathbb{Q}_p)$ and $z \in \mathscr{X}_p$ represented by a vector in $\mathbb{C}_p^n$, that we also denote by z, we have

$$(z, g) := [g^t z],$$

where $g^t \in \mathrm{SL}_n(\mathbb{Q}_p)$ denotes the transpose of g. This induces a left action of $\mathrm{SL}_n(\mathbb{Q}_p)$ on the space of $\mathbb{C}_p$-valued functions on $\mathscr{X}_p$. If $g \in \mathrm{SL}_n(\mathbb{Q}_p)$, f is a function on $\mathscr{X}_p$, and $z \in \mathscr{X}_p$

$$(g \cdot f)(z) := f(g^t z).$$

This action preserves the subspaces $\mathcal{A}$ and $\mathcal{A}_{\mathcal{L}}$.

13.2 Lifts from measures to functions on $\mathscr{X}_p$

Recall that $\mathscr{X}_p$ consists of the points in $\mathbb{P}^{n-1}(\mathbb{C}_p)$ that do not belong to a $\mathbb{Q}_p$-rational hyperplane. On the other hand, a point in $\mathbb{X} = \mathbb{Z}_p^n - p\mathbb{Z}_p^n$ gives the equation of a $\mathbb{Q}_p$-rational hyperplane. This suggests to consider the two variable function

$$\left(\mathbb{C}_p^n - \bigcup_{H \in \mathcal{H}} H\right) \times \mathbb{X} \longrightarrow \mathbb{C}_p,\ (z, x) \longmapsto \log_p(z^t \cdot x),$$

Integration with respect to the variable $x \in \mathbb{X}$ will induce a map from total mass zero measures on $\mathbb{X}$ to functions on $\mathscr{X}_p$.

Lemma 13.2.1. *Let $\lambda \in \mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p)$. The function $F\colon \mathscr{X}_p \longrightarrow \mathbb{C}_p$ given by*

$$z \longmapsto F(z) := \int_{\mathbb{X}} \log_p(z^t \cdot x) d\lambda,$$

where z in the right hand side denotes an arbitrary representative in $\mathbb{C}_p^n$ of $z \in \mathscr{X}_p$, is well-defined and belongs to $\mathcal{A}_{\mathcal{L}}$.

Proof. For every $r \geq 1$, fix V_r a set of representatives in $\mathbb{Z}^n$ of $\mathbb{X}_r = (\mathbb{Z}/p^r\mathbb{Z})^n - (p\mathbb{Z}/p^r\mathbb{Z})^n$ and define

$$f_r\colon \mathscr{X}_p \longrightarrow \mathbb{C}_p,\ z \longmapsto \sum_{v \in V_r} \lambda(U_{v/p^r}) \log_p(z^t \cdot v),$$

where $U_{v/p^r} \subset \mathbb{X}$ is as in (12.3). Observe that since $\lambda(\mathbb{X}) = 0$, $f_r(z)$ is independent of the choice of representative of z in $\mathbb{C}_p^n$, showing that f_r is a well-defined function. For the rest of the proof we will assume that the representative of z (also denoted z) is a unimodular vector. We follow the next steps:

- F is a well-defined function on $\mathscr{X}_p$. Indeed, for $z \in \mathbb{C}_p^n - \bigcup_{H \in \mathcal{H}} H$, the function $x \in \mathbb{X} \mapsto \log_p(z^t \cdot x)$ is continuous on the compact set $\mathbb{X}$. Thus, the integral defining $F(z)$ converges and we have pointwise convergence

$$F(z) = \lim_{r \to +\infty} f_r(z).$$

- The sequence $(f_{r|\mathscr{X}_p^{\leq m}})$ converges to $F_{|\mathscr{X}_p^{\leq m}}$ with respect to the sup norm for $m \geq 1$. To simplify the notation, denote by (f_r) and F the restrictions of these functions to $\mathscr{X}_p^{\leq m}$. To prove that $(f_r)_r$ converges to F with respect to the sup norm it is enough to see that $(f_r)_r$ is Cauchy with respect to this norm. Observe that, if we let $\pi\colon V_{r+1} \twoheadrightarrow V_r$ be the lift of the reduction modulo p^r map $\mathbb{X}_{r+1} \twoheadrightarrow \mathbb{X}_r$ and use that λ is a measure, we have

$$\begin{aligned} f_{r+1}(z) - f_r(z) &= \sum_{v \in V_{r+1}} \lambda(U_{v/p^{r+1}}) \log_p\left(\frac{z^t \cdot v}{z^t \cdot \pi(v)}\right) \\ &= \sum_{v \in V_{r+1}} \lambda(U_{v/p^{r+1}}) \log_p\left(1 + \frac{z^t \cdot (v - \pi(v))}{z^t \cdot \pi(v)}\right). \end{aligned}$$

Since $v \equiv \pi(v) \mod p^r$, we deduce that for every $z \in \mathscr{X}_p^{\leq m}$

$$\operatorname{ord}_p\left(\frac{z^t \cdot (v - \pi(v))}{z^t \cdot \pi(v)}\right) \geq r - m,$$

Thus, if $r > m$, we can use the power series expansion of $\log(1 + x)$ to deduce that

$$\operatorname{ord}_p(f_{r+1}(z) - f_r(z)) \geq r - m \text{ for all } z \in \mathscr{X}_p^{\leq m}$$

It follows from there that $(f_r)_r$ is Cauchy.

- $F \in \mathcal{A}_{\mathcal{L}}$. Let $m \geq 1$ and denote by (f_r) and F the restrictions of these functions to $\mathscr{X}_p^{\leq m}$. It is enough to see that F belongs to $\mathcal{A}_{\mathcal{L}}^{\leq m}$. With this aim, write

$$F = \left(\lim_{r\to+\infty}(f_r - f_{m+1})\right) + f_{m+1}.$$

We claim that $\lim_{r\to+\infty}(f_r - f_{m+1})$ is a rigid analytic function. Indeed, we can write

$$f_r(z) - f_{m+1}(z) = \sum_{v\in V_{r+1}} \lambda(U_{v/p^{r+1}}) \log_p\left(\frac{z^t \cdot v}{z^t \cdot \pi^{r-(m+1)}(v)}\right).$$

Since $v \equiv \pi^{r-(m+1)}(v) \mod p^{m+1}$, it follows from Lemma 13.1.4, that $f_r - f_{m+1}$ is rigid analytic on $\mathscr{X}_p^{\leq m}$. Then, since the sequence $(f_r - f_{m+1})$ converges with respect to the sup norm by the previous point of this proof, and $\mathcal{A}^{\leq m}$ is complete with respect to this norm, we deduce the desired claim.

On the other hand, since λ has total mass zero, we have that $f_{m+1} \in A_{\mathcal{L}}^{\leq m}$, as it can be written as a linear combination of $\log_p(f_{H,H'}(z))$ for $\mathbb{Q}_p$-rational hyperplanes $H, H' \in \mathcal{H}$. Hence, we deduce that $F \in \mathcal{A}_{\mathcal{L}}^{\leq m}$ and we are done.

□

In view of the previous lemma, we can define a lift from measures of total mass zero to log-rigid analytic functions on $\mathscr{X}_p$.

Definition 13.2.2. Let ST be the morphism given by

$$\mathrm{ST}\colon \mathbb{D}_0(\mathbb{X},\mathbb{Z}) \longrightarrow \mathcal{A}_{\mathcal{L}},\ \lambda \longmapsto \left(z \longmapsto \int_{\mathbb{X}} \log_p(z^t \cdot x) d\lambda\right).$$

The morphism ST is Γ-equivariant. Therefore, it induces a map in cohomology

$$\mathrm{ST}\colon H^{n-1}(\Gamma, \mathbb{D}_0(\mathbb{X},\mathbb{Z}_p)) \longrightarrow H^{n-1}(\Gamma, \mathcal{A}_{\mathcal{L}}).$$

Using this map, we obtain our desired log-rigid analytic class.

Definition 13.2.3. Let $\mu_0 \in H^{n-1}(\Gamma, \mathbb{D}_0(\mathbb{X},\mathbb{Z}_p))_{\mathbb{Q}}$ be as in (12.7). Define

$$J_{\mathrm{Eis}} := \mathrm{ST}(\mu_0) \in H^{n-1}(\Gamma, \mathcal{A}_{\mathcal{L}})_{\mathbb{Q}}.$$

13.3 Evaluation at totally real fields where p is inert

Let F be a totally real field of degree n where p is inert and denote by $\sigma_1, \ldots, \sigma_n$ the collection of embeddings of F into $\mathbb{R}$. Let $\mathfrak{a}$ be an integral ideal of F of norm coprime to pc. Fix $\{\tau_1, \ldots, \tau_n\}$ an oriented $\mathbb{Z}$-basis of $\mathfrak{a}^{-1}$, in the sense that the square matrix

$(\sigma_i(\tau_j))_{i,j}$ has positive determinant, and let $\tau \in F^n$ be the column vector whose ith entry is equal to τ_i. The vector τ induces an isomorphism of $\mathbb{Q}$-vector spaces

$$\mathbb{Q}^n \xrightarrow{\sim} F, \ x \longmapsto \tau^t \cdot x.$$

The action of multiplication by $F^\times$ on F, which is $\mathbb{Q}$-linear, gives an embedding

$$F \hookrightarrow \mathrm{M}_n(\mathbb{Q}), \ \alpha \longmapsto A_\alpha \tag{13.1}$$

determined by the following property: for $\alpha \in F$ and $x \in \mathbb{Q}^n$, $\alpha(\tau^t \cdot x) = \tau^t \cdot (A_\alpha x)$.

Lemma 13.3.1. *The element $\tau \in \mathbb{P}^{n-1}(\mathbb{C}_p)$ belongs to $\mathscr{X}_p$ and is fixed by $F^1 \hookrightarrow \mathrm{SL}_n(\mathbb{Q})$.*

Proof. The coordinates of τ give a $\mathbb{Q}$-basis of F. Since p is inert in F, the coordinates of τ also form a $\mathbb{Q}_p$-basis of the completion of F at p. In particular, they are independent over $\mathbb{Q}_p$. In other words, $\tau \in \mathscr{X}_p$. Finally, for every $\alpha \in F$ we have $A_\alpha^t \tau = \alpha\tau$ by the property stated below (14.2). In particular, $\tau \in \mathscr{X}_p$ is fixed by the action of $F^1 \hookrightarrow \mathrm{SL}_n(\mathbb{Q})$. □

Let U_F be the subgroup of totally positive units in $\mathcal{O}_F^\times$. We view U_F as a subgroup of Γ. Consider the following morphism in cohomology induced by evaluation at τ

$$H^{n-1}(\Gamma, \mathcal{A}_\mathcal{L}) \xrightarrow{\mathrm{ev}_\tau} H^{n-1}(U_F, \mathbb{C}_p).$$

By Dirichlet's unit theorem, $U_F \simeq \mathbb{Z}^{n-1}$. Therefore, $H_{n-1}(U_F, \mathbb{Z}) \simeq \mathbb{Z}$, and we can fix a generator of this group $c_{U_F} \in H_{n-1}(U_F, \mathbb{Z})$.

