The eigencurve at weight one Eisenstein points

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Abstract

In 1973, Serre observed that the Hecke eigenvalues of Eisenstein series can be p-adically interpolated. In other words, Eisenstein series can be viewed as specializations of a p-adic family parametrized by the weight. The notion of p-adic variations of modular forms was later generalized by Hida to include families of ordinary cuspforms. In 1998, Coleman and Mazur defined the eigencurve, a rigid analytic space classifying much more general p-adic families of Hecke eigenforms parametrized by the weight. The local nature of the eigencurve is well understood at points corresponding to cuspforms of weight $k \geq 2$, while the weight one case is far more intricate.

In this thesis, we study the geometry of the eigencurve at weight one Eisenstein points. The eigencurve is etale over the weight space if those points are regular. We show that irregular Eisenstein weight one points lie on the intersection of two Eisenstein components and a cuspidal one. The latter is etale over the weight space and meets the other two transversally. Our techniques are Galois theoretic, and build on the existence of an isomorphism between the completed local ring of the cuspidal eigencurve and a certain deformation ring of Galois representations. We determine the structure of the completed local ring of the eigencurve and show that it is Cohen-Macaulay but not Gorenstein.

The failure of etaleness of the eigencurve over the weight space implies the existence of certain weight one non-classical overconvergent generalized eigenforms. Expressing their coefficients in terms of p-adic logarithms of p-units of a number field, we obtain a new proof of Gross's formula for the derivative of the p-adic L-function.

En 1973, Serre a remarqué que les valeurs propres de l'algèbre de Hecke des séries d'Eisenstein peuvent être interpolées p-adiquement. Autrement dit, les séries d'Eisenstein peuvent être interprétées comme des spécialisations d'une famille p-adique paramétrée par leur poids. La notion de variation p-adique de formes modulaires a par la suite été généralisée par Hida, pour s'appliquer aux familles de formes paraboliques ordinaires. En 1998, Coleman et Mazur ont défini la courbe de Hecke, un espace analytique rigide classifiant des familles p-adiques beaucoup plus générales de formes propres de Hecke paramétrées par leur poids. La nature locale de la courbe de Hecke est bien comprise aux points correspondant aux formes paraboliques de poids $k \geq 2$, alors que le cas des formes de poids un est beaucoup plus délicat.

Dans cette thèse, nous étudions la géométrie de la courbe de Hecke aux points d'Eisenstein de poids un. La courbe de Hecke est étale sur l'espace des poids si ces points sont réguliers. Nous démontrons que les points d'Eisenstein de poids un irréguliers se situent sur l'intersection de deux composantes d'Eisenstein et d'une composante parabolique. Cette dernière est étale sur l'espace de poids et rencontre les deux autres transversalement. Nos méthodes proviennent de la théorie de Galois et se basent sur l'existence d'un isomorphisme entre l'anneau local de la courbe de Hecke parabolique et un anneau de déformation. Nous déterminons la structure de l'anneau local de la courbe de Hecke et nous montrons qu'il possède la propriété de Cohen-Macauley mais pas celle de Gorenstein.

Le fait que la courbe de Hecke ne soit pas étale sur l'espace de poids implique l'existence de formes propres généralisées de poids un qui sont surconvergentes, mais pas classiques. Nous obtenons une nouvelle preuve de la formule de Gross pour la dérivée de la fonction L p-adique en exprimant les coefficients de ces formes en terme des logarithmes p-adiques de p-unités d'un corps de nombre.

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I cannot thank enough my coauthors, Adel Betina and Mladen Dimitrov, for collaborating with me on a project that exceeded my original goals and for the many valuable suggestions they gave to refine my initial results. I am grateful to my external examiner, Preston Wake, for his apt remarks that improved the exposition of this material and pointed me to interesting connections with other related topics. Finally, I would like to thank Michele Fornea and Jan Vonk for many helpful conversations over the years of PhD.

Contributions

Genesis of the project

The content of this thesis is essentially based on a joint work with Adel Betina and Mladen Dimitrov devoted to understanding the geometry of the eigencurve at Eisenstein weight one points, which resulted in the article [BDP].

This topic was suggested to me by Henri Darmon. Under his and Payman Kassaei's supervision, I approached this question with Galois theoretic techniques, highly influenced by the work of Bellaïche and Dimitrov [BD16]. My strategy consisted in defining a deformation ring $\mathcal{R}^{\text{cusp}}$ classifying deformations of certain reducible representations corresponding to cuspidal modular forms. I aimed to show an isomorphism between the latter and the local ring of the cuspidal eigencurve; through this method I could establish the etaleness of the cuspidal eigencurve over the weight space.

While pursuing this goal, I became aware that Adel Betina and Mladen Dimitrov had also made significant progress on a closely related question. Their approach was analyzing the local nature of the eigencurve using congruences between cuspidal and Eisenstein families due to Lafferty [Laf]. Combining this information with the study of the ordinary deformation ring $\mathcal{R}_{\rho}^{\text{ord}}$, they proved that the cuspidal eigencurve is smooth. Unlike mine, their original strategy tied the question of understanding the geometry of the eigencurve with the derivative of the Kubota-Leopoldt p-adic L-function and in particular to the theorem of Ferrero and Greenberg [FG78] and the Gross-Stark Conjecture proved in [DDP11]. It was their intuition that combining our works would yield new proofs of these results. This lead to a collaboration that culminated in [BDP], a project that greatly surpassed my original goals. Our strategies well-complemented each other, since they were both rooted in the study of deformation functors attached to certain non-trivial extensions of finite order characters and the computation of their tangent spaces. In particular, as my coauthors observed, the ring $\mathcal{R}^{\text{cusp}}$ is a quotient of their ring $\mathcal{R}_{\rho}^{\text{ord}}$, thus singling out the subspace in the tangent space of $\mathcal{R}_{\rho}^{\text{ord}}$ classifying cuspidal lifts.

Individual Contribution

This thesis contains a comprehensive exposition of the material in [BDP], that goes beyond my individual contribution. I will explain my role in the results of *loc.cit.* My main input in this work was the construction of the cuspidal deformation ring $\mathcal{R}^{\text{cusp}}$ and the computation of its tangent space. The latter, in particular, helped tie the geometry of the eigencurve with the $\mathcal{L}$-invariants of certain Dirichlet characters. This lead to the proof of Theorem A(i) and Theorem B(i) of *loc.cit*; the exposition of the result in this thesis is consistent with the article and diverges from my original one in some ways, thanks to my coauthors' suggestions. In particular, I appreciated the input simplifying my argument of representability of the deformation functor $\mathcal{D}^{\mathrm{cusp}}$ and the suggestion of using a method due to Mazur and Wiles [MW84] to relate the cuspidal deformation ring to the local ring of the cuspidal eigencurve. The deformation rings $\mathcal{R}_\rho^{\mathrm{ord}}$, $\mathcal{R}_\rho^{\mathrm{n.ord}}$, $\mathcal{R}_\rho^{\mathrm{eis}}$ were introduced by my collaborators and their method of computing the tangent spaces using a filtration is different, though similar in spirit, from the strategy I originally adopted.

The idea of proving a second and stronger modularity result (Theorem B(ii) of *loc.cit.*) for the ordinary deformation rings $\mathcal{R}_\rho^{\mathrm{ord}}$ using Iwasawa cohomology is due to my coauthors; my only contributions towards this result were some Galois cohomology computations. The results concerning the structure of the local ring of the eigencurve, namely Theorem A(ii-iii), of *loc.cit.* were conceived and proved by my coauthors and so was Corollary 5.3. Their intuition was behind the relation between the structure of the local ring of the eigencurve and the p-adic L-function that lead to a new proof of the theorem of [FG78], which I contributed to by working out the constant term of the q-expansion of certain Eisenstein families at all cusps.

Finally, I contributed to the proof of Theorem C of *loc.cit.* by exploiting the computations of the tangent space of the deformation ring $\mathcal{R}^{\mathrm{cusp}}$ to obtain the q-expansion of certain overconvergent generalized eigenforms; I thank my coauthors for their input in the latter, and particularly for suggesting a closed formula for the general coefficients. We recently came to the realization that, as a corollary of these computations, one easily obtains a new proof of the Gross-Stark conjecture over $\mathbb{Q}$, that I included in my thesis and will appear in an updated version of [BDP].

Introduction

Modular forms are a central object of study in number theory. They are holomorphic functions of the complex upper half plane satisfying certain symmetries with respect to the action of congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. In particular, each admits a Fourier expansion $f = \sum_{n=0}^{\infty} a_n(f) q^n$ where $q = e^{2\pi i z}$; the Fourier coefficients are algebraic numbers of great arithmetic significance. Of particular interest is the relation between modular forms and representations of the absolute Galois group of $\mathbb{Q}$. This connection and more general versions of these statements are at the core of many conjectures and results, falling under the broad Langlands program, including the celebrated proof of Fermat's Last Theorem.

Given a newform f of weight $k \in \mathbb{Z}_{\geq 1}$ and level N, one can attach to it a p-adic Galois representation, whose trace encodes the coefficient $a_n(f)$ for $(n, Np) = 1$. The original construction due to Shimura for weight two consists in extracting this representation from the Tate module of an abelian subvariety A_f of the Jacobian of the modular curve of level $\Gamma_1(N)$. This approach was later generalized by Deligne, who realized these representations as appropriate quotients of the etale cohomology of certain locally constant sheaves on the modular curve for every weight $k \in \mathbb{Z}_{\geq 2}$. However, the construction of the Galois representation attached to a weight one form is different in nature and much less geometric. In the work of Serre and Deligne, given a weight one newform f, exploiting a congruence between f and an eigenform of higher weight, one constructs a compatible system of ℓ-adic Galois representation attached to f for every prime $\ell \nmid N$. Then one can show that the image of the associated residual representations is bounded independently of the prime ℓ. Thus, they all arise from a unique Artin representation, *i.e.* a finite image representation $\rho_f \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$. The non-geometric nature of this construction makes the treatment of questions related to weight one forms rather different from higher weights. In recent years, a number of mathematicians (including Buzzard, Dickinson, Shepherd-Barron and Taylor for the icosahedral case) contributed to showing the modularity of certain Artin representations using p-adic analytic methods, making substantial towards the proof of the Artin Conjecture for two-dimensional representations.

p-adic families of modular forms and Eisenstein series

The Artin Conjecture is only one example of how p-adic methods have proved crucial in approaching questions about classical modular forms, particularly in the weight one scenario. The notion of p-adic modular forms was introduced by Serre in 1973 as a p-adic limit of q-expansions of classical modular forms [Ser73]. Together with giving a geometric characterization of p-adic modular forms, Katz singled out a subspace of the huge space of p-adic modular forms, the space of overconvergent forms [Kat73]. The Hecke operator U_p acts compactly on this subspace; this observation is pivotal to studying the spectral theory of overconvergent forms for the action of the Hecke algebra.

Serre had already observed that the coefficients of the q-expansion of Eisenstein series could be p-adically interpolated. More precisely, classical Eisenstein series can be viewed as specializations of an Eisenstein family parametrized by the weight. This constitutes the first example of what would later be called a Hida family, *i.e.* a formal q-expansion $\mathcal{F}(q)$ with coefficients in an Iwasawa algebra, which specializes to classical forms for almost all weights $k \in \mathbb{Z}_{\geq 2}$. In 1998, Coleman and Mazur [CM96] constructed the eigencurve, a rigid analytic space that encompasses the previous example of p-adic families of Hecke eigenforms to include not only ordinary, but also finite slope eigenforms. Let N be a positive integer prime to p. The weight space is a rigid analytic space $\mathcal{W}$ whose $\mathbb{C}_p$-valued points are in bijection with continuous characters of

$$(\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times \to \mathbb{C}_p^\times.$$

Under this bijection, the classical weights $k \in \mathbb{Z}_{\geq 0}$ can be identified with characters $x \mapsto x^k$. The eigencurve of tame level N constructed by Coleman and Mazur is a rigid analytic space $\mathcal{C}$ whose $\mathbb{C}_p$-valued points are systems of Hecke eigenvalues (with respect to the Hecke algebra generated by the operators T_ℓ for primes $\ell \nmid Np$ and U_p) of p-adic overconvergent eigenforms of tame level N and finite slope, *i.e.* non-zero U_p-eigenvalue. By construction, the eigencurve comes equipped with a locally finite flat morphism $w\colon \mathcal{C} \to \mathcal{W}$ sending an overconvergent modular form to its weight (or rather weight-character). From this point of view, the Eisenstein family yields a section $\mathcal{W} \to \mathcal{C}$.

The construction of eigenvarieties has since been axiomatized by Buzzard [Buz07] and extended to several types of automorphic forms. However, many geometric questions remain unanswered even for the eigencurve, both at the local and global level. Among global properties, the most basic question seems still out of reach: whether the eigencurve has a finite or infinite number of components. Other properties are better understood. For instance, the eigencurve is known to be proper (or rather, satisfies the rigid analytic version of the valuative criterion for properness) [DL16]. Moreover, there is some insight on the behaviour of the eigencurve around the boundary of the weight space, suggesting that the geometry at the eigencurve at the boundary should be relatively simple. In their recent works, Andreatta, Iovita, Pilloni tackle the boundary of the eigencurve in the language of adic geometry [AIPar]. Their approach provides an integral rather than rigid analytic construction of the eigencurve, and seems promising towards a better understanding of the boundary.

Our grasp of the local geometry of the eigencurve is perhaps more satisfactory, at least for points corresponding to classical modular forms. Coleman showed that overconvergent eigenforms of weight k and slope smaller than $k-1$ are always classical. In particular, this is the case for all ordinary eigenforms of weight $k \geq 2$, a result known as Hida's Control Theorem. Coleman's beautiful result about classicality implies that the eigencurve is etale over the weight space at all points corresponding to overconvergent forms of weight $k \geq 2$, slope less than $k-1$ and regular at p.

However, this does not apply to the points corresponding to classical modular forms of weight one. In this case, the geometry of the eigencurve becomes richer. In their paper [BD16], Bellaïche and Dimitrov consider points of the eigencurve attached to classical weight one cuspforms. These points always lie in the ordinary locus $\mathcal{C}^{\mathrm{ord}}$ of $\mathcal{C}$; they are called irregular if they correspond to the unique p-stabilization of a newform whose Hecke polynomial at p has a double root, and regular otherwise. The main theorem in *loc.cit.* describes the geometry of $\mathcal{C}$ at all regular cuspidal points, showing that $\mathcal{C}$ is smooth at such points and determining under what condition the map w is etale. The failure of etaleness has interesting arithmetic bearings in its own right. It is related to the existence of generalized eigenforms for the corresponding system of Hecke eigenvalue. As shown in [DLR15a], the q-expansion of these forms can be expressed in terms of p-adic logarithms of units over a number field. When removing the regularity assumption, one expects the geometry of the eigencurve to become more involved. Indeed, the morphism w is never etale at irregular cuspidal points and not even smooth in general.

This thesis is an exposition of the results of the joint project [BDP], in collaboration with Adel Betina and Mladen Dimitrov. The object of this work is the study of the geometry of $\mathcal{C}$ in the (deceptively simple) case of classical weight one Eisenstein points. Fix an odd Dirichlet character ϕ of conductor N coprime to p and consider the Eisenstein series

$$E_1(\mathbb{1},\phi)(z) = \frac{L(\phi,0)}{2} + \sum_{n\geqslant 1} q^n \sum_{d|n} \phi(d), \text{ where } q = e^{2i\pi z}, \tag{1}$$

which is a newform of level N admitting the p-stabilizations

$$E_1(\mathbb{1},\phi)(q) - \phi(p)E_1(\mathbb{1},\phi)(q^p) \qquad \text{and} \qquad E_1(\mathbb{1},\phi)(q) - E_1(\mathbb{1},\phi)(q^p),$$

of U_p-eigenvalues 1 and $\phi(p)$. The p-stabilizations above are the weight one-specializations of two Eisenstein families parametrized by the weight space, denoted by $\mathcal{E}_{\mathbb{1},\phi}$ and $\mathcal{E}_{\phi,\mathbb{1}}$ respectively. In particular, these two families intersect at weight one if and only if $\phi(p) = 1$, *i.e.* when $E_1(\mathbb{1},\phi)$ is irregular. When $\phi(p) \neq 1$, the constant term of each one of these p-stabilizations is non-zero at some cusps in the multiplicative part of the ordinary locus of the modular curve $X(\Gamma_0(p)\cap\Gamma_1(N))$; hence these forms are not cuspidal-overconvergent and belong to a unique Eisenstein component.