Definition 13.3.2. Consider the same notation as above, and let $J \in H^{n-1}(\Gamma, \mathcal{A}_\mathcal{L})_\mathbb{Q}$. Define the evaluation of J at $[\tau] \in \mathscr{X}_p$ by the cap product

$$J[\tau] := c_{U_F} \frown \mathrm{ev}_\tau(J) \in \mathbb{C}_p.$$

Observe that, since $J_{\mathrm{Eis}} = \mathrm{ST}(\mu_0)$, it follows from the description of the map ST that $J_{\mathrm{Eis}}[\tau] \in F_p$. We also note that this definition depends, up to a sign, of the choice of generator $c_{U_F} \in H_{n-1}(U_F, \mathbb{Z})$. In the next section, we will make a precise choice of generator when comparing the local trace of these values to the local trace of p-adic logarithms of Gross–Stark units.

Section 14

Values of the log-rigid class and the Gross–Stark Conjecture

Let F be a totally real field where p is inert, let $\mathfrak{a}$ be an integral ideal of F coprime to pc, and fix $\tau \in F^n$ a vector whose entries give an oriented $\mathbb{Z}$-basis of $\mathfrak{a}^{-1}$, which yields a point $\tau \in X_p$. Recall the log-rigid analytic class J_{Eis} constructed in the previous section and its value $J_{\mathrm{Eis}}[\tau] \in F_p$ at τ. In this section, we prove

$$\mathrm{Tr}_{F_p/\mathbb{Q}_p} J_{\mathrm{Eis}}[\tau] = -L_p'(1_{[\mathfrak{a}]_p}, 0),$$

where $L_p(1_{[\mathfrak{a}]_p}, s)$ denotes a p-adic partial zeta function attached to the ideal class of $\mathfrak{a}$ in the narrow Hilbert class field of F. From this expression and the rank 1 Gross–Stark conjecture, proved in [DDP11] and [Ven15], we obtain the equality $\mathrm{Tr}_{F_p/\mathbb{Q}_p} J_{\mathrm{Eis}}[\tau] = \mathrm{Tr}_{F_p/\mathbb{Q}_p} \log_p(u_\tau)$, for $u_\tau \in \mathcal{O}_H[1/p]^\times \otimes \mathbb{Q}$ a Gross-Stark unit in the narrow Hilbert class field H of F attached to the ideal class of $\mathfrak{a}$. We conclude conjecturing $J_{\mathrm{Eis}}[\tau] = \log_p(u_\tau)$ and provide an observation that leads to this conjecture.

14.1 Gross–Stark conjecture

We state the Gross–Stark conjecture in a simple setting. For more details we refer the reader to Section 2 of [Das08]. We begin introducing the following notation. For an integral ideal $\mathfrak{f}$ of F, denote by $G_{\mathfrak{f}}$ the ray class group attached to $\mathfrak{f}$. It is obtained by taking the quotient of the set of integral ideals in F which are prime to $\mathfrak{f}$ by the relation

$$\mathfrak{b} \sim_{\mathfrak{f}} \mathfrak{c} \text{ if and only if } \mathfrak{b}\mathfrak{c}^{-1} = (\lambda)$$

for some $\lambda \in 1 + \mathfrak{f}\mathfrak{c}^{-1}$ totally positive. Then, if ε is a $\bar{\mathbb{Q}}$-valued function on $G_{\mathfrak{f}}$, we set

$$L(\varepsilon, s) := \sum_{(\mathfrak{b},\mathfrak{f})=1} \varepsilon(\mathfrak{b}) \mathrm{N}\mathfrak{b}^{-s},$$

where the sum is over integral ideals which are coprime to $\mathfrak{f}$. This sum converges for $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$ and it can be extended via analytic continuation to a meromorphic function at $\mathbb{C}$ with at most a pole at $s = 1$, that we will still denote by $L(\varepsilon, s)$.

Let c be a positive integer prime to p and denote by ε_c the function on $G_{\mathfrak{f}}$ given by $\varepsilon_c(\mathfrak{b}) = \varepsilon(\mathfrak{b}c)$. For $k \in \mathbb{Z}_{\geq 1}$, consider

$$\Delta_c(\varepsilon, 1-k) := L(\varepsilon, 1-k) - c^{nk} L(\varepsilon_c, 1-k).$$

It is result of Klingen and Siegel that $\Delta_c(\varepsilon, 1-k) \in \mathbb{Q}(\varepsilon)$, where $\mathbb{Q}(\varepsilon)$ denotes the field generated by the values of ε. Deligne–Ribet and Cassou–Noguès refined this statement by asserting that these values are p-integral and satisfy numerous congruences modulo powers of p. To state their result, fix here and from now on an embedding $\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_p$. In this way, we will view ε as a function taking values on $\bar{\mathbb{Q}}_p$ and the elements $\Delta_c(\varepsilon, 1-k) \in \bar{\mathbb{Q}}_p$.

Theorem 14.1.1. *Consider the same notation as above and suppose $\varepsilon\colon G_{\mathfrak{f}} \to \mathbb{Z}_p$ takes values in $\mathbb{Z}_p$. Let $k \in \mathbb{Z}_{\geq 1}$. Then:*

1. *We have $\Delta_c(\varepsilon, 1-k) \in \mathbb{Z}_p$.*
2. *Suppose $\mathfrak{f}$ is divisible by p^m and $\eta\colon G_{\mathfrak{f}} \to \mathbb{Z}_p$ is such that $\eta \equiv \mathrm{N}^{k-1} \mod p^m$, the two functions considered as functions on the set of prime to $\mathfrak{f}$ ideals. Then,*

$$\Delta_c(\varepsilon, 1-k) \equiv \Delta_c(\varepsilon\eta, 0) \mod p^m.$$

Proof. See Theorem (2.1) of [Rib79]. □

Let $1_{[\mathfrak{a}],p}\colon G_p \to \mathbb{Z}$ be the characteristic function of the preimage of $[\mathfrak{a}] \in G_1$ by the natural map $G_p \to G_1$. From the congruences stated above, it can be deduced that there exists a p-adic analytic function $L_p(1_{[\mathfrak{a}],p}, s)$ defined on $\mathbb{Z}_p$ characterized by the following interpolation property. For every integer $k \geq 1$ such that $k \equiv 1 \mod [F(\mu_{2p}) : F]$,

$$L_p(1_{[\mathfrak{a}],p}, 1-k) = \Delta_c(1_{[\mathfrak{a}],p}, 1-k). \tag{14.1}$$

We refer the reader to [DR80] for details on the construction of the p-adic L-function $L_p(1_{[\mathfrak{a}],p}, s)$.

Observe that $L(1_{[\mathfrak{a}],p}, s)$ is a partial zeta function with the Euler factor corresponding to p removed. This implies that $L(1_{[\mathfrak{a}],p}, 0) = 0$, and therefore $\Delta_c(1_{[\mathfrak{a}],p}, 0) = 0$. By (14.1), $L_p(1_{[\mathfrak{a}],p}, 0) = 0$ as well. The Gross–Stark conjecture gives an arithmetic interpretation for the value of derivative $L_p'(1_{[\mathfrak{a}],p}, 0)$ with respect to s at $s = 0$. For that, let H be the narrow Hilbert class field of F and consider the following subgroup of p-units in H

$$U_p := \{u \in H^\times \mid |x|_{\mathfrak{Q}} = 1 \ \forall \mathfrak{Q} \nmid p\},$$

where $\mathfrak{Q}$ runs over all archimedean and nonarchimedean places of H not dividing p. Fix $\mathfrak{P}$ a prime of H dividing p.

Proposition 14.1.2. *There exists a unique element $u \in U_p \otimes \mathbb{Q}$ satisfying*

$$\mathrm{ord}_{\mathfrak{P}}(u^{\sigma_{\mathfrak{a}}}) = L(1_{[\mathfrak{a}]}, 0) \text{ for all } \mathfrak{a} \text{ coprime to } p,$$

where $1_{[\mathfrak{a}]}$ denotes the characteristic function of $[\mathfrak{a}]$ on G_1 and $\sigma_{\mathfrak{a}} \in \mathrm{Gal}\,(H/F)$ denotes the Frobenius element associated to $\mathfrak{a}$.

Since p splits completely on H, we have $H \subset H_{\mathfrak{P}} \simeq F_p$.

Theorem 14.1.3 (Gross–Stark conjecture)**.** *Let u be as in Proposition 14.1.2. We have*

$$L_p'(1_{[\mathfrak{a}],p}, 0) = -(1 - c^n) \log_p(\mathrm{N}_{F_p/\mathbb{Q}_p} u^{\sigma_{\mathfrak{a}}}) \text{ for all } \mathfrak{a} \text{ coprime to } p.$$

Proof. See [DDP11] and [Ven15]. □

14.2 Periods of the Eisenstein class along tori attached to totally real fields

We use the differential forms representing the Eisenstein class introduced in Section 11 to prove that pullbacks of the Eisenstein class by torsion sections encode special values of zeta functions attached to totally real fields. This was proven in [BCG20, Section 12] in more generality using an adelic framework, and we specialize their results and outline the proof below for the cases that will be relevant for us. Our calculations are similar to those in Section 4.2 of [BCG23].

Recall that F is a totally real field of degree n where p is inert, $\mathfrak{a}$ is an integral ideal of F prime to pc, and $\tau \in F^n$ a column vector whose entries give a positively oriented $\mathbb{Z}$-basis of $\mathfrak{a}^{-1}$. As we saw in the previous section, τ induces a $\mathbb{Q}$-linear isomorphism

$$\beta\colon \mathbb{Q}^n \xrightarrow{\sim} F,\ x \longmapsto \tau^t \cdot x.$$

The action of multiplication by $F^\times$ on F, which is $\mathbb{Q}$-linear, gives an embedding

$$F \hookrightarrow \mathrm{M}_n(\mathbb{Q}),\ \alpha \longmapsto A_\alpha \tag{14.2}$$

determined by the following property: for all $\alpha \in F$ and $x \in \mathbb{Q}^n$, $\alpha(\tau^t \cdot x) = \tau^t \cdot (A_\alpha x)$. Let $(F \otimes \mathbb{R})^1_+$ be the subset of totally positive elements of norm 1. The embedding (14.2) induces an oriented map (see Section 12.4 of [BCG20] for more details on the orientation)

$$i_\tau\colon (F \otimes \mathbb{R})^1_+ \longrightarrow \mathscr{X}.$$

Denote by U_F the subgroup of totally positive units in $\mathcal{O}_F^\times$. Since U_F has rank $n-1$ by Dirichlet's unit theorem, it follows that

$$X(F) := U_F \backslash (F \otimes \mathbb{R})^1_+ \tag{14.3}$$

is a compact oriented manifold of dimension $n-1$. Let $r \geq 1$, consider $\chi\colon (\mathbb{Z}/p^r\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$ a finite-order character and recall $\mathbb{X}_r := (\mathbb{Z}/p^r\mathbb{Z})^n - (p\mathbb{Z}/p^r\mathbb{Z})^n$. We give a combination of pullbacks of the differential form ${}_cE_\psi$ that we will integrate along the fundamental class of $X(F)$.

Definition 14.2.1. Consider the same notation as above. Define

$$E_{\tau,\chi} := \chi(\mathrm{N}(c\mathfrak{a})) \sum_{\bar{x} \in \mathbb{X}_r} \chi(\mathrm{N}(\tau^t \cdot x))\,(x/p^r)^*\,{}_cE_\psi \in \Omega^{n-1}(\mathscr{X}), \tag{14.4}$$

where N denotes the norm map on elements in F and on ideals of F. For $s \in \mathbb{C}$, we define $E_{\tau,\chi}(s)$ as above but replacing ${}_cE_\psi$ by ${}_cE_\psi(s) = E_\psi(c^{-1}\mathbb{Z}^n, s) - c^n E_\psi(\mathbb{Z}^n, s)$.

Lemma 14.2.2. *The differential form $E_{\tau,\chi}$ on $\mathscr{X}$ is invariant under $U_F \subset \Gamma$, where the inclusion of U_F in Γ is induced by (14.2).*

Proof. For $\gamma \in \Gamma$, note that we have $\gamma^* v^* E_\psi^{(c)} = (\gamma v)^* E_\psi^{(c)}$. Then, if $\gamma \in U_F \subset \Gamma$

$$\begin{aligned}
\gamma^* E_{\tau,\chi} &:= \sum_{\bar{x} \in \mathbb{X}_r} \chi(\mathrm{N}(c\mathfrak{a})\mathrm{N}(\tau^t \cdot x))(\gamma x/p^r)^*{}_cE_\psi \\
&= \sum_{\bar{x} \in \mathbb{X}_r} \chi(\mathrm{N}(c\mathfrak{a})\mathrm{N}(\tau^t \cdot \gamma^{-1} x))(x/p^r)^*{}_cE_\psi \\
&= \sum_{\bar{x} \in \mathbb{X}_r} \chi(\mathrm{N}(c\mathfrak{a})\mathrm{N}(\varepsilon\tau^t \cdot x))(x/p^r)^*{}_cE_\psi \\
&= \sum_{\bar{x} \in \mathbb{X}_r} \chi(\mathrm{N}(c\mathfrak{a})\mathrm{N}(\tau^t \cdot x))(x/p^r)^*{}_cE_\psi,
\end{aligned}$$

where we used that $\tau^t\gamma^{-1} = \varepsilon\tau^t$, for $\varepsilon \in U_F$ the preimage of γ^{-1} by (14.2), and therefore ε has norm 1. □

Thus, $i_\tau^* E_{\tau,\chi}$ defines a closed form on $X(F)$ and we can consider

$$\int_{X(F)} i_\tau^* E_{\tau,\chi}.$$

We will express this integral in terms of L-functions attached to F. Consider the composition

$$\tilde{\chi} = \chi \circ (\mathrm{N} \mod p^r)\colon G_{p^r} \xrightarrow{\mathrm{N} \mod p^r} (\mathbb{Z}/p^r\mathbb{Z})^\times \xrightarrow{\chi} \bar{\mathbb{Q}}^\times,$$

and denote by $1_{[\mathfrak{a}],p^r}$ the characteristic function of the preimage of $[\mathfrak{a}] \in G_1$ via the natural map $G_{p^r} \to G_1$. We can then consider the L-function $L(\tilde{\chi}1_{[\mathfrak{a}],p^r}, s)$, where $\tilde{\chi}1_{[\mathfrak{a}],p^r}$ denotes the function on G_{p^r} obtained as the product of $\tilde{\chi}$ and $1_{[\mathfrak{a}],p^r}$.

Lemma 14.2.3. *We have*

$$L(\tilde{\chi}1_{[\mathfrak{a}],p^r},0)=\lim_{s\to 0}\frac{\chi(\mathrm{N}\mathfrak{a})}{2^n}\sum_{\alpha\in U_F\backslash\mathfrak{a}^{-1}}\frac{\chi(\mathrm{N}\alpha)\mathrm{sign}(\mathrm{N}\alpha)}{|\mathrm{N}\alpha|^s},$$

where $\lim_{s\to 0}$ *denotes evaluation at* $s=0$ *of the analytic continuation of the right hand side.*

Proof. The result can be deduced from equation (7.15) of [Cha07], which is originally due to Siegel ([Sie79]). □

Theorem 14.2.4. *Consider the same notation as above. Then,*

$$\int_{X(F)} i_\tau^* E_{\tau,\chi}=(1-\chi(\mathrm{N}c)c^n)L(\tilde{\chi}1_{[\mathfrak{a}],p^r},0).$$

Proof. For $s\in\mathbb{C}$ such that $\mathrm{Re}(s)\gg 0$, we have

$$\frac{1}{\chi(\mathrm{N}c\mathfrak{a})}\int_{X(F)} i_\tau^* E_{r,\chi}(s)=\int_{X(F)} i_\tau^*\left(\sum_{\bar{v}\in\mathbb{X}_r}\chi(\mathrm{N}(\tau^t\cdot v))\,(v/p^r)^*\,{}_cE_\psi(s)\right)$$
$$=\int_{X(F)} i_\tau^*\left(\sum_{\bar{v}\in\mathbb{X}_r}\chi(\mathrm{N}(\tau^t\cdot v))\left(\sum_{\lambda\in v/p^r+c^{-1}\mathbb{Z}^n}\eta(\lambda,s)-c^n\sum_{\lambda\in v/p^r+\mathbb{Z}^n}\eta(\lambda,s)\right)\right),$$

where we recall that $\eta(\lambda,s)$ was introduced in Section 11.3. Using that $\eta(v/p^r,s)=p^s\eta(v,s)$, and keeping in mind that we will later be interested in evaluating the analytic continuation of the expression above at $s=0$, it is enough to compute

$$\int_{X(F)} i_\tau^*\left(\sum_{\bar{v}\in\mathbb{X}_r}\chi(\mathrm{N}(\tau^t\cdot v))\left(\sum_{\lambda\in v+c^{-1}p^r\mathbb{Z}^n}\eta(\lambda,s)-c^n\sum_{\lambda\in v+p^r\mathbb{Z}^n}\eta(\lambda,s)\right)\right)$$
$$=\int_{X(F)} i_\tau^*\left(\sum_{\bar{v}\in\mathbb{X}_r}\chi(\mathrm{N}(\tau^t\cdot v))\left(\sum_{x\in\beta(v)+c^{-1}p^r\mathfrak{a}^{-1}}\eta(\beta^{-1}x,s)-c^n\sum_{x\in\beta(v)+p^r\mathfrak{a}^{-1}}\eta(\beta^{-1}x,s)\right)\right)$$
$$=\int_{X(F)} i_\tau^*\left(\sum_{x\in c^{-1}\mathfrak{a}^{-1}}\chi(\mathrm{N}(x))\eta(\beta^{-1}x,s)-c^n\sum_{x\in\mathfrak{a}^{-1}}\chi(\mathrm{N}(x))\eta(\beta^{-1}x,s)\right).$$

We can compute the inner sums by first taking representatives of $U_F\backslash c^{-1}\mathfrak{a}^{-1}$ and $U_F\backslash\mathfrak{a}^{-1}$, that we denote by x, and then run over all elements in U_F, denoted by u. Hence, we obtain that the previous expressions can be written as

$$\int_{X(F)} i_\tau^*\left(\sum_{U_F\backslash c^{-1}\mathfrak{a}^{-1}}\sum_{U_F}\chi(\mathrm{N}(ux))\eta(\beta^{-1}ux,s)-c^n\sum_{U_F\backslash\mathfrak{a}^{-1}}\sum_{U_F}\chi(\mathrm{N}(ux))\eta(\beta^{-1}ux,s)\right)$$
$$=\sum_{U_F\backslash c^{-1}\mathfrak{a}^{-1}}\chi(\mathrm{N}(x))\int_{(F\otimes\mathbb{R})^1_+} i_\tau^*\eta(\beta^{-1}x,s)-c^n\sum_{U_F\backslash\mathfrak{a}^{-1}}\chi(\mathrm{N}(x))\int_{(F\otimes\mathbb{R})^1_+} i_\tau^*\eta(\beta^{-1}x,s).$$

From [BCG20, Section 12.8], we have that for $x \in F$

$$\int_{(F\otimes\mathbb{R})^1_+} i_\tau^*\eta(\beta^{-1}x, s) = \pi^{-n/2}2^{s/2-n}\Gamma\left(\frac{s}{2n}+\frac{1}{2}\right)^n \frac{\operatorname{sign}(\mathrm{N}(x))}{|\mathrm{N}(x)|^s}.$$

Hence, we deduce

$$\frac{1}{\chi(\mathrm{N}c\mathfrak{a})}\int_{X(F)} E_{r,\chi} = \frac{1}{2^n}\lim_{s\to 0}\sum_{x\in U_F\backslash c^{-1}\mathfrak{a}^{-1}}\chi(\mathrm{N}(x))\frac{\operatorname{sign}(\mathrm{N}(x))}{|\mathrm{N}(x)|^s} - c^n\sum_{x\in U_F\backslash \mathfrak{a}^{-1}}\chi(\mathrm{N}(x))\frac{\operatorname{sign}(\mathrm{N}(x))}{|\mathrm{N}(x)|^s}.$$

Finally, from Lemma 14.2.3, we obtain

$$\int_{X(F)} i_{\tau,r}^* E_{r,\chi} = (1-\chi(\mathrm{N}c)c^n)L(\tilde{\chi}1_{[\mathfrak{a}],p^r}, 0)$$

as desired. □

14.3 p-adic L-functions

In this section, we state the relation between the class $\mu \in H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}_p))$ constructed in Section 12 and the p-adic L-function $L_p(1_{[\mathfrak{a}],p}, s)$ introduced above. From there, we relate $\mathrm{Tr}_{F_p/\mathbb{Q}_p} J_{\mathrm{Eis}}[\tau]$ to norms of Gross–Stark units.