A more interesting phenomenon arises when $\phi(p) = 1$. Denote by f the unique p-stabilization of $E_1(\mathbb{1},\phi)$ and let w_f be the image of the point of $\mathcal{C}$ corresponding to f under the map w. In addition to belonging to the Eisenstein components defined by $\mathcal{E}_{\mathbb{1},\phi}$ and $\mathcal{E}_{\phi,\mathbb{1}}$, the point f also belongs to the cuspidal locus $\mathcal{C}^{\mathrm{cusp}}$ of $\mathcal{C}$, since the constant coefficient of the q-expansion of f vanishes at all cusps of the multiplicative part of the ordinary locus of the modular curve of level $\Gamma_1(N)\cap\Gamma_0(p)$. Thus, while f is not cuspidal as a classical form, it is cuspidal-overconvergent in the sense of [CM96]. We focus on studying the irreducible components of $\mathcal{C}^{\mathrm{cusp}}$ passing through f.

Modularity Theorems for a p-adic family of Galois representations

We approach the study of the cuspidal components passing through f from a Galois-theoretic point of view. The underlying philosophy is reducing the question to a modularity statement, a very fruitful approach that yielded the proof of a number of groundbreaking results, the most famous of which is Fermat's Last Theorem. Let us recall here some steps of that proof, in order to carry an analogy to our setting. Fermat's Last Theorem is famously implied by the Shimura-Tanyama Conjecture, stating the modularity of the representation associated to an elliptic curve over $\mathbb{Q}$. The p-adic representations given by the Tate module of an elliptic curve form a compatible system. A clever argument, known as the "3-5 switch", allows one to reduce the problem to showing the modularity of a certain p-adic representation ρ for an appropriate prime p such that the residual mod p-representation $\bar{\rho}$ is *residually modular*. This can be formulated in a sleek way in the language of deformation rings, introduced by Mazur [Maz89]. Roughly speaking, a deformation ring is a commutative ring R classifying liftings of a given residual representation $\bar{\rho}$ satisfying certain conditions. The modularity statement above can thus be phrased as an $R = T$ isomorphism, where T is a Hecke algebra.

Analogously, our approach consists in relating the Hecke algebra corresponding to the completed local ring of the eigencurve (resp. cuspidal eigencurve) at f, denoted by $\mathcal{T}$ (resp. $\mathcal{T}^{\mathrm{cusp}}$) to certain deformation rings. An important difference is that, in our setting, instead of considering lifts of a mod p representation to characteristic zero, we investigate lifts of the representation associated to f to a formal neighbourhood of w_f over the weight space. In particular, the deformation rings we construct are endowed with the structure of Λ-modules, where Λ is the completed local ring of the weight space $\mathcal{W}$ at w_f, isomorphic to the ring of power series $\bar{\mathbb{Q}}_p[\![X]\!]$.

More precisely, our method is borrowed from Bellaïche and Dimitrov [BD16] and consists in studying the ordinary deformation ring of a representation attached to f. Nevertheless, adapting their strategy to our setting presents several technical difficulties. Indeed, the Artin representation attached to a weight one cuspform is irreducible, and, as such, unique up to change of basis. However, since f is a weight one Eisenstein series, the representation attached to f is reducible and only defined up to semisimplification, given by the Artin representation $\mathbb{1} \oplus \phi$. Since the latter is decomposable, its deformation functor is not representable in the sense of Mazur. Moreover, the representation $\mathbb{1} \oplus \phi$ is only ordinary in a degenerate sense, given that every line is fixed under the action of $G_{\mathbb{Q}_p}$. In order to circumvent these difficulties we introduce two *reducible indecomposable* representations of $G_{\mathbb{Q}}$ with values in $\mathrm{GL}_2(\bar{\mathbb{Q}}_p)$.

$$\rho = \begin{bmatrix} \phi & \eta \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \rho' = \begin{bmatrix} 1 & \phi\eta' \\ 0 & \phi \end{bmatrix}$$

where $[\eta]$ and $[\eta']$ are bases of the lines $\mathrm{H}^1(\mathbb{Q}, \phi) \simeq \mathrm{Ext}^1_{\mathbb{Q}}(\mathbb{1}, \phi)$ and $\mathrm{H}^1(\mathbb{Q}, \phi^{-1}) \simeq \mathrm{Ext}^1_{\mathbb{Q}}(\phi, \mathbb{1})$. The representations ρ and ρ' are essentially canonical, because the extensions of ϕ by $\mathbb{1}$ (resp. $\mathbb{1}$ by ϕ) are classified by one-dimensional $\bar{\mathbb{Q}}_p$-vector spaces. The ordinary deformation functors for ρ and ρ' are representable by universal ordinary deformation rings $\mathcal{R}^{\mathrm{ord}}_{\rho}$ and $\mathcal{R}^{\mathrm{ord}}_{\rho'}$; their representability is related to the fact that the only endomorphisms of ρ and ρ' are scalar, unlike those of $\mathbb{1} \oplus \phi$.

After constructing $\mathcal{R}^{\mathrm{ord}}_{\rho}$ and $\mathcal{R}^{\mathrm{ord}}_{\rho'}$, we construct a third deformation ring, denoted by $\mathcal{R}^{\mathrm{cusp}}$, classifying pairs of ordinary deformations of (ρ, ρ') sharing the same traces and Frobenius action on the unramified $G_{\mathbb{Q}_p}$-quotient. The notation $\mathcal{R}^{\mathrm{cusp}}$ is suggestive of the fact that we expect this ring to parametrize deformations of (ρ, ρ') whose traces encode the Hecke eigenvalues of *cuspidal* families specializing to f. The heuristic behind this definition can be explained as follows. Given a Λ-adic family of cuspidal eigenforms specializing to f, it should correspond to a representation $G_{\mathbb{Q}} \to \mathrm{GL}_2(\Lambda)$ which is generically irreducible (because the family is cuspidal), but residually reducible (because it specializes to the Eisenstein series f). A lemma of Ribet implies that, in this setting, changing the lattice in the corresponding representation over the fraction field of Λ, we obtain a second representation whose associated residual representation is a non-trivial extension of ϕ by $\mathbb{1}$. Hence, choosing the basis appropriately, the representation is a lift of

ρ. Similarly, because of the symmetry of the construction with respect to the characters $\mathbb{1}$ and ϕ, another choice of lattice yields a second representation lifting ρ'. The two representations share the same trace and the same Frobenius eigenvalue on the unramified quotient for the action of $G_{\mathbb{Q}_p}$, thus giving a morphism $\mathcal{R}^{\mathrm{cusp}} \to \Lambda$. The first modularity result proved in this work can be summarized as follows (Sec. 2.3.1, compare with [BDP, Thm. A(i)-B(i)]).

Theorem. *(i) There is a Λ-algebra isomorphisms $\mathcal{R}^{\mathrm{cusp}} \to \mathcal{T}^{\mathrm{cusp}}$.*

(ii) The structural morphism $\Lambda \to \mathcal{T}^{\mathrm{cusp}}$ is an isomorphism. In particular, the cuspidal eigencurve $\mathcal{C}^{\mathrm{cusp}}$ is etale at f.

Let us explain some key ingredients of the proof. Much like in Bellaïche and Dimitrov's work, the fundamental step consists in the computation of the dimension of the tangent space and the relative tangent space of the deformation functor represented by $\mathcal{R}^{\mathrm{cusp}}$. A well-known observation is that these tangent spaces can be described in terms of Galois cohomology groups for the adjoint representations of ρ and ρ'. While these representations do not have finite image, their semisimplifications do; this allows us to relate the cohomology groups above to units of the splitting field of the character ϕ, denoted by H, via class field theory. A central role, in particular, is played by the $\mathcal{L}$-invariants of the characters ϕ and ϕ^{-1}, a linear combination of p-adic logarithms of p-units of H. An application of the Baker-Brumer Theorem shows the $\bar{\mathbb{Q}}$-linear independence of $\mathcal{L}(\phi)$ and $\mathcal{L}(\phi^{-1})$ (when ϕ is not quadratic), which allows us to show the vanishing of the relative tangent space.

It remains to relate the deformation ring $\mathcal{R}^{\mathrm{cusp}}$ to the completed local ring of the eigencurve. The aim is constructing a surjective map $\mathcal{R}^{\mathrm{cusp}} \to \mathcal{T}^{\mathrm{cusp}}$, a step that is far more subtle in this setting than in [BD16], due to the fact that the representations involved are residually reducible. Standard arguments in deformation theory guarantee the existence of a representation with coefficients in the total fraction field of $\mathcal{T}^{\mathrm{cusp}}$ whose trace encodes the desired Hecke eigenvalues. Applying a method of Mazur and Wiles, we construct two $G_{\mathbb{Q}}$-stable $\mathcal{T}^{\mathrm{cusp}}$-lattices with residual representation ρ and ρ' respectively. This shows the existence of the desired surjective map. We would like to remark that both galois-theoretic and automorphic inputs are involved in the proof of the theorem. On the one hand, the computation on the tangent space of $\mathcal{R}^{\mathrm{cusp}}$ shows the uniqueness of the cuspidal component of the eigencurve and its etaleness over the weight space. On the other hand, the existence of a surjective morphism from $\mathcal{R}^{\mathrm{cusp}}$ to $\mathcal{T}^{\mathrm{cusp}}$ shows that $\mathcal{R}^{\mathrm{cusp}}$ has Krull dimension greater than or equal (and, in fact, equal) to one, which is a priori not clear solely from the point of view of representation theory.

As shown in Proposition 1.4.5, the ring $\mathcal{R}^{\mathrm{cusp}}$ can be simultaneously realized as a quotient of $\mathcal{R}_{\rho}^{\mathrm{ord}}$ and $\mathcal{R}_{\rho'}^{\mathrm{ord}}$. However, both ordinary deformation rings admit another natural quotient, denoted by $\mathcal{R}_{\rho}^{\mathrm{eis}}$ for $\mathcal{R}_{\rho}^{\mathrm{ord}}$ and $\mathcal{R}_{\rho'}^{\mathrm{eis}}$ for $\mathcal{R}_{\rho'}^{\mathrm{ord}}$, classifying *reducible* ordinary deformations of ρ and ρ' respectively. Since reducible representations capture the eigenvalues of Eisenstein series, this suggests that the modularity theorem above can be refined to include Eisenstein families as well. Denote by $\mathcal{T}_{\rho}^{\mathrm{ord}}$ (resp. $\mathcal{T}_{\rho'}^{\mathrm{ord}}$) the completed local ring of the Zariski closed subspace of $\mathcal{C}$ given by the union of $\mathcal{C}^{\mathrm{cusp}}$ and the Eisenstein component corresponding to $\mathcal{E}_{\mathbb{1},\phi}$ (resp. $\mathcal{E}_{\phi,\mathbb{1}}$). Section 2.4 is dedicated to proving the following result [BDP, Thm.B(ii)].

Theorem. *There are isomorphisms of Λ-algebras of relative complete intersection*

$$\mathcal{R}_{\rho}^{\mathrm{ord}} \to \mathcal{T}_{\rho}^{\mathrm{ord}} \qquad \textit{and} \qquad \mathcal{R}_{\rho'}^{\mathrm{ord}} \to \mathcal{T}_{\rho'}^{\mathrm{ord}}$$

with respect to the augmentation maps $\mathcal{T}_{\rho}^{\mathrm{ord}} \to \Lambda$ and $\mathcal{T}_{\rho'}^{\mathrm{ord}} \to \Lambda$ corresponding to the systems of Hecke eigenvalues of $\mathcal{E}_{\mathbb{1},\phi}$ and $\mathcal{E}_{\phi,\mathbb{1}}$ respectively.

The proof of this result relies on Wiles' Numerical Criterion, a powerful tool in commutative algebra devised to show an analogous modularity statement towards the proof of Fermat's Last Theorem. The fact that the rings are of relative complete intersection follows directly from the criterion, although one could easily prove it directly determining the structure of $\mathcal{T}_\rho^{\mathrm{ord}}$ (resp. $\mathcal{T}_{\rho'}^{\mathrm{ord}}$).

The Eisenstein Ideal and the Kubota-Leopoldt p-adic L-function

Serre's study of Eisenstein series as specializations of a p-adic family lead to a first striking application. For each classical weight $k > 2$, the constant coefficient of the weight k Eisenstein series is a (p-deprived) special value of the Riemann zeta function. From the simple observation that the Hecke eigenvalues of Eisenstein series are analytic functions of the weight, Serre managed to deduce the analyticity of the constant coefficient as well. The constant coefficient of the Eisenstein family is (a scalar multiple of) the Kubota-Leopoldt p-adic L-function $L_p(\omega_p^i, s)$, where ω_p is the Teichmuller character and $0 \leq i \leq p-1$, which interpolates the special values of the Riemann zeta function. For $i \neq 0$, this p-adic L-function can be interpreted as an element of the Iwasawa algebra $\mathbb{Z}_p[\![\mathbb{Z}_p]\!] \simeq \mathbb{Z}_p[\![X]\!]$; elements of this ring also correspond to p-adic measures over $\mathbb{Z}_p$. Through this interpolation property, Serre recovers Kummer's congruences between Bernoulli numbers and obtains many more.

Dirichlet L-functions and their p-adic analogues encode information about congruences between Eisenstein and cuspidal forms; the study of these congruences has yielded a number of remarkable applications, both in the classical and the p-adic setting. This theme was introduced in Mazur's influential paper [Maz77], devoted to the study of the torsion part of the Mordell-Weil group of the jacobian of the modular curve $X_0(N)$. Here the notion of Eisenstein ideal makes its first appearance. Geometrically, it is the ideal defining the scheme-theoretic intersection between the cuspidal locus and the line given by the system of Hecke eigenvalues of an Eisenstein series in the spectrum of the Hecke algebra. The relation between the Eisenstein ideal and the Kubota-Leopoldt p-adic L-function is at the core of the original proof of Iwasawa Main Conjecture due to Mazur and Wiles [MW84] building on an idea of Ribet. Let $\mathbb{Q}(\mu_{p^n})$ for $n \in \mathbb{Z}_{>0}$ be the p^n-th cyclotomic field, and denote $\mathbb{Q}(\mu_{p^\infty}) = \cup_{n\in\mathbb{Z}_{>0}}\mathbb{Q}(\mu_{p^n})$. Let X_n be the p-Sylow of the class groups of $\mathbb{Q}(\mu_{p^n})$. By class field theory, X_n classifies unramified pro p-abelian extension of $\mathbb{Q}(\mu_{p^n})$. The inverse limit $X_\infty = \varprojlim X_n$ comes equipped with a continuous action of $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty}), \mathbb{Q}(\mu_p)) \simeq \mathbb{Z}_p$, hence is a module over the Iwasawa algebra $\mathbb{Z}_p[\![X]\!]$. The Iwasawa Main Conjecture links the $\mathbb{Z}_p[\![X]\!]$-module X_∞ to the Kubota Leopoldt p-adic L-function $L_p(\omega^i, s)$ through an arithmetic invariant, the characteristic ideal.

In *loc.cit*, congruences between cuspidal and Eisenstein series bridge the gap between the analytic and the algebraic side. One can construct a Galois representation over the fraction field of Hida's ordinary Hecke algebra. For an appropriate choice of lattice, this yields an irreducible representation which is residually reducible modulo the Eisenstein ideal. From this representation one can extract a non-trivial cocycle, giving a pro p-unramified extension of $\mathbb{Q}(\mu_{p^\infty})$. The characteristic ideal associated to the Kubota-Leopoldt p-adic L-function is isomorphic to the quotient of the Hecke algebra by the Eisenstein ideal. This construction thus provides a large enough quotient of X_∞ to show the desired relation between the characteristic ideals.

In this work, we exploit the relation between the Eisenstein ideal and the p-adic L-function to calculate the order of vanishing of the latter at $s = 0$. Analogously to the Eisenstein family constructed by Serre for tame level 1, the constant term of the q-expansion at the cusp ∞ of the family $\mathcal{E}_{\mathbb{1},\phi}$ is also a p-adic L-function, namely $L_p(\phi\omega_p, s)$, interpolating the classical p-deprived special values. This p-adic L-function defines an element of the Iwasawa algebra $\mathcal{O}_F[\![X]\!]$, where F is a finite extension of $\mathbb{Q}_p$ containing the values of ϕ; thus, it can be viewed as a rigid analytic function over the component of the weight space containing w_f. Denote by ζ_ϕ the image of the p-adic L-function in the completed local ring $\Lambda \simeq \bar{\mathbb{Q}}_p[\![X]\!]$ of $\mathcal{W}$ at w_f. As we observed, the constant term of $\mathcal{E}_{\mathbb{1},\phi}$ vanishes at w_f, hence $L_p(\phi\omega_p, s)$ has a zero at $s = 0$.

Let $\mathfrak{p}^{\mathrm{eis}}_{\mathbb{1},\phi}$ be the kernel of the morphism $\pi^{\mathrm{eis}}_{\mathbb{1},\phi} \colon \mathcal{T} \to \Lambda$ corresponding to the system of eigenvalues of the Eisenstein family $\mathcal{E}_{\mathbb{1},\phi}$; denote by $\mathcal{J}^{\mathrm{eis}}_{\mathbb{1},\phi}$ the Eisenstein ideal attached to $\mathfrak{p}^{\mathrm{eis}}_{\mathbb{1},\phi}$, *i.e.* the image of $\mathfrak{p}^{\mathrm{eis}}_{\mathbb{1},\phi}$ in $\mathcal{T}^{\mathrm{cusp}}$. We can rephrase the results of Section 3.1 as follows. The morphism $\pi^{\mathrm{eis}}_{\mathbb{1},\phi}$ induces an isomorphism of Λ-modules

$$\mathcal{T}^{\mathrm{cusp}}/\mathcal{J}^{\mathrm{eis}}_{\mathbb{1},\phi} \simeq \Lambda/(\zeta_\phi). \tag{2}$$

The existence of this isomorphism follows from the general theory of congruence modules of Ohta [Oht03] (more specifically, from the version of Lafferty [Laf], where the regularity assumptions on the characters are lifted). Here we give a simpler yet similar in spirit proof of the statement, in the more modest Λ-adic setting. Our argument requires a careful determination of the constant coefficient of q-expansion of $\mathcal{E}_{\mathbb{1},\phi}$ at *all cusps*. The computation of the tangent space of the deformation ring $\mathcal{R}^{\mathrm{ord}}_\rho$, together with the isomorphism $\mathcal{R}^{\mathrm{ord}}_\rho \simeq \mathcal{T}^{\mathrm{ord}}_\rho$, shows that the tangent directions of the cuspidal family and the Eisenstein family $\mathcal{E}_{\mathbb{1},\phi}$ are distinct, thus proving that the quotient $\mathcal{T}^{\mathrm{cusp}}/\mathcal{J}^{\mathrm{eis}}_{\mathbb{1},\phi}$ is isomorphic to $\bar{\mathbb{Q}}_p$. This provides an independent proof of the famous result of Ferrero and Greenberg [FG78],[BDP, Prop.4.7].

Theorem (Ferrero-Greenberg). *The p-adic L-function $L_p(\phi\omega_p, s)$ has a simple zero at $s = 0$.*

It is worth noting that an analogous statement to (2) can be proved by replacing the Eisenstein family $\mathcal{E}_{\mathbb{1},\phi}$ with $\mathcal{E}_{\phi,\mathbb{1}}$. Despite the fact that the constant coefficient of the

q-expansion of $\mathcal{E}_{\phi,\mathbb{1}}$ vanishes at the cusp ∞, the constant term at the cusp 0 is, up to a non-zero scalar, the Kubota-Leopoldt L-function $L_p(\phi^{-1}\omega_p, s)$. The appearance of the L-function attached to the inverse of ϕ is, in fact, unsurprising. It reflects the fact that, twisting the family $\mathcal{E}_{\phi,\mathbb{1}}$ by ϕ^{-1}, one obtains the Eisenstein family $\mathcal{E}_{\mathbb{1},\phi^{-1}}$ whose constant coefficient at ∞ is precisely $\frac{1}{2}L_p(\phi^{-1}\omega_p, 0)$.

The Gross-Stark Conjecture and p-adic logarithms of units of a number field

Our method of proving the theorem of Ferrero and Greenberg is essentially Galois-theoretic; it is natural to wonder if this technique can be pushed further to provide an explicit formula for the derivative of the Kubota-Leopoldt p-adic L-function at $s = 0$. In 1988, Gross conjectured a formula for this derivative, which can be thought of as a p-adic analogue of the conjectures of Stark predicting the leading term of the Artin L-function at $s = 0$. Gross's formula relates the derivative of the p-adic L-function to the $\mathcal{L}$-invariant $\mathcal{L}(\phi)$, the p-adic logarithm of a p-unit of the splitting field of the character ϕ [Gro82, Conj. 3.13]. The conjecture was proved by Gross himself over $\mathbb{Q}$. Some instances of the conjecture over totally real fields were proved by Darmon, Dasgupta and Pollack in [DDP11]; finally, a complete proof of the conjecture was given by Dasgupta, Kakde and Ventullo [Ven15], [DKV18]. The techniques of [DDP11] are closely related to those of the present work.

The strategy of *loc.cit.* consists in constructing a cuspidal Hida family specializing to f as follows. Using the fact that the Kubota-Leopoldt p-adic L-function has a simple pole at $s = 1$ and taking a suitable product of Eisenstein series, one obtains a Λ-adic family, with constant leading coefficient given by the classical Dirichlet L-function $L(\phi, 0)$. Applying Hida's ordinary idempotent to an appropriate linear combination of this family and certain Eisenstein series provides a Hida family. Then one constructs a Hecke operator extracting the cuspidal part of this linear combination. This cuspidal Hida family is not an eigenform. However, its striking property is that it yields an "eigenform in a first order infinitesimal neighbourhood of weight one" (*loc.cit*). This infinitesimal cuspform plays a pivotal role in proving the Gross-Stark Conjecture.

Theorem (Gross-Stark Conjecture over $\mathbb{Q}$)**.** *The p-adic L-function $L_p(\phi\omega_p, s)$ satisfies*

$$L'_p(\phi\omega_p, 0) = -\mathcal{L}(\phi)L(\phi, 0).$$

From the etaleness of the cuspidal eigencurve at f, there is a unique Λ-adic cuspidal family of eigenforms specializing to f; thus, the first order deformation defined by Darmon, Dasgupta and Pollack is *a fortiori* encoded in the tangent space of $\mathcal{T}^{\mathrm{cusp}}$. The authors of *loc.cit.* use this "eigenform of weight $1+\varepsilon$" to construct a non trivial cocycle associated to the corresponding Galois representation.

In the present work, we obtain a new proof of the Gross-Stark Conjecture; unlike in *loc. cit*, the cohomological datum is the starting point of our investigation. Indeed, the explicit calculation of the tangent space of the cuspidal deformation ring $\mathcal{R}^{\mathrm{cusp}}$, together with the isomorphism $\mathcal{R}^{\mathrm{cusp}} \to \mathcal{T}^{\mathrm{cusp}}$ allows us to compute the derivatives of the coefficients of the unique cuspidal eigenform $\mathcal{F}$ specializing to f at weight one. This information, combined with the explicit knowledge of the coefficients for the Eisenstein families $\mathcal{E}_{\mathbb{1},\phi}$ and $\mathcal{E}_{\phi,\mathbb{1}}$, is sufficient to determine the structure of $\mathcal{T}$, the completed local ring of the eigencurve at f. This calculation has an immediate application. Denote $\mathcal{T}_{w_f} = \mathcal{T}/\mathfrak{m}_\Lambda\mathcal{T}$ the relative local ring of the eigencurve over the weight space at f. This ring is the $\bar{\mathbb{Q}}_p$-dual of the generalized eigenspace of f in the space of ordinary overconvergent weight one modular forms. The q-expansion of these generalized eigenforms can thus be computed explicitly in terms of p-units of the splitting field of ϕ. This observation was originally made by Darmon, Rotger and Lauder in [DLR15a]. Inspired by the work of Bellaïche and Dimitrov, the authors produced an overconvergent generalized eigenform attached to certain weight one theta series for a real quadratic field K, defining points at which the eigencurve is *not* etale over the weight space. The q-expansion of this form was expressed in terms of p-adic logarithms of units of a ring class field of K, perhaps suggesting an alternative approach to the explicit class field theoy of real quadratic fields to Stark's conjectures and Gross's p-adic analogues. This type of results have since been treated more systematically in [DLRar], where the authors study the generalized eigenspaces of weight one newforms.

In our setting we can easily described the generalized eigenspace attached to f. The supplement of f has a basis given by

$$\tfrac{\partial}{\partial X}\big|_{X=0}\left(\mathcal{E}_{\mathbb{1},\phi}-\mathcal{F}\right) \qquad \text{and} \qquad \tfrac{\partial}{\partial X}\big|_{X=0}\left(\mathcal{E}_{\phi,\mathbb{1}}-\mathcal{F}\right)$$

where X is a uniformizer for Λ. The coefficients of the q-expansion of these overconvergent forms are logarithms of units of H; even more explicitly, they can be expressed as combinations of the logarithms of *rational* units and the $\mathcal{L}$-invariants of the characters ϕ and ϕ^{-1} ([BDP, Thm.C], Thm.3.3.2). The Gross-Stark conjecture can thus be proved by observing that, due to the irregularity of the character ϕ, the classical weight one eigenform $E_1(\mathbb{1},\phi)$ lies in the generalized eigenspace of f. Hence, one can express $E_1(\mathbb{1},\phi)$ as a linear combination of the overconvergent forms giving a basis of the generalized eigenspace, simply by comparing their Hecke eigenvalues to those of $E_1(\mathbb{1},\phi)$. This yields a non-trivial relation between the leading term of $E_1(\mathbb{1},\phi)$, *i.e.* the $\frac{1}{2}L(0,\phi)$, and the derivative of the Kubota-Leopoldt p-adic L-function at $s=0$. Once again, this argument is an application of the general philosophy that congruences between cuspforms and Eisenstein series are a powerful tool to relate p-adic L-functions and the arithmetic of number fields.

Chapter 1

Deformation Theory

The theory of deformations of Galois representations was introduced by Mazur in [Maz89] and has since been extensively used to prove a variety of results related to automorphic forms. In this chapter, we will focus our attention on deformations of the Galois representation attached to an Eisenstein series of weight one and irregular at p. The irregularity of the representation creates some difficulties in the formulation of an (otherwise well-understood) ordinary deformation functor. We circumvent these issues through some *ad hoc* definitions and obtain an ordinary deformation ring; we then single out a quotient of the latter classifying deformations corresponding to cuspforms.

A major ingredient in describing the deformation rings involved is computing their Zariski tangent spaces, which can be interpreted in terms of Galois cohomology groups of certain adjoint representations. The representation attached to weight one Eisenstein series has finite image; let H be its splitting field. The Galois cohomology groups above are tied to p-units of H via class field theory. In particular, a special role is played by $\mathcal{L}$-invariants of odd Dirichlet characters, and their linear independence over $\bar{\mathbb{Q}}$. The connection with $\mathcal{L}$-invariants is natural in view of their relation with the derivative of the Kubota-Leopoldt p-adic L-function proved by Gross [Gro82].

1.1 Galois cohomology preliminaries

The Galois representation attached to an Eisenstein series of weight one is the direct sum of two finite order characters; in particular, it is reducible. The deformation functor attached to a reducible representation is not, in general, representable; it is, however for a non-split extension of two characters. The aim of this section is constructing such an extension and showing that it is still indecomposable as a representation of the decomposition group at p. This allows us to define an ordinary deformation functor attached to this representation.

Via class field theory, we relate certain values of Galois cocycles corresponding to the extensions above to $\mathcal{L}$-invariants. The $\bar{\mathbb{Q}}$-independence of $\mathcal{L}$-invariant will be essential to compute the dimension of the tangent space of the cuspidal deformation ring $\mathcal{R}^{\mathrm{cusp}}$.

1.1.1 Construction of a non-split Galois extension of finite order characters

Let p be a prime. Fix embeddings $\iota_p \colon \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ and $\iota_\infty \colon \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$. The choice of an embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$ determines a prime of $\bar{\mathbb{Q}}$ above p, and an isomorphism of its decomposition group with $G_{\mathbb{Q}_p}$. The inclusion ι_∞ provides a choice of complex conjugation $\tau \in G_{\mathbb{Q}}$. Let $\phi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a primitive character of conductor N for $(N,p) = 1$ and such that $\phi(-1) = -1$. We will assume throughout this chapter that

$$\phi(p) = 1 \quad \text{(irregularity condition)}. \tag{1.1}$$

Via the isomorphism $\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \simeq (\mathbb{Z}/N\mathbb{Z})^\times$, we view ϕ as an odd Dirichlet character of conductor N. The choice of embeddings ι_p, ι_∞ allows us to view ϕ as a Galois character valued in $\bar{\mathbb{Q}}_p$.

Let H be the splitting field of the character ϕ. It is a cyclic extension of $\bar{\mathbb{Q}}$. Since

ϕ is an odd character, H is totally imaginary, so that $[H:\mathbb{Q}] = 2r$ for some $r \in \mathbb{Z}_{>0}$. Moreover, the condition $\phi(p) = 1$ implies that p splits completely in H. Denote by σ a generator of $G = \mathrm{Gal}(H/\mathbb{Q})$ and v_0 the place of H determined by the embedding $\iota_p \colon \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$. Since p splits completely in H, the Galois group acts simply transitively on the set of places of $\mathbb{Q}$ above p. Thus, we can denote such places as $v_i = v_0 \circ \sigma^{-i}$ for $0 \leq i < 2r$. Let H_v be the completion of H at the place v and let $\mathcal{O}$ and $\mathcal{O}_v$ be the ring of integers of H and H_v respectively.

Denote $\mathrm{Hom}(-, \bar{\mathbb{Q}}_p)$ the functor of continuous group homomorphisms.

Lemma 1.1.1. *The* $\bar{\mathbb{Q}}_p[G]$*-module* $\mathrm{Hom}(G_H, \bar{\mathbb{Q}}_p)$ *is isomorphic to*

(i) $\ker\left(\bigoplus_{i=0}^{2r-1} \mathrm{Hom}(\mathcal{O}_{v_i}^\times, \bar{\mathbb{Q}}_p) \to \mathrm{Hom}(\mathcal{O}^\times, \bar{\mathbb{Q}}_p)\right)$

(ii) $\ker\left(\bigoplus_{i=0}^{2r-1} \mathrm{Hom}(H_{v_i}^\times, \bar{\mathbb{Q}}_p) \to \mathrm{Hom}(\mathcal{O}[1/p]^\times, \bar{\mathbb{Q}}_p)\right)$.

Proof. By global class theory, the global Artin homomorphism $\theta_H \colon \mathbb{A}_H^\times / H^\times \to G_H^{\mathrm{ab}}$ is surjective with kernel given by the connected component of the identity, isomorphic to the product $\prod_{w|\infty} H_w^\times$, because H is totally complex. Let S be a finite set of finite places in H and let U_S be the subgroup of $\mathbb{A}_H^\times$ given by $U_S = \prod_{v \notin S} \mathcal{O}_v^\times \times \prod_{v \in S} H_v^\times$ (with the convention that $\mathcal{O}_w = H_w$ for every place above infinity). The cokernel of the map

$$\psi_S \colon U_S \to \mathbb{A}_H^\times / H^\times$$

is finite. Since for $S' \subset S$, we have $U_{S'} \subset U_S$, it suffices to see it for $S' = \emptyset$, in which case $\mathbb{A}_H^\times / H^\times U_S$ is isomorphic to the class group of $\mathcal{O}$, which is finite. Let $\mathcal{O}_{H,S}^\times = H^\times \cap \psi_S(U_S)$. Thus, $\mathrm{Hom}(\mathbb{A}_H^\times / H^\times, \bar{\mathbb{Q}}_p)$ is equal the kernel of $\mathrm{Hom}(U_S, \bar{\mathbb{Q}}_p) \to \mathrm{Hom}(\mathcal{O}_{H,S}^\times, \bar{\mathbb{Q}}_p)$. By continuity

$$\mathrm{Hom}(U_S, \bar{\mathbb{Q}}_p) = \bigoplus_{v \in S} \mathrm{Hom}(H_v^\times, \bar{\mathbb{Q}}_p) \oplus \bigoplus_{v \notin S, v | p} \mathrm{Hom}(\mathcal{O}_v^\times, \bar{\mathbb{Q}}_p).$$

In particular, the places above at infinity give no contribution. It follows that $\mathrm{Hom}(G_H, \bar{\mathbb{Q}}_p)$ is isomorphic to the kernel of

$$\left(\bigoplus_{v \in S} \mathrm{Hom}(H_v^\times, \bar{\mathbb{Q}}_p) \oplus \bigoplus_{v \notin S, v | p} \mathrm{Hom}(\mathcal{O}_v^\times, \bar{\mathbb{Q}}_p) \right) \to \mathrm{Hom}(\mathcal{O}_{H,S}^\times, \bar{\mathbb{Q}}_p).$$

Taking $S = \emptyset$ and $S = \{v \mid p\}$, we obtain the desired results. $\square$

It follows that there is an exact sequence of $\bar{\mathbb{Q}}_p[G]$-modules

$$0 \to \mathrm{Hom}(G_H, \bar{\mathbb{Q}}_p) \to \bigoplus_{i=0}^{2r-1} \mathrm{Hom}(\mathcal{O}_{v_i}^\times, \bar{\mathbb{Q}}_p) \to \mathrm{Hom}(\mathcal{O}^\times, \bar{\mathbb{Q}}_p) \to 0. \tag{1.2}$$

The exactness of the sequence at the last stage is equivalent to the Leopoldt Conjecture for H, which is known in our case because H is abelian over $\mathbb{Q}$. Hence $\mathrm{Hom}(G_H, \bar{\mathbb{Q}}_p)$ is a $\bar{\mathbb{Q}}_p$-vector space of dimension

$$\sum_{i=0}^{2r-1} \dim_{\bar{\mathbb{Q}}_p} \mathrm{Hom}(\mathcal{O}_{v_i}^\times, \bar{\mathbb{Q}}_p) - \dim_{\bar{\mathbb{Q}}_p} \mathrm{Hom}(\mathcal{O}^\times, \bar{\mathbb{Q}}_p) = 2r - (r-1) = r+1,$$

because p is totally split in H and H is totally imaginary. We now describe the structure of $\mathrm{Hom}(G_H, \bar{\mathbb{Q}}_p)$ as a $\bar{\mathbb{Q}}_p[G]$-module.