For $\chi\colon \mathbb{Z}_p^\times \to \bar{\mathbb{Q}}_p^\times$ a continuous function, define the U_F-equivariant map

$$\varphi_\chi\colon \mathbb{D}(\mathbb{X}, \mathbb{Z}_p) \longrightarrow \bar{\mathbb{Q}}_p,\ \ \lambda \longmapsto \int_{\mathbb{X}} \chi\left(\mathrm{N}(c\mathfrak{a})\mathrm{N}_{F_p/\mathbb{Q}_p}(\tau^t\cdot x)\right) d\lambda.$$

It induces a map in cohomology

$$\varphi_\chi\colon H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}_p)) \longrightarrow H^{n-1}(U_F, \bar{\mathbb{Q}}_p).$$

Fix the generator $c_{U_F} \in H_{n-1}(U_F, \mathbb{Z}) \simeq H_{n-1}(X(F), \mathbb{Z}) \simeq \mathbb{Z}$ corresponding to the positive orientation of $X(F)$ in (14.3). We can then consider the cap product

$$c_{U_F} \frown \varphi_\chi(\mu) \in \bar{\mathbb{Q}}_p.$$

To make the notation more transparent, we will write

$$c_{U_F} \frown \varphi_\chi(\mu) = \int_{\mathbb{X}} \chi\left(\mathrm{N}(c\mathfrak{a})\mathrm{N}_{F_p/\mathbb{Q}_p}(\tau^t\cdot x)\right) d\mu(c_{U_F}).$$

When χ is a finite order character, this quantity relates to special values of partial L-functions in the following way.

Lemma 14.3.1. *Let $\chi\colon \mathbb{Z}_p^\times \twoheadrightarrow (\mathbb{Z}_p/p^r\mathbb{Z}_p)^\times \to \bar{\mathbb{Q}}^\times \subset \bar{\mathbb{Q}}_p^\times$ be a character and let $\tilde{\chi} = \chi\circ(\mathrm{N} \bmod p^r)$. Then,*

$$\int_{\mathbb{X}} \chi\left(\mathrm{N}(c\mathfrak{a})\mathrm{N}_{F_p/\mathbb{Q}_p}(\tau^t\cdot x)\right) d\mu(c_{U_F}) = \Delta_c(1_{[\mathfrak{a}],p}\tilde{\chi}, 0).$$

Proof. Consider the U_F-equivariant morphism

$$\varphi_{\chi,r}\colon \mathbb{D}(\mathbb{X}_r,\mathbb{Z}_p) \longrightarrow \bar{\mathbb{Q}}_p,\ \lambda_r \longmapsto \sum_{\bar{v}\in\mathbb{X}_r} \chi\left(\mathrm{N}(c\mathfrak{a})\mathrm{N}(\tau^t\cdot v)\right)\lambda_r(\bar{v}).$$

Since χ factors through $\mathbb{Z}_p/p^r\mathbb{Z}_p$, it follows that

$$c_{U_F} \frown \varphi_\chi(\mu) = c_{U_F} \frown \varphi_{\chi,r}(\mu_r), \tag{14.5}$$

where $\mu_r \in H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{Z}[1/m]))$ is the class described in Definition 12.2.1. In particular, $c_{U_F} \frown \varphi_{\chi,r}(\mu_r) \in \bar{\mathbb{Q}}$. Fix an embedding $\bar{\mathbb{Q}} \subset \mathbb{C}$. Then, the right-hand side of (14.5) can be computed using a representative of the image of μ_r in $H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{R}))$. By Proposition 12.3.2, a representative of μ_r is given by

$$\varphi_r\colon \Gamma^n \longrightarrow \mathbb{D}(\mathbb{X}_r,\mathbb{R}),\ (\gamma_0,\ldots,\gamma_{n-1}) \longmapsto \left(\bar{x} \longmapsto \int_{\Delta(\gamma_0 z,\ldots,\gamma_{n-1}z)} (x/p^r)^*{}_cE_\psi\right),$$

where $z \in \mathscr{X}$ denotes an arbitrary point and $\Delta(\gamma_0 z,\ldots,\gamma_{n-1}z)$ is the geodesic simplex in $\mathscr{X}$ with vertices $\{\gamma_i z\}_i$. Hence, (14.5) can be written as

$$\int_{X(F)} \iota_\tau^* \sum_{\bar{x}\in\mathbb{X}_r} \chi(\mathrm{N}(c\mathfrak{a})\mathrm{N}(\tau^t\cdot x))(x/p^r)^*{}_cE_\psi = \int_{X(F)} \iota_\tau^* E_{\tau,\chi},$$

where $X(F)$ is given in (14.3) and $E_{\tau,\chi}$ in (14.4). By Theorem 14.2.4, the result follows. □

Define the following p-adic analytic function

$$\mathscr{L}_p\colon (\mathbb{Z}/(p-1)\mathbb{Z})\times\mathbb{Z}_p \longrightarrow F_p,\ s \longmapsto \int_{\mathbb{X}} \left(\mathrm{N}(c\mathfrak{a})\mathrm{N}_{F_p/\mathbb{Q}_p}(\tau^t\cdot x)\right)^{-s} d\mu(c_{U_F}),$$

We restrict this function to $\{0\}\times\mathbb{Z}_p \simeq \mathbb{Z}_p$, and we denote the restriction with the same symbol. Using the previous lemma, we can relate $\mathscr{L}_p(s)$ to the p-adic L-function $L_p(1_{[\mathfrak{a}],p}, s)$.

Theorem 14.3.2. *Let $k \in \mathbb{Z}_{\geq 1}$ be such that $k \equiv 1 \mod [F(\mu_{2p}) : F]$. We have $\mathscr{L}_p(1-k) = L_p(1_{[\mathfrak{a}],p}, 1-k)$. In particular, $\mathscr{L}_p(s) = L_p(1_{[\mathfrak{a}],p}, s)$ for every $s \in \mathbb{Z}_p$.*

Proof. It is enough to prove

$$\mathscr{L}_p(1-k) = L_p(1_{[\mathfrak{a}],p}, 1-k) \mod p^r$$

for every $r \geq 1$. For that, fix $r \geq 1$ and let $\varepsilon\colon \mathbb{Z}_p \twoheadrightarrow (\mathbb{Z}/p^r\mathbb{Z})^\times \to \mathbb{Z}$ be a locally constant function such that $\varepsilon(x) \equiv x^{k-1} \mod p^r$ for every $x \in \mathbb{Z}_p^\times$. By Theorem 14.1.1, we have

$$L_p(1_{[\mathfrak{a}],p}, 1-k) \equiv \Delta_c(1_{[\mathfrak{a}],p}\cdot\tilde{\varepsilon}, 0) \mod p^r.$$

On the other hand, it follows from the expression of $L(1-k)$ that

$$\mathscr{L}_p(1-k) \equiv \int_{\mathbb{X}} \varepsilon\left(\mathrm{N}(c\mathfrak{a})\mathrm{N}_{F_p/\mathbb{Q}_p}(\tau^t \cdot x)\right) d\mu(c_{U_F}) \mod p^r.$$

In view of the previous two congruences, it is enough to prove

$$\int_{\mathbb{X}} \varepsilon\left(\mathrm{N}(c\mathfrak{a})\mathrm{N}_{F_p/\mathbb{Q}_p}(\tau^t \cdot x)\right) d\mu(c_{U_F}) = \Delta_c(1_{[\mathfrak{a}],p} \cdot \tilde{\varepsilon}, 0).$$

For that, let $\chi_1, \ldots, \chi_m \colon (\mathbb{Z}/p^r\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$ be characters and $b_1, \ldots, b_m \in \bar{\mathbb{Q}}$ be such that for every $x \in \mathbb{Z}_p$, we have $\varepsilon(x) = \sum_j b_j \chi_j(x)$. Then,

$$\begin{aligned}\int_{\mathbb{X}} \varepsilon\left(\mathrm{N}(c\mathfrak{a})\mathrm{N}_{F_p/\mathbb{Q}_p}(\tau^t \cdot x)\right) d\mu(c_{U_F}) &= \sum_j b_j \int_{\mathbb{X}} \chi_j\left(\mathrm{N}(c\mathfrak{a})\mathrm{N}_{F_p/\mathbb{Q}_p}(\tau^t \cdot x)\right) d\mu(c_{U_F}) \\ &= \sum_j b_j \Delta_c(1_{[\mathfrak{a}],p}\tilde{\chi}_j, 0) \\ &= \Delta_c(1_{[\mathfrak{a}],p}\tilde{\varepsilon}, 0),\end{aligned}$$

where we used Lemma 14.3.1 in the second to last equality. □

As a consequence, we obtain the desired relation between $J_{\mathrm{Eis}}[\tau]$ and norms of Gross–Stark units.

Corollary 14.3.3. *Let $u \in U_p \otimes \mathbb{Q}$ be the Gross–Stark unit introduced in Proposition 14.1.2 and denote by $u_\tau := u^{1-c^n} \in U_p \otimes \mathbb{Q}$. We have,*

$$\mathrm{Tr}_{F_p/\mathbb{Q}_p} J_{\mathrm{Eis}}[\tau] = \mathrm{Tr}_{F_p/\mathbb{Q}_p} \log_p(u_\tau).$$

Proof. Observe that, by viewing $\mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p) \subset \mathbb{D}(\mathbb{X}, \mathbb{Z}_p)$, we can consider $c_{U_F} \frown \varphi_{(\cdot)^{-s}}(\mu_0) \in \bar{\mathbb{Q}}_p$. Moreover, since μ_0 is a lift of μ, we have that for every $s \in \mathbb{Z}_p$,

$$c_{U_F} \frown \varphi_{(\cdot)^{-s}}(\mu_0) = \mathscr{L}_p(s).$$

Then, it follows from the definition of $J_{\mathrm{Eis}} = \mathrm{ST}(\mu_0)$, and the fact that μ_0 takes values on measures of total mass zero, that $\mathrm{Tr}_{F_p/\mathbb{Q}_p} J_{\mathrm{Eis}}[\tau] = -\mathscr{L}_p'(0)$. Hence, the result follows from Theorem 14.3.2 and Theorem 14.1.3. □

14.4 Conjectural relation between $J_{\mathrm{Eis}}[\tau]$ and Gross–Stark units

In view of Corollary 14.3.3, we make the following conjecture.