Proposition 1.1.2. *There is an isomorphism of $\bar{\mathbb{Q}}_p[G]$-modules*

$$\mathrm{Hom}(G_H, \bar{\mathbb{Q}}_p) \simeq \bigoplus_{\psi = \mathbb{1} \text{ or } \psi \text{ odd}} \bar{\mathbb{Q}}_p \psi.$$

In particular, $\mathrm{H}^1(\mathbb{Q}, \psi) = 0$ *for every even character* $\psi \neq \mathbb{1}$ *of* G *and*

$$\dim_{\bar{\mathbb{Q}}_p} \mathrm{H}^1(\mathbb{Q}, \psi) = 1$$

if ψ *is odd or trivial.*

Proof. The $\bar{\mathbb{Q}}_p[G]$-module $\bigoplus_{i=0}^{2r-1} \mathrm{Hom}(\mathcal{O}_{v_i}^\times, \bar{\mathbb{Q}}_p) = \bigoplus_{i=0}^{2r} \bar{\mathbb{Q}}_p(\log_p \circ v_i)$ is isomorphic to $\bar{\mathbb{Q}}_p[G]$ as a left $\bar{\mathbb{Q}}_p[G]$-module via the morphism $\sum_{i=0}^{2r-1} a_i \sigma^i \mapsto \sum_{i=0}^{2r-1} \log_p \circ v_i$. As a representation of G, $\bar{\mathbb{Q}}_p[G] \simeq \bigoplus_\psi \bar{\mathbb{Q}}_p \psi$ where the sum runs over all the characters of G, since G is cyclic. As a $\bar{\mathbb{Q}}_p[G]$-module, $\mathrm{Hom}(\mathcal{O}^\times, \bar{\mathbb{Q}}_p)$ decomposes as $\bigoplus_{\psi \neq \mathbb{1} \text{ even}} \bar{\mathbb{Q}}_p \psi$ (compare with [BD16, Sec. 3.2]), hence the result follows from the exact sequence (1.2).

The inflation-restriction exact sequence for $G_H \subset G_{\mathbb{Q}}$ gives

$$0 \to \mathrm{H}^1(G, \bar{\mathbb{Q}}_p) \to \mathrm{H}^1(\mathbb{Q}, \psi) \to \mathrm{H}^1(H, \bar{\mathbb{Q}}_p)^G \to \mathrm{H}^2(G, \bar{\mathbb{Q}}_p).$$

Since G is finite and $\bar{\mathbb{Q}}_p$ is a field of characteristic 0, the cohomology groups $\mathrm{H}^k(G, \bar{\mathbb{Q}}_p)$ vanish for $k > 0$. Thus, $\mathrm{H}^1(\mathbb{Q}, \psi)$ can be identified with the ψ^{-1}-eigenspace of $\mathrm{H}^1(H, \bar{\mathbb{Q}}_p) = \mathrm{Hom}(G_H, \bar{\mathbb{Q}}_p)$; the claim follows. □

Remark 1.1.3. The analogous statement to Lemma 1.1.1 does, in fact, hold when replacing the functor $\mathrm{Hom}(-, \bar{\mathbb{Q}}_p)$ with $\mathrm{Hom}(-, \mathbb{Q}_p)$. The choice of using coefficients in $\bar{\mathbb{Q}}_p$ is motivated by the previous proposition.

Relation with the $\mathcal{L}$-invariant

Recall that the embedding $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ determines a place v_0 of H and embeddings $G_{\mathbb{Q}_p} = G_{H_{v_0}} \subset G_H$ and $I_{\mathbb{Q}_p} = I_{H_{v_0}} \subset G_H$ yielding canonical restriction maps

$$\mathrm{res}_p \colon \mathrm{H}^1(H, \bar{\mathbb{Q}}_p) \to \mathrm{H}^1(\mathbb{Q}_p, \bar{\mathbb{Q}}_p) \quad \text{and} \quad \mathrm{res}_{I_{\mathbb{Q}_p}} : \mathrm{H}^1(H, \bar{\mathbb{Q}}_p) \to \mathrm{Hom}(I_{\mathbb{Q}_p}, \bar{\mathbb{Q}}_p). \tag{1.3}$$

Proposition 1.1.4. *Let ψ be a character of G, either odd or trivial. Then the image of the map $\mathrm{res}_{I_{\mathbb{Q}_p}} \colon \mathrm{H}^1(\mathbb{Q}, \psi) = \mathrm{H}^1(H, \bar{\mathbb{Q}}_p)^{\psi^{-1}} \to \mathrm{Hom}(I_{\mathbb{Q}_p}, \bar{\mathbb{Q}}_p)$ is a one-dimensional $\bar{\mathbb{Q}}_p$-vector space, independent of the choice of the character ψ.*

Proof. By local class field theory, the image of $\mathrm{Hom}(G_{H_{v_0}}, \bar{\mathbb{Q}}_p) = \mathrm{Hom}(G_{\mathbb{Q}_p}, \bar{\mathbb{Q}}_p) \to \mathrm{Hom}(I_{H_{v_0}}, \bar{\mathbb{Q}}_p) = \mathrm{Hom}(I_{\mathbb{Q}_p}, \bar{\mathbb{Q}}_p)$ is isomorphic to $\mathrm{Hom}(\mathcal{O}_{v_0}^\times, \bar{\mathbb{Q}}_p)$. In particular, it is one-dimensional over $\bar{\mathbb{Q}}_p$. Thus, it suffices to show that the restriction of $\mathrm{res}_{I_{\mathbb{Q}_p}}$ to the ψ^{-1}-eigenspace of $\mathrm{Hom}(G_H, \bar{\mathbb{Q}}_p)$ is injective. By compatibility between local and global class field theory, and the exact sequence (1.2), there is a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(G_H, \bar{\mathbb{Q}}_p)^{\psi^{-1}} \arrow[r, "\theta_H^\wedge"] \arrow[d, "\mathrm{res}_{I_{\mathbb{Q}_p}}"] & \left(\bigoplus_{i=0}^{2r-1} \mathrm{Hom}(\mathcal{O}_{v_i}^\times, \bar{\mathbb{Q}}_p)\right)^{\psi^{-1}} \arrow[d, "\pi_0"] \\ \mathrm{Hom}(I_{\mathbb{Q}_p}, \bar{\mathbb{Q}}_p) & \mathrm{Hom}(\mathcal{O}_{v_0}^\times, \bar{\mathbb{Q}}_p) \arrow[l] \end{tikzcd}$$

where the top horizontal map is induced by the Artin reciprocity map θ_H by duality and π_0 is the projection on the v_0-component. The horizontal maps are injective.

Let η be an element in the ψ^{-1}-component of $\mathrm{Hom}(G_H, \bar{\mathbb{Q}}_p)$ such that $\mathrm{res}_{I_{\mathbb{Q}_p}}(\eta) = 0$. Since G acts transitively over the places v_i for $i = 0, \ldots 2r-1$, it follows that $\theta_H^\wedge(\eta) = 0$; in other words, π_0 is injective on the ψ^{-1}-eigenspace. Thus, $\mathrm{res}_{I_{\mathbb{Q}_p}}$ is injective on $\mathrm{H}^1(\mathbb{Q}, \psi)$. □

For any odd or trivial character ψ of G, we can choose a cocycle η_ψ whose class generates $\mathrm{H}^1(\mathbb{Q}, \psi)$. In particular, we can fix the homomorphism $\eta_{\mathbb{1}} = \log_p \circ \chi$, where χ is the p-adic cyclotomic character as generator of $\mathrm{H}^1(\mathbb{Q}, \bar{\mathbb{Q}}_p) = \mathrm{Hom}(G_{\mathbb{Q}}, \bar{\mathbb{Q}}_p)$ and $\log_p$ is the standard choice of p-adic logarithm satisfying $\log_p(p) = 0$; with such choice $\eta_{\mathbb{1}}(\mathrm{Frob}_\ell) = \log_p(\ell)$ for every prime $\ell \neq p$. By the previous proposition, we can normalize the cocycles η_ψ in such a way that

$$\eta_\psi|_{I_{\mathbb{Q}_p}} = (\log_p \circ \chi)|_{I_{\mathbb{Q}_p}}. \tag{1.4}$$

for every ψ. Note that the cocycles η_ψ are unramified for all primes $\ell \nmid Np$, since η_ψ defines a $\mathbb{Z}_p$-extension of H which can only be ramified at primes above p by class field theory. The kernel of $\mathrm{res}_{I_{\mathbb{Q}_p}}$ is spanned by $(\eta_{\mathbb{1}} - \eta_\psi)$ for ψ odd.

Denote

$$L_{\bar{\mathbb{Q}}} := \bigoplus_{\psi \text{ odd or } \psi=\mathbb{1}} \bar{\mathbb{Q}}\eta_\psi$$

the $\bar{\mathbb{Q}}$-linear subspace of $L_{\bar{\mathbb{Q}}} \otimes_{\bar{\mathbb{Q}}} \bar{\mathbb{Q}}_p = \mathrm{H}^1(H, \bar{\mathbb{Q}}_p)$.

Denoting $\mathrm{ord}_p : \mathbb{Q}_p^\times \to \mathbb{Z}$ the valuation, we consider the $\bar{\mathbb{Q}}_p$-linear maps

$$\log_{v_0} : \mathcal{O}_H[\tfrac{1}{p}]^\times \otimes \bar{\mathbb{Q}}_p \longrightarrow \bar{\mathbb{Q}}_p \qquad \mathrm{ord}_{v_0} : \mathcal{O}_H[\tfrac{1}{p}]^\times \otimes \bar{\mathbb{Q}}_p \longrightarrow \bar{\mathbb{Q}}_p$$

$$u \otimes x \mapsto \log_p(\iota_p(u))x \qquad u \otimes x \mapsto \mathrm{ord}_p(\iota_p(u))x$$

Given any odd character ψ of G, the ψ^{-1}-eigenspace of $\mathcal{O}_H[\frac{1}{p}]^\times \otimes \bar{\mathbb{Q}}_p$ is one-dimensional; let u_ψ be its basis. Note that $\mathrm{ord}_{v_0}(u_\psi) \neq 0$ since otherwise, by ψ^{-1}-equivariance, one would have $\mathrm{ord}_{v_i}(u_\psi) = \mathrm{ord}_{v_0}(\sigma^{-i}(u_\psi)) = 0$ for all $0 \leqslant i \leqslant 2r-1$, which is impossible since the ψ^{-1}-eigenspace of $\mathcal{O}_H^\times \otimes \bar{\mathbb{Q}}_p$ is zero.

Following [DDP11, (7)] we recall the definition of the $\mathcal{L}$-invariants and relate them to cohomology classes in $\mathrm{H}^1(H, \bar{\mathbb{Q}}_p)$ [BDP, Prop. 2.5].

Definition 1.1.5. The $\mathcal{L}$-invariant of ψ is

$$\mathcal{L}(\psi) := -\frac{\log_{v_0}(u_\psi)}{\mathrm{ord}_{v_0}(u_\psi)} \in \bar{\mathbb{Q}}_p. \tag{1.5}$$

Proposition 1.1.6. *$(\eta_\psi - \eta_{\mathbb{1}})(\mathrm{Frob}_p) = \mathcal{L}(\psi^{-1})$.*

Proof. Recall that, by Lemma 1.1.1, there is an exact sequence of $\bar{\mathbb{Q}}_p[G]$-modules

$$0 \to \mathrm{Hom}(G_H, \bar{\mathbb{Q}}_p) \to \bigoplus_{i=0}^{2r-1} \mathrm{Hom}(H_{v_i}^\times, \bar{\mathbb{Q}}_p) \to \mathrm{Hom}(\mathcal{O}_H[\tfrac{1}{p}]^\times, \bar{\mathbb{Q}}_p). \tag{1.6}$$

where $\xi : G_H \to \bar{\mathbb{Q}}_p$ is sent to the collection of maps $\xi_i : H_{v_i}^\times \to \bar{\mathbb{Q}}_p$, $0 \leqslant i < 2r$, defined by taking the restriction to $H_{v_i}^\times \subset \widehat{H_{v_i}^\times} \simeq G_{H_{v_i}}^{\mathrm{ab}}$. Then $(\eta_\psi - \eta_{\mathbb{1}})(\mathrm{Frob}_p) = (\eta_{\psi,0} - \eta_{\mathbb{1},0})(\varpi_0)$, where ϖ_0 denotes a uniformizer of H_{v_0}. Denoting by e the exponent of the Hilbert class group of H, there exists $x_0 \in \mathcal{O}_H[\frac{1}{p}]^\times$ whose valuation at v_0 is e and 0 at any other place of H. We can write $x_0 = \varpi_0^e y$ with $y \in \mathcal{O}_{v_0}^\times$; we have

$$(\eta_{\psi,0} - \eta_{\mathbb{1},0})(x_0) = (\eta_{\psi,0} - \eta_{\mathbb{1},0})(\varpi_0^e y) = e \cdot (\eta_{\psi,0} - \eta_{\mathbb{1},0})(\varpi_0) = e \cdot (\eta_\psi - \eta_{\mathbb{1}})(\mathrm{Frob}_p) \quad (1.7)$$

Since $x_0 \in \mathcal{O}_H[\frac{1}{p}]^\times$ and $\eta_\psi - \eta_{\mathbb{1}} \in \mathrm{Hom}(G_H, \bar{\mathbb{Q}}_p)$ the sequence (1.6) implies that

$$(\eta_{\psi,0} - \eta_{\mathbb{1},0})(x_0) = -\sum_{i=1}^{2r-1} (\eta_{\psi,i} - \eta_{\mathbb{1},i})(x_0). \quad (1.8)$$

Since by definition η_ψ belongs to the ψ^{-1}-eigenspace for the G-action, it is entirely determined by $\eta_{\psi,0}$. More precisely we have that $\eta_{\psi,i} = \psi(\sigma)^i(\eta_{\psi,0} \circ \sigma^{-i})$ for all $0 \leqslant i < 2r$. Combining this with (1.8) and observing that $\sigma^i(x_0) \in \mathcal{O}_{v_0}^\times$ for every $1 \leqslant i < 2r$, we obtain

$$\begin{aligned} & e \cdot (\eta_\psi - \eta_{\mathbb{1}})(\mathrm{Frob}_{v_0}) = (\eta_{\psi,0} - \eta_{\mathbb{1},0})(x_0) = -\sum_{i=1}^{2r-1} (\eta_{\psi,i} - \eta_{\mathbb{1},i})(x_0) = \\ & = -\sum_{i=1}^{2r-1} (\psi(\sigma)^i \eta_{\psi,0}(\sigma^{-i}(x_0)) - \eta_{\mathbb{1},0}(\sigma^{-i}(x_0))) = -\sum_{i=0}^{2r-1} (\psi(\sigma)^i - 1) \log_p(\iota_p(\sigma^{-i}(x_0))) \\ & = -\sum_{i=0}^{2r-1} \psi(\sigma)^i \log_p(\iota_p(\sigma^{-i}(x_0))) + \sum_{i=0}^{2r-1} \log_p(\iota_p(\sigma^{-i}(x_0))) \end{aligned} \quad (1.9)$$

because the restrictions of η_ψ and $\eta_{\mathbb{1}}$ to $I_{H_{v_0}}$ are given by $\log_p$. Observe first that

$$\sum_{i=0}^{2r-1} \log_p(\iota_p(\sigma^i(x_0)) = \log_p(\iota_p(\mathrm{N}_{H/\mathbb{Q}}(x_0)) \in \log_p(\iota_p(\pm p^{\mathbb{Z}})) = \{0\}.$$

Finally note that $\sum_{i=0}^{2r-1} \psi(\sigma)^i \log_p(\iota_p(\sigma^{-i}(x_0))) = \log_{v_0}(u_{\psi^{-1}})$, where

$$u_{\psi^{-1}} = \sum_{i=0}^{2r-1} \sigma^{-i}(x_0) \otimes \psi(\sigma)^i$$

clearly belongs to the ψ-eigenspace of $\mathcal{O}_H[\frac{1}{p}]^\times \otimes \bar{\mathbb{Q}}_p$, and $\mathrm{ord}_{v_0}(u_{\psi^{-1}}) = \mathrm{ord}_{v_0}(x_0 \otimes 1) = e$, hence $\log_{v_0}(u_{\psi^{-1}}) = -e \cdot \mathcal{L}(\psi^{-1})$ by definition (1.5). Combining this with (1.9) yields the claim. $\square$

An application of the Baker-Brumer Theorem

Recall the statement of the Baker-Brumer Theorem [Bru67].

Theorem 1.1.7 (Baker-Brumer). *Fix an embedding $\iota_p \colon \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Let $x_1, x_2, \ldots x_n$ be algebraic numbers. If the logarithms*

$$\log_p(\iota_p(x_i))$$

for $i = 1, \ldots, n$ are linearly independent over $\mathbb{Q}$, then they are also linearly independent over $\bar{\mathbb{Q}}$.

We apply the previous theorem to obtain the following result [BDP, Prop.2.6].

Proposition 1.1.8. *(i) The $\mathcal{L}(\psi)$ are linearly independent over $\bar{\mathbb{Q}}$, when ψ runs over all odd characters of G.*

(ii) The restriction to $L_{\bar{\mathbb{Q}}}$ of the map res_p defined in (1.3) is injective.

Proof. (i) Suppose that $\sum_{\psi \text{ odd}} m_\psi \mathcal{L}(\psi) = 0$ for some $m_\psi \in \bar{\mathbb{Q}}$. As in the proof of Proposition 1.1.6, we denote by e the exponent of the Hilbert class group of H, and fix an element $x_0 \in \mathcal{O}_H[\frac{1}{p}]^\times$ with valuation e at v_0 and 0 at any other place of H. It follows that

$$\sum_{\psi \text{ odd}} m_\psi \left(\sum_{i=0}^{2r-1} (\psi(\sigma)^i - 1) \log_p(\iota_p(\sigma^{-i}(x_0))) \right) = 0 \tag{1.10}$$

Note that the $i = 0$ summand vanishes. Letting $m_{\mathbb{1}} = -\sum_{\psi \text{ odd}} m_\psi$ the formula can be written as

$$\sum_{i=1}^{2r-1} \left(\sum_{\psi \text{ odd or } \psi=\mathbb{1}} m_\psi \psi(\sigma)^i \right) \log_p(\iota_p(\sigma^{-i}(x_0))) = 0 \tag{1.11}$$

The values $\{\log_p(\iota_p(\sigma^{-i}(x_0)))\}_{1 \leqslant i \leqslant 2r-1}$ are linearly independent over $\mathbb{Q}$. To see this suppose $\log_p(\iota_p(x)) = 0$ for some element $x = \prod_{1 \leqslant i \leqslant 2r-1} \sigma^{-i}(x_0)^{n_i}$ with $n_i \in \mathbb{Z}$. Since $\iota_p(x) \in \bar{\mathbb{Z}}_p^\times$ this implies that x is a root of unity in H leading to $n_i = \mathrm{ord}_{v_i}(x) = 0$ for all $1 \leqslant i \leqslant 2r - 1$. By Theorem 1.1.7, the elements $\{\log_p(\iota_p(\sigma^i(x_0)))\}_{1 \leqslant i \leqslant 2r-1}$ are also linearly independent over $\bar{\mathbb{Q}}$, leading to

$$\sum_{\psi \text{ odd or } \psi=1} \psi(\sigma)^i m_\psi = 0 \tag{1.12}$$

for any $1 \leqslant i \leqslant 2r-1$. Moreover, since $m_{\mathbb{1}} = -\sum_{\psi \text{ odd}} m_\psi$, (1.12) holds for $i = 0$ as well.

Let $\psi_1, \psi_2, \ldots \psi_r$ be a numbering of the odd characters of G. The condition (1.12) can be written as $(m_{\mathbb{1}}, m_{\psi_1}, \ldots, m_{\psi_r}) \cdot M = (0, 0, \ldots, 0)$, where

$$M = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \psi_1(\sigma) & \psi_1(\sigma)^2 & \ldots & \psi_1(\sigma)^{2r-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \psi_r(\sigma) & \psi_r(\sigma)^2 & \ldots & \psi_r(\sigma)^{2r-1} \end{bmatrix}$$

Since $2r \geqslant r+1$ for every $r \geq 1$, M contains as a sub-matrix the Vandermonde matrix of $(1, \psi_1(\sigma), \ldots, \psi_r(\sigma))$ which as well-known is invertible, implying that $m_\psi = 0$ for every ψ.

(ii) It suffices to notice that the kernel of the restriction map $L_{\bar{\mathbb{Q}}_p} \to \mathrm{H}^1(I_{\mathbb{Q}_p}, \bar{\mathbb{Q}}_p)$ is spanned by $\{(\eta_{\mathbb{1}} - \eta_\psi)\}_{\psi \text{ odd}}$. Combining Proposition 1.1.6 with (i) yields the desired result.

□

1.2 Deformations and pseudodeformations

In this section we define deformation functors for certain Galois representations and show their representability. We follow the strategy of [Maz89], with the difference that we fix residual representations with coefficients in $\bar{\mathbb{Q}}_p$ instead of a finite field. The representations attached to Eisenstein series are reducible, which implies that they are only determined by their trace up to semisimplification. To account for this issue, we recall the notion of pseudorepresentations, and relate the deformation rings of representations and pseudorepresentations.

1.2.1 Universal deformation ring

Let $\mathfrak{C}_{\bar{\mathbb{Q}}_p}$ be the category of complete local noetherian $\bar{\mathbb{Q}}_p$-algebras with residue field $\bar{\mathbb{Q}}_p$. Denote by $\mathfrak{C}^{\mathrm{art}}_{\bar{\mathbb{Q}}_p}$ the full subcategory of local artinian $\bar{\mathbb{Q}}_p$-algebras with residue field $\bar{\mathbb{Q}}_p$. For every A in $\mathfrak{C}_{\bar{\mathbb{Q}}_p}$ with maximal ideal $\mathfrak{m}_A$, there is a canonical $\bar{\mathbb{Q}}_p$-algebra isomorphism $A/\mathfrak{m}_A \to \bar{\mathbb{Q}}_p$. Moreover, every object A of $\mathfrak{C}_{\bar{\mathbb{Q}}_p}$ is isomorphic to the inverse limit of artinian rings in $\mathfrak{C}^{\mathrm{art}}_{\bar{\mathbb{Q}}_p}$, since $A = \varprojlim_n A/\mathfrak{m}_A^n$. An object A of $\mathfrak{C}^{\mathrm{art}}_{\bar{\mathbb{Q}}_p}$ is a finite $\bar{\mathbb{Q}}_p$-vector space, endowed with p-adic topology. An arbitrary object A of $\mathfrak{C}_{\bar{\mathbb{Q}}_p}$ is endowed with the inverse limit topology of its artinian quotients.

Universal cyclotomic character

Let $\psi\colon G_{\mathbb{Q}} \to \bar{\mathbb{Q}}_p^\times$ be a continuous character. Consider the functor $\mathcal{D}_\psi\colon \mathfrak{C}_{\bar{\mathbb{Q}}_p} \to \mathbf{Sets}$ sending an object A of $\mathfrak{C}_{\bar{\mathbb{Q}}_p}$ to the set of continuous characters $\psi_A\colon G_{\mathbb{Q}} \to A^\times$ such that

$\psi_A \bmod \mathfrak{m}_A = \psi$, where $\mathfrak{m}_A$ is the maximal ideal of A. Denote $\Lambda = \bar{\mathbb{Q}}_p[\![X]\!]$.

Lemma 1.2.1. *The functor $\mathcal{D}_\psi$ is representable by Λ.*

Proof. Let A be an object in $\mathfrak{C}_{\bar{\mathbb{Q}}_p}$ and let $\psi_A\colon G_\mathbb{Q} \to A^\times$ be a lift of ψ. Since A is $\bar{\mathbb{Q}}_p$-algebra, we can view ψ as a character $\psi\colon G_\mathbb{Q} \to A^\times$. Then $\psi_A\psi^{-1}$ is a lift of the trivial character to A. Thus, without loss of generality, we can assume that ψ is trivial, so ψ_A is a continuous homomorphism of $G_\mathbb{Q}$ to $\ker(A^\times \to \bar{\mathbb{Q}}_p^\times) = 1+\mathfrak{m}_A$. In particular, ψ_A factors through the abelianization $G_\mathbb{Q}^{\mathrm{ab}} \simeq \hat{\mathbb{Z}}^\times$. Moreover, since $1+\mathfrak{m}_A$ is torsion-free and ψ_A is continuous, the homomorphism factors through the maximal torsion-free pro-p quotient of $\hat{\mathbb{Z}}^\times$, *i.e.* $1+\mathbf{q}\mathbb{Z}_p$, where $\mathbf{q} = p$ if $p \neq 2$ and $p = 4$ otherwise, which is isomorphic to $\mathbb{Z}_p$. Since the continuous homomorphisms of $1+\mathbf{q}\mathbb{Z}_p$ to $1+\mathfrak{m}_A$ are determined by sending a topological generator $1+\mathbf{q}$ to any element of $1+\mathfrak{m}_A$, they are in bijection with $\mathrm{Hom}_{\mathfrak{C}_{\bar{\mathbb{Q}}_p}}(\Lambda, A)$, it follows that $\mathcal{D}_\psi$ is representable by Λ. $\square$

We now describe the universal character. Denote by $\chi\colon G_\mathbb{Q} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \to \mathbb{Z}_p^\times$ the p-adic cyclotomic character. There is an exact sequence of abelian groups

$$0 \to 1+\mathbf{q}\mathbb{Z}_p \to \mathbb{Z}_p^\times \to (\mathbb{Z}/\mathbf{q}\mathbb{Z})^\times \to 0 \tag{1.13}$$

The Teichmuller character $\omega_p\colon (\mathbb{Z}/\mathbf{q}\mathbb{Z})^\times \to \mathbb{Z}_p^\times$ provides a section for the projection $\mathbb{Z}_p^\times \to (\mathbb{Z}/\mathbf{q}\mathbb{Z})^\times$. Following the notation of [CM96], we denote $\langle\!\langle x \rangle\!\rangle = x\omega_p^{-1}(\bar{x})$. Let κ_Λ the character obtained by composition

$$G_\mathbb{Q} \xrightarrow{\chi} \mathbb{Z}_p^\times \xrightarrow{\langle\!\langle \cdot \rangle\!\rangle} 1+\mathbf{q}\mathbb{Z}_p \to \Lambda^\times \tag{1.14}$$

where the last map is the continuous group homomorphism determined by sending $1+\mathbf{q}$ to $1+X$. The reduction of κ_Λ modulo (X) is the trivial character. From the proof of 1.2.1, κ_Λ is the universal character lifting the trivial character. For any character $\psi\colon G_\mathbb{Q} \to \bar{\mathbb{Q}}_p^\times$, the universal object for the functor $\mathcal{D}_\psi$ is $\psi\kappa_\Lambda$.

Recall that we fixed a generator $\eta_{\mathbb{1}}$ of $\mathrm{H}^1(\mathbb{Q},\bar{\mathbb{Q}}_p)$ satisfying $\eta_{\mathbb{1}} = \log_p \circ \chi$.

Lemma 1.2.2. *The character κ_Λ satisfies $\kappa_\Lambda = 1 + \frac{X}{\log_p(1+\mathbf{q})}\eta_{\mathbb{1}} \bmod (X^2)$.*

Proof. The character κ_Λ is given by

$$\kappa_\Lambda(\sigma) = (1+X)^{\frac{\log_p \langle\!\langle \chi(\sigma) \rangle\!\rangle}{\log_p(1+\mathbf{q})}} = (1+X)^{\frac{\eta_{\mathbb{1}}(\sigma)}{\log_p(1+\mathbf{q})}},$$

hence the formula follows by taking the first derivative. □

Universal local character

Let $\psi \colon G_{\mathbb{Q}_p} \to \bar{\mathbb{Q}}_p^\times$ be a continuous character. Denote $\mathcal{D}_\psi^{\mathrm{loc},p}$ the functor classifying, for every object A in $\mathfrak{C}_{\bar{\mathbb{Q}}_p}$, the set of lifts of the character ψ to A.

Proposition 1.2.3. *The functor $\mathcal{D}_\psi^{\mathrm{loc},p}$ is representable by the ring of formal power series in two variables over $\bar{\mathbb{Q}}_p$, denoted by $\mathcal{R}_\psi^{\mathrm{loc},p}$.*

Proof. Up to twisting by a character, we can assume that the residual character ψ is trivial. Any lift of the trivial character factors through the maximal pro-p torsion-free quotient of the abelianization of $G_{\mathbb{Q}_p}^{\mathrm{ab}}$; by local class field theory, the latter is isomorphic to the profinite completion of $\mathbb{Q}_p^\times$. Thus, the maximal pro-p torsion-free quotient is isomorphic to $\mathbb{Z}_p^2$. A lift $\psi_A \colon G_{\mathbb{Q}} \to A^\times$ of the trivial character is thus determined by the image of a basis of $\mathbb{Z}_p^2$ in $1+\mathfrak{m}_A$. Hence, it corresponds to a homomorphism $\mathcal{R}_\psi^{\mathrm{loc},p} \to A$. □

Indecomposable reducible representations and representability of deformation functors

Consider the representation $\phi \oplus \mathbb{1} \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$, under the assumptions of Section 1.1. It is a reducible semisimple Artin representation with splitting field H. The continuous extensions of $\mathbb{1}$ by ϕ are classified by

$$\mathrm{Ext}^1_{\bar{\mathbb{Q}}_p[G_{\mathbb{Q}}]}(\mathbb{1}, \phi) = \mathrm{H}^1(\mathbb{Q}, \phi) \simeq \mathrm{H}^1(H, \bar{\mathbb{Q}}_p)^{\phi^{-1}}.$$

where the notation $\mathrm{H}^i(-, \bar{\mathbb{Q}}_p)$ simply denotes the Galois cohomology with respect to the *trivial action* of the Galois group. It is a one-dimensional $\bar{\mathbb{Q}}_p$-vector space by Proposition 1.1.2. Similarly, the extensions of ϕ by $\mathbb{1}$ are classified by the one-dimensional space

$$\mathrm{Ext}^1_{\bar{\mathbb{Q}}_p[G_{\mathbb{Q}}]}(\phi, \mathbb{1}) = \mathrm{H}^1(\mathbb{Q}, \phi^{-1}) \simeq \mathrm{H}^1(H, \bar{\mathbb{Q}}_p)^{\phi}.$$

We fix cocycles $\eta = \eta_\phi \in \mathrm{Z}^1(\mathbb{Q}, \phi)$ and $\eta' = \eta_{\phi^{-1}} \in \mathrm{Z}^1(\mathbb{Q}, \phi^{-1})$ such that their restriction to the inertia $I_{\mathbb{Q}_p}$ is $\log_p \circ \chi$ as in (1.4). Since $\phi(\tau)$ and $\phi^{-1}(\tau)$ are not trivial up to modifying η, η' by a coboundary, we can further assume that $\eta(\tau) = \eta'(\tau) = 0$.

Let $V = V' = \bar{\mathbb{Q}}_p^2$. Denote $\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}(V)$ and $\rho'\colon G_{\mathbb{Q}} \to \mathrm{GL}(V')$ the representations

$$\rho = \begin{bmatrix} \phi & \eta \\ 0 & \mathbb{1} \end{bmatrix}, \qquad \text{and} \qquad \rho' = \begin{bmatrix} \mathbb{1} & \phi\eta' \\ 0 & \phi \end{bmatrix}$$

in the standard bases e_1, e_2 of M and e'_1, e'_2 of M'.

Definition 1.2.4. Given an object A in $\mathfrak{C}_{\bar{\mathbb{Q}}_p}$, with maximal ideal $\mathfrak{m}_A$, a *lift* of ρ (resp. ρ') to A is a continuous representation $\rho_A\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(A)$ (resp. ρ'_A) such that $\rho_A \bmod \mathfrak{m}_A = \rho$ (resp. $\rho'_A \bmod \mathfrak{m}_A = \rho'$). Two lifts of ρ (resp. ρ') to A are equivalent if they are conjugate by an element in the kernel of $\mathrm{GL}_2(A) \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$.

A *deformation* of ρ (resp. ρ') to A is an equivalence class of lifts of ρ (resp. ρ') to A.