Conjecture 14.4.1. *Let $u \in U_p \otimes \mathbb{Q}$ be the Gross–Stark unit introduced in Proposition 14.1.2 and denote by $u_\tau := u^{1-c^n} \in U_p \otimes \mathbb{Q}$. We have,*

$$J_{\mathrm{Eis}}[\tau] = \log_p(u_\tau).$$

Remark 14.4.2. Suppose $n = 2$ and consider $\mathcal{J}_{\mathrm{DR}} \in H^1(\mathrm{SL}_2(\mathbb{Z}), \mathcal{A}^\times)^-$ a lift of (the restriction to $\mathrm{SL}_2(\mathbb{Z})$ of) $J_{\mathrm{DR}} \in H^1(\mathrm{SL}_2(\mathbb{Z}[1/p]), \mathcal{A}^\times/\mathbb{C}_p^\times)^-$ constructed in [DPV24]. By comparing the constructions of J_{DR} and J_{Eis}, we deduce

$$J_{\mathrm{Eis}} = \log_p(\mathcal{J}_{\mathrm{DR}}) \in H^1(\mathrm{SL}_2(\mathbb{Z}), \mathcal{A}_{\mathcal{L}}).$$

Then, Theorem B of [DPV24] implies that Conjecture 14.4.1 holds if $n = 2$.

We conclude with some observations to support the conjecture for general $n \geq 2$. Recall the class $\mu \in H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}_p))^-$ constructed in Section 12 and denote by $\mu_{|U_F}$ its restriction to $U_F \subset \Gamma$. Consider the U_F-equivariant morphism

$$\bar{\mathcal{E}} \colon \mathbb{D}(\mathbb{X}, \mathbb{Z}_p) \longrightarrow F_p/\mathbb{Z}_p \log_p(\mathcal{O}_F^\times), \ \lambda \longmapsto \int_{\mathbb{X}} \log_p(\tau^t \cdot x) d\lambda,$$

where $\mathbb{Z}_p \log_p(\mathcal{O}_F^\times)$ denotes the $\mathbb{Z}_p$-span of $\log_p(\mathcal{O}_F^\times)$ in F_p (i.e. its completion in F_p). Corollary 14.3.3 implies that

$$c_{U_F} \frown \bar{\mathcal{E}}(\mu_{|U_F}) = u_\tau \mod \mathbb{Z}_p \log_p(\mathcal{O}_F^\times). \tag{14.6}$$

We would like to obtain an expression for the Gross–Stark u_τ without the ambiguity $\mathbb{Z}_p \log_p(\mathcal{O}_F^\times)$. Observe that, if we consider measures of total mass zero, we can define the U_F-equivariant morphism

$$\mathcal{E} \colon \mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p) \longrightarrow F_p, \ \lambda \longmapsto \int_{\mathbb{X}} \log_p(\tau^t \cdot x) d\lambda.$$

Moreover, it follows from Proposition 12.4.2 that $\mu_{|U_F}$ lifts to a class in $H^{n-1}(U_F, \mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p))$. However, the lift is not unique. Indeed, the long exact sequence

$$\cdots \longrightarrow H^{n-2}(U_F, \mathbb{Z}_p) \xrightarrow{\delta} H^{n-1}(U_F, \mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p)) \longrightarrow H^{n-1}(U_F, \mathbb{D}(\mathbb{X}, \mathbb{Z}_p)) \longrightarrow \cdots$$

shows that a lift of $\mu_{|U_F}$ is well-defined up to the image of δ. Since $U_F \simeq \mathbb{Z}^{n-1}$ by Dirichlet's unit theorem, we have a natural isomorphism

$$H^{n-2}(U_F, \mathbb{Z}_p) \simeq U_F \otimes \mathbb{Z}_p.$$

This leads to the following proposition.

Proposition 14.4.3. *The map*

$$H^{n-2}(U_F, \mathbb{Z}_p) \xrightarrow{\sim} \mathbb{Z}_p \log_p(U_F), \;\; u \longmapsto c_{U_F} \frown \mathcal{E}(\delta(u))$$

is an isomorphism. More precisely, it is equal to $\log_p \colon U_F \otimes \mathbb{Z}_p \xrightarrow{\sim} \mathbb{Z}_p \log_p(U_F)$.

In other words, the process of lifting $\mu_{|U_F}$ to a class valued in total mass zero measures allows to compute its image under $\mathcal{E}$ and construct an element in F_p. However, since the lift is only well-defined up to $U_F \otimes \mathbb{Z}_p$, the elements we construct in F_p are only well-defined up to $\mathbb{Z}_p \log_p(U_F)$. Therefore, we obtain the same ambiguity as the one we encountered in (14.6) using the proof of the Gross–Stark conjecture.

On the other hand, if we work with cohomology classes for Γ, instead of U_F, we obtain that $\mu \in H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}_p))^-$ has a unique lift (up to torsion) $\mu_0 \in H^{n-1}(\Gamma, \mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p))^-$. Indeed, as explained in Section 12.4, this follows from the long exact sequence

$$H^{n-2}(\Gamma, \mathbb{Z}_p)^- \longrightarrow H^{n-1}(\Gamma, \mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p))^- \longrightarrow H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}_p))^- \longrightarrow H^{n-1}(\Gamma, \mathbb{Z}_p)^-,$$

Proposition 12.4.2, and the fact that $H^{n-2}(\Gamma, \mathbb{Z}_p)^-$ is torsion by [LS19]. Then, the restriction $\mu_{0|U_F}$ is a preferred lift of $\mu_{|U_F}$ to $H^{n-1}(U_F, \mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p))$. Hence, using $\mu_{0|U_F}$ and the map $\mathcal{E}$, we are able to produce a canonical element in F_p

$$c_{U_F} \frown \mathcal{E}(\mu_{0|U_F}) = J_{\mathrm{Eis}}[\tau] \in F_p.$$

The fact that this construction is unique suggests that the quantity we produced could be a preferred lift of $\mathrm{Tr}_{F_p/\mathbb{Q}_p} \log_p(u_\tau)$, and this motivates us to state Conjecture 14.4.1 above.

We summarize this discussion with the following commutative diagram

$$\begin{array}{ccccc}
\text{torsion} & \longrightarrow & H^{n-1}(\Gamma, \mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p))^- & \longrightarrow & H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}_p))^- \\
\downarrow & & \downarrow & & \downarrow \\
U_F \otimes \mathbb{Z}_p & \xrightarrow{\delta} & H^{n-1}(U_F, \mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p)) & \longrightarrow & H^{n-1}(U_F, \mathbb{D}(\mathbb{X}, \mathbb{Z}_p)) \\
\downarrow{\scriptstyle c_{U_F} \frown \mathcal{E} \circ \delta(\cdot)} & & \downarrow{\scriptstyle c_{U_F} \frown \mathcal{E}(\cdot)} & & \downarrow{\scriptstyle c_{U_F} \frown \bar{\mathcal{E}}(\cdot)} \\
\mathbb{Z}_p \log_p(U_F) & \longrightarrow & F_p & \longrightarrow & F_p/(\mathbb{Z}_p \log(U_F)).
\end{array}$$

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Chapter III

Discussion

Section 15

Discussion on the Gross–Kohnen–Zagier theorem

In this section, we outline the history of the Gross–Kohnen–Zagier theorem and compare some of its previous proofs with the one we presented in Chapter I. This motivates future research directions, which we outline. Finally, we explore the possibility of extending the techniques of Chapter I to other settings where the role of Heegner points is replaced by different objects of arithmetic significance.

15.1 Brief history of the GKZ theorem

We begin with a brief outline of how the Gross–Zagier formula led to the Gross–Kohnen–Zagier theorem, as it motivates several of the research directions we will describe below. Let E be an elliptic curve over $\mathbb{Q}$ of conductor N and consider a modular curve parametrization of E. This parametrization, together with the theory of complex multiplication, allows for the construction of Heegner points $P_D \in E(\mathbb{Q})$, for $K = \mathbb{Q}(\sqrt{-D})$ a quadratic imaginary field satisfying the Heegner hypothesis and $-D$ a fundamental discriminant.

In [GZ86], Gross and Zagier proved the so-called *Gross–Zagier formula*, which gives a criterion for the point P_D to have infinite order in terms of an analytic quantity. A consequence of their result suitable for our exposition is the following.

Theorem 15.1.1. *Let $-D$ be a fundamental discriminant coprime to $2N$ satisfying the Heegner hypothesis, let $L(E,s)$ be the L-function of E, and $L(E,\chi_D,s)$ the L-function twisted by $\left(\frac{-D}{\cdot}\right)$. Finally, denote by $\langle\cdot,\cdot\rangle$ the height pairing on E. Then*

$$\langle P_D, P_D\rangle \doteq L'(E,1)L(E,\chi_D,1). \tag{15.1}$$

where, here and from now on, $\doteq$ indicates equality up to a non-zero scalar factor.

Suppose now that $\mathrm{ord}_{s=1}L(E,s)=1$. By a result of Waldspurger, there are infinitely many quadratic imaginary fields $K=\mathbb{Q}(\sqrt{-D})$ such that:

- K satisfies the Heegner hypothesis,
- $L(E,\chi_D,1)\neq 0$.

Thus, by the Gross–Zagier formula (15.1), P_D is of infinite order. In particular, this implies that $\mathrm{rk}(E(\mathbb{Q}))\geq 1$. The BSD Conjecture, unknown for rank 1 by the time the Gross–Zagier formula was proven, predicts that $\mathrm{rk}(E(\mathbb{Q}))=1$. On the other hand, the reasoning above proves that P_D is non-torsion for infinitely many D. Thus, the points $\{P_D\}_D$ should be collinear. In [GKZ87], Gross, Kohnen, and Zagier verified this prediction by studying the position of Heegner points in $E(\mathbb{Q})$ using the height pairing $\langle\cdot,\cdot\rangle$. We give their main result in a simplified setting, but a similar expression holds more generally.

Theorem 15.1.2. *Suppose that the conductor of E is prime and let D, D' be coprime fundamental discriminants. Then,*

$$\langle P_D,P_D\rangle \doteq L'(E,1)c(D)^2,$$

$$\langle P_D,P_{D'}\rangle \doteq L'(E,1)c(D)c(D').$$

Here $c(D)$ is the Dth Fourier coefficient of the modular form of weight $3/2$ in the Kohnen space corresponding to E via the Shimura correspondence.

In particular, an application of Cauchy–Schwarz in the case of equality confirms the anticipated collinearity of Heegner points. In addition, this result guided Gross, Kohnen, and Zagier to the statement that generating series of Heegner points are modular forms of weight $3/2$. The collinearity of Heegner points became evidence for the fact that $\mathrm{rk}(E(\mathbb{Q}))=1$. This fact was later proven by Kolyvagin in [Kol88a] and [Kol88b] using the theory of Euler systems.