Definition 1.2.5. Let $\mathcal{D}_\rho$ (resp. $\mathcal{D}_{\rho'}$) be the functor

$$\mathfrak{C}_{\bar{\mathbb{Q}}_p} \to \mathbf{Sets}$$

sending $A \in \mathrm{Ob}(\mathfrak{C}_{\bar{\mathbb{Q}}_p})$ to the set of deformations of ρ (resp. ρ') to A. Let $\mathcal{D}^0_\rho$ (resp. $\mathcal{D}^0_{\rho'}$) be the subfunctor of $\mathcal{D}_\rho$ (resp. $\mathcal{D}_{\rho'}$) sending $A \in \mathrm{Ob}(\mathfrak{C}_{\bar{\mathbb{Q}}_p})$ to the set of equivalence classes of lifts in $\mathcal{D}_\rho$ (resp. $\mathcal{D}_{\rho'}$) of determinant ϕ.

Let $\bar{\mathbb{Q}}_p[\varepsilon]$ be the ring of dual numbers over $\bar{\mathbb{Q}}_p$. Denote

$$t_\rho = \mathcal{D}_\rho(\bar{\mathbb{Q}}_p[\varepsilon]), \qquad t^0_\rho = \mathcal{D}^0_\rho(\bar{\mathbb{Q}}_p[\varepsilon]), \qquad t_{\rho'} = \mathcal{D}_{\rho'}(\bar{\mathbb{Q}}_p[\varepsilon]), \qquad t^0_{\rho'} = \mathcal{D}^0_{\rho'}(\bar{\mathbb{Q}}_p[\varepsilon])$$

the Zariski tangent spaces of the corresponding functors. This terminology refers to the fact that, if a functor $\mathcal{D}\colon \mathfrak{C}_{\bar{\mathbb{Q}}_p} \to \mathbf{Sets}$ is representable by a ring R, $\mathcal{D}(\bar{\mathbb{Q}}_p[\varepsilon])$ is indeed isomorphic to the Zariski tangent space of R at its maximal ideal. Note that for the deformation functors defined above the tangent space is always endowed with the structure of $\bar{\mathbb{Q}}_p$-vector space and is finite if the functor is representable.

It is a standard technique in deformation theory to interpret these tangent spaces in terms of Galois cohomology. Let $\mathrm{ad}(\rho)$ be the adjoint representation of ρ and let $\mathrm{ad}^0(\rho)$ be the subrepresentation of $\mathrm{ad}(\rho)$ on the endomorphisms of trace zero. There are isomorphisms of $\bar{\mathbb{Q}}_p$-vector spaces

$$\mathrm{H}^1(\mathbb{Q}, \mathrm{ad}(\rho)) \to t_\rho, \qquad\qquad \mathrm{H}^1(\mathbb{Q}, \mathrm{ad}^0(\rho)) \to t^0_\rho \tag{1.15}$$

given by $\Theta \mapsto (1 + \varepsilon\Theta)\rho$. The analogous statement holds for ρ'.

Proposition 1.2.6. *The functors $\mathcal{D}_\rho$ and $\mathcal{D}^0_\rho$ (resp. $\mathcal{D}_{\rho'}$ and $\mathcal{D}^0_{\rho'}$) are representable.*

Proof. The proof is an application of Schlessinger's criterion for representability; the argument is essentially the same as [Maz89]. What remains to verify is the fact that the Zariski tangent space t_ρ is finite and that the representation ρ has only scalar automorphisms. For the finiteness of the tangent space, by (1.15), it suffices to show that $\mathrm{H}^1(\mathbb{Q}, \mathrm{ad}(\rho))$ is finite dimensional over $\bar{\mathbb{Q}}_p$. The representation ρ is a non-split extension of $\mathbb{1}$ by ϕ; in particular, its semisimplification $\rho^{\mathrm{ss}} = \mathbb{1} \oplus \phi$ is an Artin representation. Thus, the semisimplification of the adjoint representation $\mathrm{ad}(\rho)$ is $\mathrm{ad}(\rho)^{\mathrm{ss}} = \mathbb{1}^2 \oplus \phi \oplus \phi^{-1}$. Writing a filtration of $\mathrm{ad}(\rho)$ with quotients given by finite image characters, we get an upper bound of the dimension of t_ρ

$$\dim_{\bar{\mathbb{Q}}_p} t_\rho = \dim_{\bar{\mathbb{Q}}_p} \mathrm{H}^1(\mathbb{Q}, \mathrm{ad}(\rho)) \leq \dim_{\bar{\mathbb{Q}}_p} (\mathrm{H}^1(\mathbb{Q}, \bar{\mathbb{Q}}_p)^2 \oplus \mathrm{H}^1(\mathbb{Q}, \phi) \oplus \mathrm{H}^1(\mathbb{Q}, \phi^{-1})) = 4$$

by Proposition 1.1.2.

Since ρ is a *non-split* extension of distinct characters, the only matrices in $\mathrm{M}_2(\bar{\mathbb{Q}}_{\mathrm{p}})$ commuting with the image of ρ are scalars; thus the condition $\mathrm{End}_{\bar{\mathbb{Q}}_{\mathrm{p}}}(\rho) = \bar{\mathbb{Q}}_{\mathrm{p}}$ is satisfied. This suffices to prove that the functor $\mathcal{D}_\rho$ is representable. The analogous argument shows that $\mathcal{D}^0_\rho$, $\mathcal{D}_{\rho'}$, $\mathcal{D}^0_{\rho'}$ are also representable. □

Remark 1.2.7. In [Maz89], Mazur shows the representability of deformation functors for absolutely irreducible representation of a profinite group Π in characteristic p. In order to ensure this, he imposes what he denotes as Φ_p condition, *i.e.* the finiteness of $\mathrm{Hom}(\Pi_0, \mathbb{F}_p)$, for every Π_0 open subgroup of Π. In our context, we can waive this condition and show that the tangent spaces t_ρ and $t_{\rho'}$ are finite in a more direct manner. In particular, we do not need to replace the absolute Galois group $G_{\mathbb{Q}}$ with the Galois group $G_{\mathbb{Q},S}$ of the maximal extension of $\mathbb{Q}$ unramified outside a finite set of primes S.

Denote by $\mathcal{R}_\rho$, $\mathcal{R}^0_\rho$, $\mathcal{R}_{\rho'}$, $\mathcal{R}^0_{\rho'}$ the objects in $\mathfrak{C}_{\bar{\mathbb{Q}}_p}$ that represent the functors $\mathcal{D}_\rho$, $\mathcal{D}^0_\rho$, $\mathcal{D}_{\rho'}$, $\mathcal{D}^0_{\rho'}$ respectively; we refer to them as universal deformation rings. The determinant of the universal representation for ρ and ρ' is a lift of ϕ to $\mathcal{R}^\times_\rho$ and $\mathcal{R}^\times_{\rho'}$ respectively. By universal property of Λ, we obtain unique homomorphism

$$\Lambda \to \mathcal{R}_\rho \qquad \text{and} \qquad \Lambda \to \mathcal{R}_{\rho'},$$

inducing the determinant maps by composition with $\phi\kappa_\Lambda$. Thus, $\mathcal{R}^0_\rho \simeq \mathcal{R}_\rho / \mathfrak{m}_\Lambda \mathcal{R}_\rho$ and

$\mathcal{R}^0_{\rho'} \simeq \mathcal{R}_{\rho'}/\mathfrak{m}_\Lambda \mathcal{R}_{\rho'}$

Remark 1.2.8. Note that $\rho' \otimes \phi^{-1} = \left[\begin{smallmatrix} \phi^{-1} & \eta' \\ 0 & \mathbb{1} \end{smallmatrix}\right]$, which is the representation obtained by replacing ϕ with its inverse. As explained in [Maz89, Section 1.3], the representations ρ' and $\left[\begin{smallmatrix} \phi^{-1} & \eta' \\ 0 & \mathbb{1} \end{smallmatrix}\right]$ are twist-equivalent, which implies that their deformation rings are canonically isomorphic. Thus, all arguments that apply to $\mathcal{R}_\rho$ naturally transfer to $\mathcal{R}_{\rho'}$.

1.2.2 Reducible deformation ring

Let $V_{\mathcal{R}_\rho}$ (resp. $V_{\mathcal{R}_{\rho'}}$) the universal object of $\mathcal{D}_\rho$ (resp. $\mathcal{D}_{\rho'}$). It is a free rank 2 $\mathcal{R}_\rho$-module (resp. $\mathcal{R}_{\rho'}$-module) with a continuous action of $G_\mathbb{Q}$. Since the complex conjugation τ has order 2, which is invertible in $\mathcal{R}_\rho$, we have a decomposition of $V_{\mathcal{R}_\rho}$ as

$$V_{\mathcal{R}_\rho} = V^+_{\mathcal{R}_\rho} \oplus V^-_{\mathcal{R}_\rho}.$$

where $V^\pm_{\mathcal{R}_\rho} = \{m \in V_{\mathcal{R}_\rho} \mid \tau m = \pm m\}$. In particular, $V^\pm_{\mathcal{R}_\rho}$ is a projective, finitely generated $\mathcal{R}_\rho$-module, hence is free. If $\mathfrak{m}_{\mathcal{R}_\rho}$ is the maximal ideal of $\mathcal{R}_\rho$, we have $V^\pm_{\mathcal{R}_\rho}/\mathfrak{m}_{\mathcal{R}_\rho} V^\pm_{\mathcal{R}_\rho} \subset (V_{\mathcal{R}_\rho}/\mathfrak{m}_{\mathcal{R}_\rho} V_{\mathcal{R}_\rho})^\pm$ because $V^\pm_{\mathcal{R}_\rho}$ is a direct summand of $V_{\mathcal{R}_\rho}$. The latter is one dimensional, so the rank of $V^\pm_{\mathcal{R}_\rho}$ is at most one. Since $V_{\mathcal{R}_\rho}$ is the direct sum of $V^+_{\mathcal{R}_\rho}$ and $V^-_{\mathcal{R}_\rho}$, this implies that each must be free of rank one.

Let $e_\pm$ be the generator of $V^\pm_{\mathcal{R}_\rho}$ such that

$$e_- \otimes 1_{\bar{\mathbb{Q}}_p} = e_1 \qquad \text{and} \qquad e_+ \otimes 1_{\bar{\mathbb{Q}}_p} = e_2.$$

We denote

$$\rho_{\mathcal{R}_\rho,\tau} \colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{R}_\rho), \qquad \rho_{\mathcal{R}_\rho,\tau} = \left[\begin{smallmatrix} a_\tau & b_\tau \\ c_\tau & d_\tau \end{smallmatrix}\right] \tag{1.16}$$

the representation corresponding to $V_{\mathcal{R}_\rho}$ in the basis (e_-, e_+). We call the (e_-, e_+) a τ-

adapted basis for $V_{\mathcal{R}_\rho}$. This basis is unique up to conjugation by a matrix of the form $\left[\begin{smallmatrix}\lambda & 0\\ 0 & \mu\end{smallmatrix}\right] \in \ker(\mathrm{GL}_2(\mathcal{R}_\rho) \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p))$.

In particular, the continuous functions

$$a_\tau, d_\tau \colon G_{\mathbb{Q}} \to \mathcal{R}_\rho \qquad\qquad X_\tau \colon G_{\mathbb{Q}} \times G_{\mathbb{Q}} \to \mathcal{R}_\rho,\ X_\tau(\sigma, \sigma') = b_\tau(\sigma)c_\tau(\sigma')$$

are independent of the choice of (e_+, e_-).

Following [CM96, Sec.5], we denote by I_ρ^{red} the ideal generated by $X_\tau(\sigma, \sigma')$ for $\sigma, \sigma' \in G_{\mathbb{Q}}$; we refer to I_ρ^{red} as the *reducibility ideal* of $\mathcal{R}_\rho$. Similarly, $V_{\mathcal{R}_{\rho'}}$ also decomposes as a direct sum of two rank 1 $\mathcal{R}_{\rho'}$-eigenspaces for the action of τ, denoted by $V^{\pm}_{\mathcal{R}_{\rho'}}$. Given a choice of generators $e'_\pm$ of $V_{\mathcal{R}_{\rho'}}$, we have a representation

$$\rho'_{\mathcal{R}_{\rho'},\tau} \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{R}_{\rho'}), \qquad \rho_{\mathcal{R}_\rho,\tau} = \left[\begin{smallmatrix} a'_\tau & b'_\tau \\ c'_\tau & d'_\tau \end{smallmatrix}\right] \tag{1.17}$$

in the basis (e'_-, e'_+). Again, the reducibility ideal $I_{\rho'}^{\mathrm{red}}$ is generated by the products $b'_\tau(\sigma)c'_\tau(\sigma')$ for $\sigma, \sigma' \in G_{\mathbb{Q}}$. Denote by C_ρ the ideal of $\mathcal{R}_\rho$ (resp. $\mathcal{R}_{\rho'}$) generated by $c_\tau(\sigma)$ (resp. $c'_\tau(\sigma)$) for $\sigma \in G_{\mathbb{Q}}$). We have the following result (see also [BDP, Lemma 1.4]).

Lemma 1.2.9. *$C_\rho = I_\rho^{\mathrm{red}}$ and $C_{\rho'} = I_{\rho'}^{\mathrm{red}}$.*

Proof. From the definition is clear that $I_\rho^{\mathrm{red}} \subset C_\rho$, so it suffices to show the opposite inclusion. The representation ρ is a non-split extension of ϕ by the trivial character. In particular, there exists an element σ_0 such that $\eta(\sigma_0) \in \bar{\mathbb{Q}}_p^\times$. Thus $b_\tau(\sigma_0) \in \mathcal{R}_\rho^\times$; since $b(\sigma_0)c(\sigma) \in I_\rho^{\mathrm{red}}$ for every $\sigma \in G_{\mathbb{Q}}$, it follows that $c(\sigma) \in I_\rho^{\mathrm{red}}$. Therefore $C_\rho \subset I_\rho^{\mathrm{red}}$. Similarly, $I_{\rho'}^{\mathrm{red}} = C_{\rho'}$. $\square$

Definition 1.2.10. Let $\mathcal{D}_\rho^{\mathrm{red}}$ (resp. $\mathcal{D}_{\rho'}^{\mathrm{red}}$) be the subfunctor of $\mathcal{D}_\rho$ (resp. $\mathcal{D}_{\rho'}$) sending $A \in \mathrm{Ob}(\mathfrak{C}_{\bar{\mathbb{Q}}_p})$ to the set of equivalence classes of reducible lifts of ρ (resp. ρ') to A.

Proposition 1.2.11. *The functors $\mathcal{D}_\rho^{\mathrm{red}}$ and $\mathcal{D}_{\rho'}^{\mathrm{red}}$ are representable by rings $\mathcal{R}_\rho^{\mathrm{red}}$ and*

$\mathcal{R}_{\rho'}^{\mathrm{red}}$.

Proof. Let A be an object of $\mathfrak{C}_{\bar{\mathbb{Q}}_p}$ and let ρ_A be a representative of an equivalence class in $\mathcal{D}_\rho^{\mathrm{red}}(A)$. By the universal property of $\mathcal{R}_\rho$, there is a morphism $\varphi_A \colon \mathcal{R}_\rho \to A$ such that $\varphi_A \circ \rho_{\mathcal{R}_\rho,\tau}$ is equivalent to ρ_A. Since $\varphi_A \circ \rho_{\mathcal{R}_\rho,\tau} = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, the only possible $G_{\mathbb{Q}}$-stable subspaces for $\varphi_A \circ \rho_{\mathcal{R}_\rho,\tau}$ are $\langle e_\pm \otimes 1_A \rangle$. Therefore, ρ_A is reducible if and only if I_ρ^{red} is in the kernel of φ_A. Thus, $\mathcal{D}_\rho^{\mathrm{red}}$ is representable by $\mathcal{R}_\rho^{\mathrm{red}} = \mathcal{R}_\rho / I_\rho^{\mathrm{red}}$. Similarly, $\mathcal{D}_\rho^{\mathrm{ord}}$ is representable by $\mathcal{R}_{\rho'}^{\mathrm{red}} = \mathcal{R}_{\rho'} / I_{\rho'}^{\mathrm{red}}$.

□

1.2.3 Universal pseudodeformation ring

We recall the definition of pseudorepresentation. Originally defined by Wiles for odd two-dimensional representations [Wil88] and then by Taylor for general groups [Tay91], pseudorepresentations are, roughly speaking, functions satisfying the same relations as traces of representations. The theory of pseudorepresentations offers advantages over that of representations from the point of view of representability of deformation functors, particularly when dealing with residually reducible representations.

Definition 1.2.12. Let G be a group, let d be a positive integer and R be a ring such that $d!$ is invertible in R. A pseudorepresentation of dimension d is a map $\mathbf{T} \colon G \to R$ satisfying

- $\mathbf{T}(1_G) = d$;
- $\mathbf{T}(g_1 g_2) = \mathbf{T}(g_2 g_1)$ for all g_1, g_2 in G;
- *d-dimensional pseudorepresentation identity*: let $\sigma \in S_{d+1}$ a permutation with cycle decomposition $\sigma = c_1 c_2 \cdots c_r$. If c_i is the cycle $c_i = (j_1, j_2, \ldots, j_s)$, let $\mathbf{T}^{c_i} \colon G^{d+1} \to$

R be the function

$$\mathbf{T}^{c_1}(g_1, g_2, \ldots, g_{d+1}) = \mathbf{T}(g_{j_1} g_{j_2} \cdots g_{j_s}).$$

Define $\mathbf{T}^{\sigma} = \mathbf{T}^{c_1}\mathbf{T}^{c_2} \cdots \mathbf{T}^{c_r}$. Then, for every $g_1, g_2, \ldots, g_{d+1} \in G$,

$$\sum_{\sigma \in S_{d+1}} \varepsilon(\sigma)\mathbf{T}^{\sigma}(g_1, g_2, \ldots, g_{d+1}) = 0,$$

where $\varepsilon(\sigma)$ is the sign of σ.

If $\rho\colon G \to \mathrm{GL}_{\mathrm{d}}(R)$ is a representation, then $\mathrm{Tr}(\rho)$ is a pseudorepresentation.

Definition 1.2.13. Let $\mathcal{D}^{\mathrm{ps}}_{\phi+1}$ be the functor

$$\mathfrak{C}_{\bar{\mathbb{Q}}_p} \to \mathbf{Sets}$$

sending $A \in \mathrm{Ob}(\mathfrak{C}_{\bar{\mathbb{Q}}_p})$ to the set of continuous two-dimensional pseudorepresentations $\mathbf{T}_A\colon G_{\mathbb{Q}} \to A$ such that $\mathbf{T}_A \bmod \mathfrak{m}_A = \phi + 1$, where $\mathfrak{m}_A$ is the maximal ideal of A.

Proposition 1.2.14. *The functor $\mathcal{D}^{\mathrm{ps}}_{\phi+1}$ is representable by a ring $\mathcal{R}^{\mathrm{ps}}_{\phi+1}$.*

Proof. All conditions of Schlessinger's representability criterion are trivially satisfied, except for the fact that the Zariski tangent space $t^{\mathrm{ps}}_{\phi+1} = \mathcal{D}^{\mathrm{ps}}_{\phi+1}(\bar{\mathbb{Q}}_p[\varepsilon])$ is finite-dimensional. Since the pseudorepresentation $\phi + \mathbb{1}$ is the sum of two finite order characters, and ϕ is odd, a (finite) upper bound of the dimension can be proven as in [SW99, Lemma 2.10]. □

We refer to $\mathcal{R}^{\mathrm{ps}}_{\phi+1}$ as the universal pseudodeformation ring.

Remark 1.2.15. A more general notion of pseudorepresentation was introduced by Chenevier in [Che14b], overcoming the issue that the pseudodeformation functor is ill-behaved in characteristic 2. However, in the characteristic zero setting, the two notions are equivalent. Hence, for our purposes Definition 1.2.13 will suffice.

The traces of the universal deformations $\rho_{\mathcal{R}_\rho,\tau}$ and $\rho'_{\mathcal{R}_{\rho'},\tau}$ are pseudodeformations of $\phi+1$. By universal property of $\mathcal{R}^{\mathrm{ps}}_{\phi+1}$, this induces morphism

$$\mathcal{R}^{\mathrm{ps}}_{\phi+1} \to \mathcal{R}_\rho \qquad\qquad \text{and} \qquad\qquad \mathcal{R}^{\mathrm{ps}}_{\phi+1} \to \mathcal{R}_{\rho'} \tag{1.18}$$

The fact that $\mathrm{Ext}_{\bar{\mathbb{Q}}_p[G_\mathbb{Q}]}(\mathbb{1},\phi)$ and $\mathrm{Ext}_{\bar{\mathbb{Q}}_p[G_\mathbb{Q}]}(\phi,\mathbb{1})$ are one-dimensional implies the following result [Kis09, Lemma 1.4.3].

Proposition 1.2.16. *The morphism $\mathcal{R}^{\mathrm{ps}}_{\phi+1} \to \mathcal{R}_\rho$ (resp. $\mathcal{R}^{\mathrm{ps}}_{\phi+1} \to \mathcal{R}_{\rho'}$) is surjective.*

1.3 Ordinary deformation ring

We now introduce the functors parametrizing ordinary deformations of ρ. Since ordinary representations have a stable line for the action of the decomposition group of p, we introduce a functor classifying the pairs of representation with a filtration. The unique $G_{\mathbb{Q}_p}$-stable line of ρ is $L = \langle e_1\rangle$; thus any ordinary lift of ρ would be stable for a line lifting L. This prompts the following definition.

Let A be an object of $\mathfrak{C}_{\bar{\mathbb{Q}}_p}$. Let ρ_A be a lift of ρ to A and let $L_A \subset A^2$ be a free direct summand such that $L_A \otimes \bar{\mathbb{Q}}_p = L$. We say that two pairs (ρ_A, L_A) and $(\tilde{\rho}_A, \tilde{L}'_A)$ are equivalent if there exists $g \in \ker(\mathrm{GL}_2(A) \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p))$ such that

$$\tilde{\rho}_A = g\rho_A g^{-1} \qquad\qquad \text{and} \qquad\qquad \tilde{L}_A = gL_A.$$

Similarly, we denote $L' = \langle e'_1\rangle$ the unique $G_{\mathbb{Q}_p}$-stable line for ρ' and consider the pairs (ρ'_A, L'_A) where ρ_A is a lift of ρ' and L'_A is a free direct summand of A^2 such that $L'_A \otimes \bar{\mathbb{Q}}_p = L'$. We say that two such pairs are equivalent if they are equal up to conjugation as above. We denote by $[\rho_A, L_A]$ the equivalence class of the pair (ρ_A, L_A).

Definition 1.3.1. Let $\mathcal{D}^{\mathrm{fil}}_\rho \colon \mathfrak{C}_{\bar{\mathbb{Q}}_p} \to$ **Sets** (resp. $\mathcal{D}^{\mathrm{fil}}_{\rho'}$) be the functor sending an object A

in $\mathfrak{C}_{\bar{\mathbb{Q}}_p}$ to the set of equivalence classes of pairs (ρ_A, L_A), where ρ_A is a lift of ρ (resp. ρ') to A and L_A is a free direct summand of A^2 such that $L_A \otimes \bar{\mathbb{Q}}_p = L$ (resp. $L_A \otimes \bar{\mathbb{Q}}_p = L'$).

Lemma 1.3.2. *The functors $\mathcal{D}_\rho^{\mathrm{fil}}$ and $\mathcal{D}_{\rho'}^{\mathrm{fil}}$ are representable by $\mathcal{R}_\rho[\![Y]\!]$ and $\mathcal{R}'_\rho[\![Y]\!]$.*

Proof. For every $A \in \mathfrak{C}_{\bar{\mathbb{Q}}_p}$, there is a map $F_A \colon \mathrm{Hom}_{\mathfrak{C}_{\bar{\mathbb{Q}}_p}}(\mathcal{R}_\rho[\![Y]\!], A) \to \mathcal{D}_\rho^{\mathrm{fil}}(A)$ defined as follows. Denote by ι the inclusion $\iota \colon \mathcal{R}_\rho \to \mathcal{R}_\rho[\![Y]\!]$ and fix $\rho_{\mathcal{R}_\rho}$ a representative of the equivalence class of the universal deformation of ρ. A morphism $\varphi_A \colon \mathcal{R}_\rho[\![Y]\!] \to A$, defines a representation $\rho_A = \varphi_A \circ \iota \circ \rho_{\mathcal{R}_\rho} \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(A)$. Denote $L_A = \langle e_1 + \varphi_A(Y) e_2 \rangle$. Since $\varphi_A(Y) \in \mathfrak{m}_A$, the line L_A satisfies $L_A \otimes \bar{\mathbb{Q}}_p = L$; we let $F_A(\varphi_A)$ be the equivalence class of (ρ_A, L_A). The surjectivity of F is clear. Indeed, given a pair of (ρ_A, L_A) of representation with filtration as above, by the universal property of $\mathcal{R}_\rho$, we can conjugate it by an element in the kernel of $\mathrm{GL}_2(A) \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ and see that the pair is in the image of F_A.

It remains to verify the injectivity of F_A. Suppose $F_A(\varphi_A) = F_A(\tilde{\varphi}_A)$ for two morphisms $\varphi_A, \tilde{\varphi}_A \in \mathrm{Hom}_{\mathfrak{C}_{\bar{\mathbb{Q}}_p}}(\mathcal{R}_\rho[\![Y]\!], A)$. By the universal property of $\mathcal{R}_\rho$, this implies that $\varphi_A \circ \iota = \tilde{\varphi}_A \circ \iota$. If $g \in \ker(\mathrm{GL}_2(A) \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)))$ commutes with the image of $\varphi_A \circ \iota \circ \rho_{\mathcal{R}_\rho}$, then g is a scalar matrix, so any line is stable under g. Thus, $\varphi_A(Y) = \tilde{\varphi}_A(Y)$, which implies $\phi_A = \tilde{\phi}_A$. $\square$

Let $\rho_A \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(A)$ be a lift of ρ for some object A in $\mathfrak{C}_{\bar{\mathbb{Q}}_p}$. Denote $V_A = A^2$ the free A-module with the action of $G_{\mathbb{Q}}$ given by ρ_A. We say that V_A is ordinary if there is an exact sequence of $A[G_{\mathbb{Q}_p}]$-modules

$$0 \to V_A^{\mathrm{sub}} \to V_A \to V_A^{\mathrm{quo}} \to 0 \tag{1.19}$$

such that $V_A^{\mathrm{sub}}, V_A^{\mathrm{quo}}$ are free A-modules of rank one and V_A^{quo} is unramified. There is a map $V_A^{\mathrm{sub}} \otimes \bar{\mathbb{Q}}_p \to V_{\mathcal{R}_\rho}^{\mathrm{sub}} = \langle e_1 \rangle$; it is injective because V_A^{sub} is a direct summand, and surjective because both modules are one-dimensional over $\bar{\mathbb{Q}}_p$.

Definition 1.3.3. Let $\mathcal{D}^{\mathrm{n.ord}}_{\rho}\colon \mathfrak{C}_{\bar{\mathbb{Q}}_p} \to \mathbf{Sets}$ (resp. $\mathcal{D}^{\mathrm{n.ord}}_{\rho'}$) be the functor sending an object A of $\mathfrak{C}_{\bar{\mathbb{Q}}_p}$ to the equivalence class of pairs $[\rho_A, L_A] \in \mathcal{D}^{\mathrm{fil}}_{\rho}(A)$ (resp. $\mathcal{D}^{\mathrm{fil}}_{\rho'}(A)$) such that L_A is a $G_{\mathbb{Q}_p}$-stable subspace.

Given a pair (ρ_A, L_A) whose equivalence class is in $\mathcal{D}^{\mathrm{n.ord}}_{\rho}(A)$, we denote by

$$\vartheta_A\colon G_{\mathbb{Q}_p} \to A^{\times}$$

the character acting on the quotient A^2/L_A. Clearly $\vartheta_A \bmod \mathfrak{m}_A = \mathbb{1}$. Hence

$$(\rho_A, L_A) \mapsto \vartheta_A$$

defines a natural transformation $\mathcal{D}^{\mathrm{n.ord}}_{\rho} \to \mathcal{D}^{\mathrm{loc},p}_{\mathbb{1}}$.

Definition 1.3.4. Let $\mathcal{D}^{\mathrm{ord}}_{\rho}\colon \mathfrak{C}_{\bar{\mathbb{Q}}_p} \to \mathbf{Sets}$ (resp. $\mathcal{D}^{\mathrm{ord}}_{\rho'}$) be the functor sending an object A of $\mathfrak{C}_{\bar{\mathbb{Q}}_p}$ to the set of equivalence classes $[\rho_A, L_A] \in \mathcal{D}^{\mathrm{n.ord}}_{\rho}(A)$ (resp. $\mathcal{D}^{\mathrm{n.ord}}_{\rho'}(A)$) such that $G_{\mathbb{Q}_p}$-character ϑ_A is unramified.

The following proposition shows the representability of the functors above, as well as the relation between their universal deformation rings and $\mathcal{R}_{\rho}$ and $\mathcal{R}_{\rho'}$ [BDP, Lemma 1.2].

Proposition 1.3.5. *(i) The functors $\mathcal{D}^{\mathrm{n.ord}}_{\rho}$, $\mathcal{D}^{\mathrm{ord}}_{\rho}$ (resp. $\mathcal{D}^{\mathrm{n.ord}}_{\rho'}$, $\mathcal{D}^{\mathrm{ord}}_{\rho'}$) are representable by rings $\mathcal{R}^{\mathrm{n.ord}}_{\rho}$, $\mathcal{R}^{\mathrm{ord}}_{\rho}$ (resp. $\mathcal{R}^{\mathrm{n.ord}}_{\rho'}$, $\mathcal{R}^{\mathrm{ord}}_{\rho'}$);*

(ii) The map $\mathcal{R}_{\rho} \to \mathcal{R}^{\mathrm{ord}}_{\rho}$ (resp. $\mathcal{R}_{\rho'} \to \mathcal{R}^{\mathrm{ord}}_{\rho'}$) is surjective.

Proof. (i) By definition $\mathcal{D}^{\mathrm{n.ord}}_{\rho}(A) \subset \mathcal{D}^{\mathrm{fil}}_{\rho}(A)$. By Lemma 1.3.2, the functor $\mathcal{D}^{\mathrm{fil}}_{\rho}$ is representable by $\mathcal{R}_{\rho}[\![Y]\!]$. Let $\rho_{\mathcal{R}_{\rho}} = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ be any representative of the equivalence class of the universal deformation of ρ. Fix a morphism $\varphi_A\colon \mathcal{R}_{\rho}[\![Y]\!] \to A$. This morphism yields a pair $(\varphi_A \circ \iota \circ \rho_{\mathcal{R}_\rho}, e_1 + \phi(A)e_2)$, which belongs to $\mathcal{D}_\rho^{\mathrm{n.ord}}(A)$ if and only if the line $\langle e_1 + \varphi_A(Y)e_2 \rangle$ is $G_{\mathbb{Q}_p}$-stable. Since conjugating by the matrix $\left[\begin{smallmatrix} 1 & 0 \\ -Y & 1 \end{smallmatrix}\right]$ yields

$$\begin{bmatrix} 1 & 0 \\ -Y & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} = \begin{bmatrix} a + bY & b \\ c + (d-a)Y - bY^2 & d - bY \end{bmatrix}, \tag{1.20}$$

we see that the line $\langle e_1 + \varphi_A(Y)e_2 \rangle$ is $G_{\mathbb{Q}_p}$-stable if and only if φ factors through the quotient of $\mathcal{R}_\rho[\![Y]\!]$ modulo the ideal

$$I_\rho^{\mathrm{n.ord}} = \langle c(\sigma) + (d(\sigma) - a(\sigma))Y - b(\sigma)Y^2, \forall \sigma \in G_{\mathbb{Q}_p} \rangle.$$

Similarly, the equivalence class of the pair $(\varphi_A \circ \iota \circ \rho_{\mathcal{R}_\rho}, e_1 + \phi(A)e_2)$ belongs to $\mathcal{D}_\rho^{\mathrm{ord}}(A)$ if and only if φ_A factors through the quotient modulo

$$I_\rho^{\mathrm{ord}} = \langle c(\sigma) + (d(\sigma) - a(\sigma))Y - b(\sigma)Y^2 \ \forall \sigma \in G_{\mathbb{Q}_p},\ 1 - d(\sigma') + b(\sigma')Y \ \forall \sigma' \in I_p \rangle.$$

(ii) Fix an element σ_0 of $I_{\mathbb{Q}_p}$ such that $\eta(\sigma_0) \neq 0$. We have

$$d(\sigma_0) - b(\sigma_0)Y = 1. \tag{1.21}$$

But $b(\sigma_0) \bmod \mathfrak{m}_{\mathcal{R}_\rho} = \eta(\sigma_0) \in \bar{\mathbb{Q}}_p^\times$, so $b(\sigma_0)$ is invertible, hence $Y = (1 - d(\sigma_0))b(\sigma_0)^{-1}$ belongs to the image of $\mathcal{R}_\rho \to \mathcal{R}_\rho[\![Y]\!]/I^{\mathrm{ord}}$.