The Gross–Kohnen–Zagier theorem for Heeger divisors on Shimura curves was first proven by Borcherds in [Bor99]. There, Borcherds theory of singular theta lifts provides a lift from weakly holomorphic modular forms of weight $1/2$ to meromorphic forms on the Shimura curve with known zeros or poles. The divisors of these lifts are linear combinations of Heegner points, given by the principal part of weakly holomorphic modular forms weight $1/2$. This provides relations of Heegner divisors on the Jacobian of the Shimura curve, which translate to the fact that certain Heegner divisors appear as the Fourier coefficients of a modular form of weight $3/2$ by Serre duality, giving the Gross–Kohnen–Zagier theorem.

15.2 Future research

In Chapter I, we obtained a new proof of the Gross–Kohnen–Zagier theorem for the case of Shimura curves as a consequence of the following expression. Let E be an elliptic curve defined over $\mathbb{Q}$ that appears in the Jacobian of a Shimura curve X. Similar to above, E is equipped with a collection of Heegner points $\{P_D\}_D$, where the indexing set is over discriminants satisfying the Heegner hypothesis (for the Shimura curve parametrization). For p a rational prime dividing the discriminant of the Shimura curve, we constructed a p-adic family of theta series Θ_k of weight $3/2+k$. The family satisfies that $\Theta_0 = 0$ and

$$\sum_{D\geq 1} \log_p(P_D)q^D = \mathrm{pr}_E\left(e_{\mathrm{ord}}\Theta_0'\right), \tag{15.2}$$

where we follow the same notation as in the General Introduction. In the proof of this equality, we did not use the collinearity of Heegner points, the computations of their height pairings, nor the theory of Borcherds lifts. This raises the question of whether we can go in the reverse direction and deduce properties of Heegner points from the fact that they package in a modular generating series. In addition, we would like to explore analogs of (15.2) in different settings.

Collinearity

As we mentioned, a consequence of the main formula proved in Chapter I, together with a study of the Hecke equivariance of theta lifts is that

$$\sum_{D\geq 1} \log_p(P_D) = \mathrm{pr}_E\left(e_{\mathrm{ord}}\Theta_0'\right).$$

Here, $e_{\mathrm{ord}}\Theta_0' \in S_{3/2}(\Gamma_0(4N),\mathbb{Q}_p)$, for N the conductor of the elliptic curve, and

$$\mathrm{pr}_E\colon S_{3/2}(\Gamma_0(4N),\mathbb{Q}_p) \longrightarrow\!\!\!\!\!\rightarrow S_{3/2}(\Gamma_0(4N,\mathbb{Q}_p)_E$$

denotes a Hecke projection to the subspace $S_{3/2}(\Gamma_0(4N),\mathbb{Q}_p)_E$ spanned by eigenforms with the same Hecke eigenvalues as the weight 2 eigenform corresponding to E. The Shimura correspondence implies that this subspace is 2-dimensional. We could refine the projector pr_E so that it has image in a one-dimensional subspace of $S_{3/2}(\Gamma_0(4N),\mathbb{Q}_p)_E$, for example in its Kohnen subspace. Denote by F_E a generator of this subspace. Then, (up to replacing $\{P_D\}$ by their images by this refined operator), we would have

$$\sum_{D\geq 1} \log_p(P_D) = cF_E,$$

for $c\in\mathbb{Q}_p$. This would give a new proof that the Heegner points $\{P_D\}_D$ are collinear and the relations between them are given by the Fourier coefficients of F_E.

Calculation of height pairings

Knowing that Heegner points are collinear and that their precise relations are given by Fourier coefficients of half-integral weight modular forms can be used to simplify calculations of height pairings between them. Bruinier and Yang exploited this idea in [BY09] to give a new proof of the Gross–Zagier formula for the case of modular curves which uses minimal information on finite intersections between Heegner divisors. An informal summary of their strategy is that to compute $\langle P_D, P_D \rangle$ it is enough to compute instead $\langle P_{D_1}, P_{D_2} \rangle$ for a suitable choice of discriminants D_1, and D_2. Indeed, the two quantities differ by an explicit factor involving Fourier coefficients of half-integral weight modular forms. Then, they choose D_1 and D_2 such that the height pairing has only archimedean contribution.

It would be interesting to explore if a similar strategy could yield a simpler approach to compute height pairings, or p-adic analogs of them, in our setting.

Gross–Kohnen–Zagier theorems in other settings

We aim to investigate whether identities similar to (15.2), or more generally similar to the main theorem of Chapter I (Theorem 1.2.1), can be found in other settings, were the role of Heegner points is replaced by different objects. This seems especially interesting in the cases where these objects do not have a known algebraic construction, as identities analog to (15.2) could shed light on their arithmetic properties. A compelling setting would be the study of a Gross–Kohnen–Zagier theorem for the so-called plectic points.

Plectic points are a p-adic generalization of Heegner points, meaningful for elliptic curves of arbitrary rank. They were introduced by Fornea and Gehrmann in [FG23], where they conjectured that plectic points are related to algebraic points on elliptic curves of arbitrary rank. Together with these two authors, we are working on generalizing (15.2) to this setting. There, Heegner points are replaced by plectic points, and the derivative of the family of theta series is replaced by a higher-order derivative of a family Hilbert theta series. This would imply bounds on the span of plectic points in line with the predictions of the BSD Conjecture, generalizing the collinearity of Heegner points and extending the current results on their algebraicity.

Section 16

Discussion on rigid classes for $\mathrm{SL}_n(\mathbb{Z}[1/p])$

In this section, we outline the theory of rigid analytic classes initiated by Darmon and Vonk in [DV21] and [DV22] and discuss how our construction of J_{Eis} in Chapter II could fit in this theory. This leads to a discussion towards a generalization of the theory of rigid analytic classes to $\mathrm{SL}_n(\mathbb{Z}[1/p])$, which would require to refine the construction of J_{Eis} to a class for this p-arithmetic group. Throughout the section, $n \geq 2$, $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ and $\Gamma^p = \mathrm{SL}_n(\mathbb{Z}[1/p])$.

16.1 Rigid analytic cocycles for $\mathrm{SL}_2(\mathbb{Z}[1/p])$

For $n = 2$, Darmon and Vonk introduced the theory of *rigid analytic theta classes* for Γ^p in [DV22, Section 3]. These are classes in $H^1(\Gamma^p, \mathcal{A}^\times/\mathbb{C}_p^\times)$, where $\mathcal{A}$ is the ring of rigid analytic functions on the p-adic upper half plane $\mathscr{H}_p$. This cohomology group admits an action by Hecke operators and there is a Hecke equivariant short exact sequence

$$0 \longrightarrow \mathbb{Q} \longrightarrow H^1(\Gamma^p, \mathcal{A}^\times/\mathbb{C}_p^\times)_{\mathbb{Q}} \longrightarrow H^1(\Gamma_0(p), \mathbb{Q}) \longrightarrow 0. \tag{16.1}$$

There is also an involution induced by $w = \mathrm{diag}(1, -1) \in \mathrm{GL}_2(\mathbb{Z})/\Gamma$, which yields an isomorphism on the -1 eigenspaces (see [DPV21, Lemma 2.1])

$$H^1(\Gamma^p, \mathcal{A}^\times/\mathbb{C}_p^\times)_{\mathbb{Q}}^- \xrightarrow{\sim} H^1(\Gamma_0(p), \mathbb{Q})^-. \tag{16.2}$$

Therefore, by (16.1), (16.2) and the Eichler–Shimura theorem, rigid cocycles for Γ^p have a structure similar to that of modular forms of weight 2 and level $\Gamma_0(p)$. In contrast, rigid cocycles can be evaluated at real quadratic points, as mentioned in Chapter II.

Indeed, let $\tau \in \mathscr{H}_p$ be a real quadratic point and consider the real quadratic field $F = \mathbb{Q}(\tau)$. The stabilizer of τ in Γ^p is isomorphic to $\langle \pm\gamma_\tau \rangle$, for γ_τ of infinite order. To simplify the exposition, suppose that such stabilizer is contained in Γ. Let $J \in H^1(\Gamma^p, \mathcal{A}^\times/\mathbb{C}_p^\times)$ be a rigid analytic theta class. Observe that the short exact sequence of Γ-modules

$$1 \longrightarrow \mathbb{C}_p^\times \longrightarrow \mathcal{A}^\times \longrightarrow \mathcal{A}^\times/\mathbb{C}_p^\times \longrightarrow 1$$

induces a long exact sequence in cohomology

$$\cdots \longrightarrow H^1(\Gamma, \mathbb{C}_p^\times) \longrightarrow H^1(\Gamma, \mathcal{A}^\times) \longrightarrow H^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times) \longrightarrow H^2(\Gamma, \mathbb{C}_p^\times) \longrightarrow \cdots .$$

Since the groups $H^i(\Gamma, \mathbb{C}_p^\times)$ for $i \in \{1,2\}$ are finite and killed by 12, (the 12th power of) the restriction of J to Γ admits a unique lift $\mathcal{J} \in H^1(\Gamma, \mathcal{A}^\times)$ up to torsion.

Definition 16.1.1. Consider the same notation as above. Define the value of J at $\tau \in \mathscr{H}_p$ as $J[\tau] := \mathcal{J}(\gamma_\tau)(\tau) \in \mathbb{C}_p^\times$.

Suppose that f is a Hecke eigenclass in $H^1(\Gamma_0(p), \mathbb{Q})^-$ and let $J_f \in H^1(\Gamma^p, \mathcal{A}^\times/\mathbb{C}_p^\times)^-$ be the corresponding class via the isomorphism (16.2). Then, it is conjectured that the value $J_f[\tau] \in \mathbb{C}_p^\times$ has arithmetic significance. More precisely:

- If f corresponds to a cuspidal eigenform, and we denote by $E(\mathbb{C}_p) \simeq \mathbb{C}_p^\times/{q_f}^{\mathbb{Z}}$ the $\mathbb{C}_p$-points of the elliptic curve attached to it, then $J_f[\tau] \in \mathbb{C}_p^\times/{q_f}^{\mathbb{Z}}$ is conjectured to be a global point defined over an abelian extension of F.

- If f is the Eisenstein class in $H^1(\Gamma_0(p), \mathbb{Q})$, then J_f, usually denoted J_{DR}, satisfies that $J_{\mathrm{DR}}[\tau] \in \mathbb{C}_p^\times/p^{\mathbb{Z}}$ is equal to a Gross–Stark unit in an abelian extension of F.

In fact, the previous two points are a restatement of the explanations given in the interlude (Table i), once we apply the $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism due to Van der Put [vdP82]

$$\mathcal{A}^\times/\mathbb{C}_p^\times \simeq \mathbb{D}_0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Z}),$$

where $\mathbb{D}_0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Z})$ denotes $\mathbb{Z}$-valued measures on $\mathbb{P}^1(\mathbb{Q}_p)$ of total mass zero.

The short exact sequence (16.1) has the following refinement. Suppose $f \in H^1(\Gamma_0(p), \mathbb{Q})$ is an eigenclass and let $J_f \in H^1(\Gamma^p, \mathcal{A}^\times/\mathbb{C}_p^\times)_{\mathbb{Q}}$ be as above. Then, using the theory of p-adic families of modular forms it can be proven that J_f admits a lift $\mathcal{J}_{f,\Lambda} \in H^1(\Gamma^p, \mathcal{A}^\times/\Lambda)$, where we remark that $\mathcal{J}_{f,\Lambda}$ is a class for the p-arithmetic group Γ^p, and

- Λ is commensurable with $q_f^{\mathbb{Z}}$, if f corresponds to a cuspidal eigenform,

- $\Lambda = p^{\mathbb{Z}}$ if f is the Eisenstein class.

From there, we could alternatively define the value $J_f[\tau] = \mathcal{J}_{f,\Lambda}(\gamma_\tau)(\tau) \in \mathbb{C}_p^\times/\Lambda$. This definition justifies the importance of interpreting the values of J_f modulo certain periods.

16.2 Future research

Suppose $n \geq 2$. In Chapter II, we constructed a class

$$J_{\mathrm{Eis}} \in H^{n-1}(\Gamma, \mathcal{A}_{\mathcal{L}})^{-},$$

where $\mathcal{A}_{\mathcal{L}}$ denotes the space of log-rigid analytic functions on Drinfeld's symmetric domain $\mathscr{X}_p \subset \mathbb{P}^{n-1}(\mathbb{C}_p)$. When $n = 2$, the class J_{Eis} is related to $J_{\mathrm{DR}} \in H^1(\Gamma^p, \mathcal{A}^\times/\mathbb{C}_p^\times)^-$ via the expression $J_{\mathrm{Eis}} = \log_p(\mathcal{J}_{\mathrm{DR}})$, where $\mathcal{J}_{\mathrm{DR}} \in H^1(\Gamma, \mathcal{A}^\times)$ is a lift of (a power of) the restriction of J_{DR} to Γ. Moreover, for $\tau \in \mathscr{X}_p$ a point represented by a vector in $\mathbb{C}_p^n$ whose coordinates generate a fractional ideal of a totally real field F where p is inert, we defined $J_{\mathrm{Eis}}[\tau] \in F_p$ and conjectured $J_{\mathrm{Eis}}[\tau] = \log_p(u_\tau)$, where $u_\tau \in F^{\mathrm{ab}}$.

We aim to investigate whether a general theory of rigid analytic cocycles for Γ^p can be developed, and how our construction of J_{Eis} integrates into such a framework. It would also be interesting to examine whether this theory could be used to construct invariants associated with totally real fields of degree n, even in cases where p is not inert.

Rigid analytic theta cocycles for $\Gamma^p := \mathrm{SL}_n(\mathbb{Z}[1/p])$

In view of the definition of rigid analytic theta classes when $n = 2$, the group

$$H^{n-1}(\Gamma^p, \mathcal{A}^\times/\mathbb{C}_p^\times),$$

where $\mathcal{A}$ denotes the rigid analytic functions on $\mathscr{X}_p$, appears to be a natural candidate for the space of rigid analytic theta cocycles for Γ^p. In particular, it would be desirable to have a generalization of the short exact sequence (16.1), describing the space of rigid analytic theta cocycles in terms of cohomology classes for congruence subgroups of Γ.

When $n = 2$, this short exact sequence can be deduced from the $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphisms

$$\mathcal{A}^\times/\mathbb{C}_p^\times \simeq \mathbb{D}_0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Z}) \simeq C^1_{\mathrm{har}}(\mathbb{Z}),$$

where $C^1_{\mathrm{har}}(\mathbb{Z})$ denotes the space of harmonic cochains on the Bruhat–Tits tree (the definition of harmonic cochain and the second isomorphism can be found in Definition 5.6, Definition 5.10 and (5.5) of [Dar04]). For general $n \geq 2$, Gekeler provides a generalization of these isomorphisms [Gek20] . Indeed, we have a $\mathrm{GL}_n(\mathbb{Q}_p)$-equivariant isomorphism

$$\mathcal{A}^\times/\mathbb{C}_p^\times \simeq \mathbb{D}_0(\mathbb{P}^{n-1}(\mathbb{Q}_p), \mathbb{Z}).$$

Moreover, these spaces have interpretation in terms of harmonic cochains (Theorem 4.16 of [Gek20]):

- There is a $\mathrm{GL}_n(\mathbb{Q}_p)$-equivariant isomorphism between $\mathbb{D}_0(\mathbb{P}^{n-1}(\mathbb{Q}_p), \mathbb{Z})$ and a space of harmonic cochains on the Bruhat–Tits building of $\mathrm{PGL}_n(\mathbb{Q}_p)$ (maps from oriented 1-simplices to $\mathbb{Z}$ satisfying certain conditions, see Section 3.2 of [Gek20]).
- There is an isomorphism between $\mathbb{D}_0(\mathbb{P}^{n-1}(\mathbb{Q}_p), \mathbb{Z})$ and harmonic cochains in a *tree* obtained as a subcomplex of the Bruhat–Tits building of $\mathrm{PGL}_n(\mathbb{Q}_p)$.

The study of these spaces of harmonic cochains could lead to the desired understanding of the structure of $H^{n-1}(\Gamma^p, \mathcal{A}^\times/\mathbb{C}_p^\times)$. The paper [Geh22] also seems relevant for this aim.

Refinements in the construction of the Eisenstein class

Based on the theory for the $n = 2$ case, we expect that the class $J_{\mathrm{Eis}} \in H^{n-1}(\Gamma, \mathcal{A}_{\mathcal{L}})$ of Chapter II can be refined to a class

$$J_{\mathrm{DR}} \in H^{n-1}(\Gamma^p, \mathcal{A}^\times/p^{\mathbb{Z}})$$

such that $J_{\mathrm{Eis}} = \log_p(J_{\mathrm{DR}})$, where J_{DR} is viewed as a class for Γ in this last equality. Following [DPV24], a strategy to achieve this would be to refine the construction of $\mu_0 \in H^{n-1}(\Gamma, \mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p))$ given in Section 12 (more precisely in (12.7)) to a class

$$\mu_{\mathrm{DR}} \in H^{n-1}(\Gamma^p, \mathbb{D}_0(\mathbb{X}^p, \mathbb{Z})^p),$$

where $\mathbb{D}_0(\mathbb{X}^p, \mathbb{Z})^p$ denotes the space of p-invariant measures λ on $\mathbb{X}^p := \mathbb{Q}_p^n - \{0\}$ such that $\lambda(\mathbb{X}) = 0$. Then, the Γ^p-equivariant multiplicative lift

$$\mathrm{ST}^\times \colon \mathbb{D}_0(\mathbb{X}^p, \mathbb{Z}) \longrightarrow \mathcal{A}^\times/p^{\mathbb{Z}}, \ \lambda \longmapsto \left(z \longmapsto \times\!\!\!\!\!\!\int_{\mathbb{X}} z^t \cdot x \, d\lambda \right)$$

would allow to define $J_{\mathrm{DR}} := \mathrm{ST}^\times(\mu_{\mathrm{DR}})$. This is ongoing joint work with Peter Xu.

It would also be interesting to explore if the classes μ_0 for different primes (not necessarily inert in F) can be used to extract information on tame refinements of the Gross–Stark conjecture. This could potentially lead to a proof of Conjecture 9.2.3, as a consequence of the Gross tower of fields conjecture (see [Das08] and [DK23]).

In a different direction, J_{Eis} was used to construct invariants attached to totally real fields where p is inert, that conjecturally belong to abelian extensions of such fields. More generally, these invariants can be defined for totally real fields F such that $F \otimes \mathbb{Q}_p$ is a field. One of the reasons for this is that it seems that $\mathscr{X}_p$ only contains special points attached to totally real fields satisfying this condition. This leads to the next two questions:

- Can the class μ_0 be used to conjecturally construct Gross–Stark units in abelian extensions of totally real fields F of degree n, where $F \otimes \mathbb{Q}_p$ is not a field?
- Is there a p-adic symmetric space with special points attached to totally real fields of degree n where $F \otimes \mathbb{Q}_p$ is not a field?

Section 17

Values of rigid classes and Fourier coefficients of p-adic families

This last section discusses the relation between values of rigid classes and Fourier coefficients of derivatives of p-adic families modular forms. After a summary of some known results for the case of classes for $\mathrm{SL}_2(\mathbb{Z}[1/p])$ due to Darmon, Pozzi, and Vonk, we discuss future directions involving the generalization of these results to $\mathrm{SL}_n(\mathbb{Z}[1/p])$. We conclude by outlining a possible framework that would encompass Chapter I and Chapter II. In this section, $n \geq 2$, $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ and $\Gamma^p = \mathrm{SL}_n(\mathbb{Z}[1/p])$.

17.1 Results of Darmon, Pozzi, and Vonk

For $n = 2$, the construction of rigid analytic theta classes for Γ^p is explicit and achieved via analytic methods. This allows for the computation of values of rigid classes to high precision, providing numerical evidence for the conjectures stated in Section 16.1. On the other hand, the absence of an algebraic theory underlying these constructions complicates the possibility of proving these conjectures in full generality. Despite this fact, there has been substantial progress in the study of values of rigid classes via the connections between:

- Values of rigid classes.
- Fourier coefficients of derivatives of p-adic families of modular forms.
- Deformations of Galois representations attached to p-adic families of modular forms.

In particular, the algebraic nature of the values of rigid classes results from the algebraic properties of the corresponding Galois deformations. These techniques have been especially fruitful when the corresponding Galois deformations are deformations of Artin representations. The works of Darmon, Pozzi, and Vonk develop this strategy to study the values of the Dedekind–Rademacher cocycle

$$J_{\mathrm{DR}} \in H^1(\Gamma^p, \mathcal{A}^\times/\mathbb{C}_p^\times),$$

where we recall that J_{DR} corresponds to the Eisenstein class in $H^1(\Gamma_0(p), \mathbb{Q})^-$ via (16.2). More precisely, the main result of [DPV24] is the following one (already given in Theorem 9.1.3).

Theorem 17.1.1 (Darmon–Pozzi–Vonk)**.** *Let $\tau \in \mathscr{H}_p$ be a real quadratic point such that p is inert in $\mathbb{Q}(\tau)$. Then,*

$$\log_p(J_{\mathrm{DR}}[\tau]) = \log_p(u_\tau),$$

for $u_\tau \in \mathcal{O}_H[1/p]^\times \otimes \mathbb{Q}$ a Gross–Stark in an abelian extension H of F.

As we mentioned, one of the ingredients in their proof is to relate values of rigid classes to Fourier coefficients of derivatives of p-adic families of modular forms. We now outline their main results on this subject, as it is relevant for the theme of this thesis. For that, we need to introduce another rigid class. Let $X_0(p)$ be the closed modular curve of level $\Gamma_0(p)$ and consider

$$\varphi_w \in H^1(\Gamma_0(p), \mathbb{Q}) \simeq H_1(X_0(p), \{0, \infty\}, \mathbb{Q})$$

the class corresponding to the projection of the path from 0 to ∞ in $\mathscr{H}$ via Poincaré duality. It can be verified that $\varphi_w \in H^1(\Gamma_0(p), \mathbb{Q})^-$. Then, the *winding class* is the rigid analytic theta class

$$J_w \in H^1(\Gamma^p, \mathcal{A}^\times/\mathbb{C}_p^\times)^-$$

mapping to φ_w via (16.2). We will also be interested in the translates $T_m J_w$ of J_w by the mth Hecke operator for every $m \geq 1$.

Let $\tau \in \mathscr{H}_p$ be a real quadratic point such that p is inert in $F := \mathbb{Q}(\tau)$. We can consider the values $J_{\mathrm{DR}}[\tau]$ and $T_m J_w[\tau] \in \mathbb{C}_p^\times$ for every $m \geq 1$. It can be deduced from their definition that they belong to $F_p^\times$, where F_p denotes the p-adic completion of F. Darmon, Pozzi, and Vonk relate them to Fourier coefficients of derivatives of p-adic families of modular forms:

1. In [DPV21], they consider a p-adic family of Hilbert–Eisenstein series of parallel weight $k \in \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$ attached to an odd function $\psi\colon \mathrm{Cl}^+(F) \to \mathbb{C}$ on a narrow ring class field of F determined by $\tau \in \mathscr{H}_p$

$$E_k(1, \psi)(z_1, z_2).$$

The diagonal restriction $G_k := E_k(1,\psi)(z,z)$ is a p-adic family of elliptic modular forms of weight $2k$ satisfying that its specialization $G_1 = 0$ and, if we denote by G_1' the value at $k = 1$ of its derivative with respect to k, we have

$$\log_p(\mathrm{N}_{F_p/\mathbb{Q}_p} J_{\mathrm{DR}}[\tau]) + \sum_{m\geq 1} \log_p(\mathrm{N}_{F_p/\mathbb{Q}_p} T_m J_w[\tau]) q^m = e_{\mathrm{ord}}(G_1'). \tag{17.1}$$

2. In [DPV24], they refine the results of the previous point by considering also *cuspidal* deformations of certain Hilbert–Eisenstein series. In this way, they construct a p-adic elliptic modular form ∂f_ψ, which has similar role as G_1' above, and prove

$$\log_p(J_{\mathrm{DR}}[\tau]) + \sum_{m\geq 1} \log_p(T_m J_w[\tau]) q^m = e_{\mathrm{ord}}(\partial f_\psi). \tag{17.2}$$

Moreover, the Fourier coefficients of ∂f_ψ can be explicitly computed via the study of Galois deformations, and $a_0(e_{\mathrm{ord}}\partial f_\psi) = a_0(\partial f_\psi) = \log_p(u_\tau)$, where u_τ is as in Theorem 17.1.1.

17.2 Future research

In Chapter II, we constructed an analog for the values $\log_p(J_{\mathrm{DR}}[\tau])$ for the case of totally real fields where p is inert. Thus, it would be interesting to explore if techniques of Darmon, Pozzi, and Vonk generalize to $n \geq 2$. In particular, this could lead to a proof of Conjecture 9.2.3. Moreover, the similarities between the main formula of Chapter I, (namely (15.2) or Theorem 1.2.1), and expression (17.1) above motivate the search for a general framework where the two equalities appear as special cases. For the rest of the section, $n \geq 2$.

Generalization of the work of Darmon, Pozzi, and Vonk

We would like to generalize (17.1) and (17.2) to the setting of rigid analytic classes for Γ^p. We comment on the possible generalization of each side of the equality (17.1), which is ongoing joint work with Romain Branchereau and Peter Xu:

- In Chapter II, we presented a generalization of $\log_p(J_{\mathrm{DR}}[\tau])$. On the other hand, we should construct an analog for the winding cocycle J_w. Following the principle that rigid analytic classes for Γ^p should be related to cohomology classes for congruence subgroups of Γ, the classes studied in the work [Bra24] of Branchereau coming from the projection of embeddings

$$\mathrm{GL}_{n-1}(\mathbb{R})^+/\mathrm{SO}_{n-1} \longrightarrow \mathscr{X} = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n$$

to quotients of $\mathscr{X}$ by congruence subgroups seem promising candidates to generalize $\varphi_w \in H^1(\Gamma_0(p), \mathbb{Q})$ attached to the path $\{0, +\infty\}$. Indeed, observe that $\{0, +\infty\}$ can be seen as the image of an inclusion of $\mathrm{GL}_1(\mathbb{R})^+$ into $\mathscr{H}$.

- To generalize (17.1), a candidate for G_k is the diagonal restriction of a p-adic family of Hilbert–Eisenstein series attached to a totally real field of degree n where p is inert. More precisely, a suitable Hecke translate of this family to ensure that it vanishes at $k = 1$ (i.e. in weight n). Moreover, Theorem 1.1 of [Bra24] relates Fourier coefficients of diagonal restrictions of Hilbert–Eisenstein series to the Eisenstein class studied in Section 10.2 and the generalizations of the winding class mentioned above, suggesting that a generalization of (17.1) could follow by taking a p-adic limit of the results of [Bra24].

General relations between values of cocycles and p-adic families

Recall the main result of Chapter I (Theorem 1.2.1) which, following the same notation as in the General Introduction, implies

$$\sum_{D \geq 1} \log_p(P_D) q^D = \mathrm{pr}_E(e_{\mathrm{ord}}(\Theta'_0)).$$

On the other hand, the identity (17.1) reads as

$$\log_p(\mathrm{N}_{F_p/\mathbb{Q}_p} J_{\mathrm{DR}}[\tau]) + \sum_{m \geq 1} \log_p(\mathrm{N}_{F_p/\mathbb{Q}_p} T_m J_w[\tau]) q^m = e_{\mathrm{ord}}(G'_1). \tag{17.3}$$

The left-hand side of these equations can be interpreted as an average of values of rigid analytic classes at tori (either attached to imaginary or real quadratic fields). Indeed, this is clear for the second equation, while for the first one we refer the reader to the Interlude (see Table i).

On the other hand, the right-hand side of these equations is the derivative of a p-adic family of theta lifts. For the first case, we considered a family of definite ternary theta series. For the second one, observe that the diagonal restriction of a Hilbert–Eisenstein series is a Kudla–Millson theta lift (see [Bra23]).

Therefore, both equalities relate generating series of values of rigid classes to derivatives of a p-adic families of theta series. It would be instructive to explore if these two equalities arise as the special case of a more general theory, as it could provide a more conceptual approach to prove them. With this aim, we outline a possible approach:

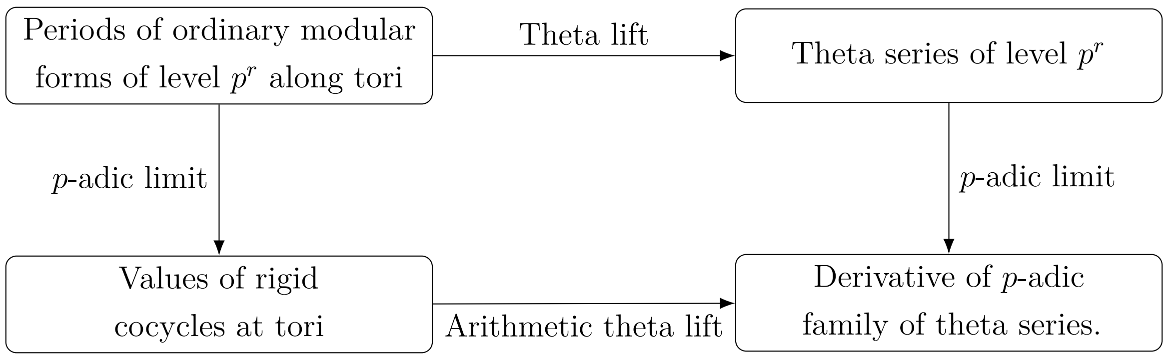
1. Values of rigid analytic classes are determined by periods along tori of p^r-level specializations of ordinary p-adic families of modular forms.

2. In certain scenarios, Fourier coefficients of theta lifts are equal to periods of modular forms along tori.

3. We would then expect that periods of p^r-level specializations of an ordinary family along tori appear as the p^r-level specialization of an ordinary family of theta lifts.

4. The p-adic limiting process of periods of modular forms used to define the values of rigid cocycles (as the integral with respect to certain measure), corresponds to the p-adic limiting process of taking the derivative of the p-adic family of theta series.

The next diagram illustrates the previous relations.



Final conclusion

In this thesis, we explored two connections between p-adic families of theta series and arithmetic:

1. Derivatives of p-adic families of definite ternary theta series encode Abel–Jacobi images of Heegner points on Shimura curves, yielding a new proof of the Gross–Kohnen–Zagier theorem.

2. Periods of p-adic families of Eisenstein series can be combined into a group cohomology class for $\mathrm{SL}_n(\mathbb{Z})$ and lead to a log-rigid analytic class for $\mathrm{SL}_n(\mathbb{Z})$. Moreover, they can be evaluated at tori attached to totally real fields of degree n where p is inert, thus providing invariants attached to these fields. The local traces of these invariants coincide with those of p-adic logarithms of Gross–Stark units in abelian extensions of totally real fields.

These findings provide new instances in which p-adic families of modular forms encode arithmetic information. Moreover, they set the stage for further exploration of general identities between p-adic families of theta series and values of rigid classes, as well as for developing the theory of rigid classes for $\mathrm{SL}_n(\mathbb{Z}[1/p])$. Combining these two directions would relate the values of our log-rigid class to Fourier coefficients of p-adic families of modular forms, and ultimately to deformations of Artin representations. This would shed light on the arithmetic nature of the invariants we constructed for totally real fields in which p is inert.

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