□

Remark 1.3.6. The previous proposition implies that given an ordinary lift of ρ of the form $\rho_A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with a filtration as in (1.19), the ordinary subspace can be read off the representation, or even the trace of a representation by Proposition 1.2.16. Given an element $\sigma_0 \in I_{\mathbb{Q}_p}$ with $\eta(\sigma_0) \neq 0$, and a Frobenius element Frob_p, the $G_{\mathbb{Q}_p}$-unramified character ϑ_A acting on the quotient is characterized by

$$\vartheta_A(\mathrm{Frob}_p) = d(\mathrm{Frob}_p) - b(\mathrm{Frob}_p)(d(\sigma_0) - 1))b(\sigma_0)^{-1}.$$

Instead, in Proposition 1.5.1, we will show that for a nearly ordinary representation lifting ρ, the $G_{\mathbb{Q}_p}$-stable line is not unique in general.

Remark 1.3.7. An ordinary lift of ρ (resp. ρ') to A admits two distinct filtrations of A^2: one is given by the $G_{\mathbb{Q}_p}$-stable subspace in (1.19) and the other is induced by the eigenspace for the complex conjugation τ. Since the representation ρ (resp. ρ') is reducible, we have $V_{\mathcal{R}_\rho}^{\mathrm{sub}} = V_{\mathcal{R}_\rho}^{-} = \langle e_1 \rangle$. However, the two filtrations will be different in general.

1.4 Cuspidal and Eisenstein deformation rings

In this section, we define two functors represented by quotients of the ordinary deformation rings $\mathcal{R}_\rho^{\mathrm{ord}}$ and $\mathcal{R}_{\rho'}^{\mathrm{ord}}$.

The functors $\mathcal{D}_\rho^{\mathrm{eis}}$ and $\mathcal{D}_{\rho'}^{\mathrm{eis}}$ classify certain reducible ordinary deformations of ρ and ρ' with a given unramified quotient (see also [BDP, Sec. 1.3]).

The functor $\mathcal{D}^{\mathrm{cusp}}$ parametrizes pairs of deformations of ρ and ρ' satisfying certain conditions (see also [BDP, Sec. 1.4]). The definition exploits the symmetry between the deformation functors $\mathcal{D}_\rho^{\mathrm{ord}}$ and $\mathcal{D}_{\rho'}^{\mathrm{ord}}$ due to the interchangeability of the characters $\mathbb{1}$ and ϕ. Although the link between the functor $\mathcal{D}^{\mathrm{cusp}}$ and cuspidal deformations of ρ is not a priori clear, we will later show that the universal ring representing this functor is isomorphic to the completed local ring of the eigencurve, thus justifying the notation.

Definition 1.4.1. Let $\mathcal{D}_\rho^{\mathrm{eis}} \colon \mathfrak{C}_{\bar{\mathbb{Q}}_p} \to \mathbf{Sets}$ (resp. $\mathcal{D}_{\rho'}^{\mathrm{eis}}$) be the functor sending an object A in $\mathfrak{C}_{\bar{\mathbb{Q}}_p}$ to the set of equivalence classes of pairs (ρ_A, L_A) in $\mathcal{D}_\rho^{\mathrm{ord}}(A)$ (resp. $\mathcal{D}_{\rho'}^{\mathrm{ord}}(A)$) such that the line L_A is $G_{\mathbb{Q}}$-stable.

Proposition 1.4.2. *The functors $\mathcal{D}_{\rho}^{\mathrm{eis}}$ (resp. $\mathcal{D}_{\rho'}^{\mathrm{eis}}$) is representable by $\mathcal{R}_{\rho}^{\mathrm{eis}}$ (resp. $\mathcal{R}_{\rho'}^{\mathrm{eis}}$). The morphism $\mathcal{R}_{\rho} \to \mathcal{R}_{\rho}^{\mathrm{eis}}$ (resp. $\mathcal{R}_{\rho'} \to \mathcal{R}_{\rho'}^{\mathrm{eis}}$) is surjective.*

Proof. Choose a representative of the equivalence class of the universal representation $\rho_{\mathcal{R}_{\rho}} = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. The same argument as in the proof of Proposition 1.3.5 shows that $\mathcal{D}_{\rho}^{\mathrm{eis}}$ is representable by the quotient of $\mathcal{R}_{\rho}[\![Y]\!]$ modulo the ideal

$$I_{\rho}^{\mathrm{eis}} = \langle c(\sigma) + (d(\sigma) - a(\sigma))Y - b(\sigma)Y^2 \ \forall \sigma \in G_{\mathbb{Q}}, \ 1 - d(\sigma') + b(\sigma')Y \ \forall \sigma' \in I_p \rangle.$$

Analogously for $\mathcal{D}_{\rho'}^{\mathrm{eis}}$. □

Remark 1.4.3 (Eisenstein deformations and reducible ordinary deformations)**.** The definition of the Eisenstein deformation ring is designed to encode representations for which the trace evaluated at Frob_{ℓ} gives the ℓ-th coefficient of an Eisenstein series for every prime ℓ such that $(\ell, Np) = 1$. It is natural to compare Eisenstein deformations with reducible ordinary deformations of ρ, represented by $\mathcal{R}_{\rho}^{\mathrm{ord,red}} = \mathcal{R}_{\rho}^{\mathrm{red}} \hat{\otimes}_{\mathcal{R}_{\rho}} \mathcal{R}_{\rho}^{\mathrm{ord}}$. From the proofs of Proposition 1.3.5 and Lemma 1.2.9, we see that

$$\mathcal{R}_{\rho}^{\mathrm{red}} \otimes_{\mathcal{R}_{\rho}} \mathcal{R}_{\rho}^{\mathrm{ord}} \simeq \mathcal{R}_{\rho}[\![Y]\!]/(C_{\rho}, I_{\rho}^{\mathrm{ord}}). \tag{1.22}$$

By the universal property of $\mathcal{R}_{\rho}^{\mathrm{eis}}$, there is a natural surjective map

$$\mathcal{R}_{\rho}^{\mathrm{ord,red}} \to \mathcal{R}_{\rho}^{\mathrm{eis}}. \tag{1.23}$$

However, we will see that it is not an isomorphism (Remark 1.5.10).

Definition 1.4.4. Let $\mathcal{D}^{\mathrm{cusp}} \colon \mathfrak{C}_{\bar{\mathbb{Q}}_p} \to \mathbf{Sets}$ be the functor sending an object $A \in \mathfrak{C}_{\bar{\mathbb{Q}}_p}$ to the set of pairs of equivalence classes of representations with filtration $([\rho_A, L_A], [\rho'_A, L'_A]) \in$

$\mathcal{D}_{\rho}^{\mathrm{ord}}(A) \times \mathcal{D}_{\rho'}^{\mathrm{ord}}(A)$ such that

$$\mathrm{Tr}(\rho_{\mathrm{A}}) = \mathrm{Tr}\rho'_{\mathrm{A}} \qquad \text{and} \qquad \vartheta_A = \vartheta'_A.$$

We show that the functor $\mathcal{D}^{\mathrm{cusp}}$ is representable and relate its universal ring to the rings $\mathcal{R}_{\rho}^{\mathrm{ord}}$ and $\mathcal{R}_{\rho'}^{\mathrm{ord}}$ [BDP, Lemma 1.8].

Proposition 1.4.5. *(i) The functor $\mathcal{D}^{\mathrm{cusp}}$ is representable by a ring $\mathcal{R}^{\mathrm{cusp}}$.*

(ii) The morphisms $\mathcal{R}_{\rho}^{\mathrm{ord}} \to \mathcal{R}^{\mathrm{cusp}}$ and $\mathcal{R}_{\rho'}^{\mathrm{ord}} \to \mathcal{R}^{\mathrm{cusp}}$ are surjective.

Proof. (i) The functor $\mathcal{D}^{\mathrm{cusp}}$ can be described as $\mathcal{D}_{\rho}^{\mathrm{ord}} \times_{\mathcal{D}_{\psi+1}^{ps} \times \mathcal{D}_{\mathbb{1}}^{\mathrm{loc},p}} \mathcal{D}_{\rho'}^{\mathrm{ord}}$. Hence it is representable by $\mathcal{R}_{\rho}^{\mathrm{ord}} \hat{\otimes}_{\mathcal{R}_{\psi+1}^{ps} \hat{\otimes} \mathcal{R}_{\mathbb{1}}^{\mathrm{loc},p}} \mathcal{R}_{\rho'}^{\mathrm{ord}}$. More explicitly, $\mathcal{R}^{\mathrm{cusp}}$ can be realized as the quotient of $\mathcal{R}_{\rho}^{\mathrm{ord}} \otimes_{\bar{\mathbb{Q}}_p} \mathcal{R}_{\rho'}^{\mathrm{ord}}$ modulo the ideal generated by

$$\begin{aligned} &\mathrm{Tr}\rho_{\mathcal{R}_{\rho}^{\mathrm{ord}}}(\sigma) \hat{\otimes} 1 - 1 \hat{\otimes} \mathrm{Tr}\rho'_{\mathcal{R}_{\rho'}^{\mathrm{ord}}}(\sigma), && \forall \sigma \in G_{\mathbb{Q}}. \\ &\vartheta_{\mathcal{R}_{\rho}^{\mathrm{ord}}}(\sigma) \hat{\otimes} 1 - 1 \hat{\otimes} \vartheta'_{\mathcal{R}_{\rho'}^{\mathrm{ord}}}(\sigma) && \forall \sigma \in G_{\mathbb{Q}_p}. \end{aligned}$$

(ii) It suffices to verify this on tangent spaces, *i.e.* show that the morphism of $\bar{\mathbb{Q}}_p$-vector spaces $\mathcal{D}^{\mathrm{cusp}}(\bar{\mathbb{Q}}_p[\varepsilon]) \to \mathcal{D}_{\rho}^{\mathrm{ord}}(\bar{\mathbb{Q}}_p[\varepsilon])$ is injective. Let $(\rho_\varepsilon, \rho'_\varepsilon)$ be an element in the kernel of this morphism. This implies that $\rho_\varepsilon = \rho$. Since, by definition of $\mathcal{D}^{\mathrm{cusp}}$, we have $\mathrm{Tr}(\rho_\varepsilon) = \mathrm{Tr}(\rho_\varepsilon)$, it follows that $\mathrm{Tr}(\rho'_\varepsilon) = 1 + \phi = \mathrm{Tr}(\rho')$. Then Proposition 1.2.16, this implies that $\rho'_\varepsilon = \rho'$.

∎

Note that $\mathcal{R}^{\mathrm{cusp}}$ has a well-defined Λ-algebra structure induced by $([\rho_A, L_A], [\rho'_A, L'_A]) \mapsto \det \rho_A = \det \rho'_A$. The equality of the determinants follows from the fact that

$$\det \rho_A(\sigma) = \frac{1}{2}(\mathrm{Tr}\rho_A(\sigma)^2 - \mathrm{Tr}\rho_A(\sigma^2))$$

and the equality $\mathrm{Tr}\rho_A = \mathrm{Tr}\rho'_A$.

Let us summarize the relations between the deformation rings defined so far. We have a commutative diagram of Λ-algebra morphisms

$$\begin{tikzcd}
 & & & \mathcal{R}_\rho^{\mathrm{eis}} \\
 & \mathcal{R}_\rho \arrow[r] & \mathcal{R}_\rho^{\mathrm{ord}} \arrow[ur] \arrow[dr] & \\
\mathcal{R}_{\phi+1}^{\mathrm{ps}} \arrow[ur] \arrow[dr] & & & \mathcal{R}^{\mathrm{cusp}} \\
 & \mathcal{R}_{\rho'} \arrow[r] & \mathcal{R}_{\rho'}^{\mathrm{ord}} \arrow[ur] \arrow[dr] & \\
 & & & \mathcal{R}_{\rho'}^{\mathrm{eis}}
\end{tikzcd}$$

where all the maps are *surjective*. This can be interpreted geometrically as follows. The space of deformations of the pseudorepresentation $1+\phi$ contains the deformation spaces of ρ and ρ' as closed subspaces. In each of them, one can cut out an ordinary locus. The ordinary locus then contains a subspace parametrizing certain reducible deformations of ρ that one expect to classify representations attached to Eisenstein families. The space of cuspidal deformations lies in the intersection of the ordinary deformation spaces of ρ and ρ'.

Note that the projection $\mathcal{R}_\rho^{\mathrm{ord}} \to \mathcal{R}_\rho^{\mathrm{eis}}$ factors through $\mathcal{R}_\rho^{\mathrm{ord,red}}$; the relation between the latter and $\mathcal{R}^{\mathrm{cusp}}$ is not a priori clear.

1.5 The tangent space of the universal deformation rings

We are now going to give an explicit description of the Zariski tangent spaces of the universal deformation rings defined in the previous sections (we follow the treatment in [BDP, Sec.2]). Recall that the adjoint representation of ρ, denoted by $\mathrm{ad}(\rho)$, is given by the vector space $\mathrm{End}_{\bar{\mathbb{Q}}_\mathrm{p}}(\mathrm{V})$, where V is the underlying space of ρ, endowed with the action of $g \in G_{\mathbb{Q}}$ given by

$$(g * \Theta)(v) = g(\Theta(g^{-1}v)) \quad \forall v \in V, \quad \forall \Theta \in \mathrm{End}_{\bar{\mathbb{Q}}_\mathrm{p}}(\mathrm{V}).$$

The choice of the basis e_1, e_2 of V lets us identify $\mathrm{End}_{\bar{\mathbb{Q}}_\mathrm{p}}(\mathrm{V})$ with $\mathrm{M}_2(\bar{\mathbb{Q}}_\mathrm{p})$ and the action of $g \in G_{\mathbb{Q}}$ with the conjugation by $\rho(g)$. Let $E_{i,j}$ for $1 \leq i, j \leq 2$ be the standard basis of $\mathrm{M}_2(\bar{\mathbb{Q}}_\mathrm{p})$, so that

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \alpha E_{1,1} + \beta E_{1,2} + \gamma E_{2,1} + \delta E_{2,2}.$$

We denote by $\mathrm{ad}^0(\rho)$ the subspace of $\mathrm{ad}(\rho)$ corresponding to matrices of trace 0. There is an isomorphism of $\bar{\mathbb{Q}}_p[G_{\mathbb{Q}}]$-modules

$$\mathrm{ad}(\rho) \to \mathrm{ad}^0(\rho) \oplus \bar{\mathbb{Q}}_p, \qquad \Theta \mapsto (\Theta - \mathrm{Tr}(\Theta)/2, \mathrm{Tr}(\Theta)/2).$$

Since $\mathrm{ad}(\rho)$ is canonically isomorphic to $V \otimes V^*$ as a $G_{\mathbb{Q}}$-representation, and V and V^* are reducible representation, $\mathrm{ad}(\rho)$ is reducible as well. More precisely, in the basis $E_{1,2}, E_{1,1}, E_{2,2}, E_{2,1}$ of $M_2(\bar{\mathbb{Q}}_p)$, the adjoint representation $\mathrm{ad}(\rho) \colon G_{\mathbb{Q}} \to \mathrm{GL}_4(\bar{\mathbb{Q}}_p)$ is given by the matrix

$$\mathrm{ad}(\rho) = \begin{bmatrix} \phi & -\eta & \eta & -\eta^2 \\ 0 & 1 & 0 & \phi^{-1}\eta \\ 0 & 0 & 1 & -\phi^{-1}\eta \\ 0 & 0 & 0 & \phi^{-1} \end{bmatrix}. \tag{1.24}$$

The cocycle conditions for an element $\Theta = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right] \in \mathrm{Z}^1(\mathbb{Q}, \mathrm{ad}(\rho))$ can be summarized as

$$(\mathbf{d}_{\mathbb{1}}\alpha)(\sigma,\sigma') = \phi(\sigma)^{-1}\eta(\sigma)\gamma(\sigma') \tag{1.25}$$

$$(\mathbf{d}_{\phi}\beta)(\sigma,\sigma') = \phi(\sigma)(\delta(\sigma') - \alpha(\sigma')) - \phi(\sigma)^{-1}\eta(\sigma)\gamma(\sigma')(\eta(\sigma) + \phi(\sigma)\eta(\sigma')) \tag{1.26}$$

$$(\mathbf{d}_{\phi^{-1}}\gamma)(\sigma,\sigma') = 0 \tag{1.27}$$

$$(\mathbf{d}_{\mathbb{1}}\delta)(\sigma,\sigma') = \phi(\sigma)\gamma(\sigma)\eta(\sigma'), \tag{1.28}$$

where we denoted

$$(\mathbf{d}_{\psi}\xi)(\sigma,\tau) = \xi(\sigma\tau) - \xi(\sigma) - \psi(\sigma)\xi(\tau)$$

for any function $\xi\colon G_{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$ and character ψ of $G_{\mathbb{Q}}$. The subspace of coboundaries $\mathrm{B}^1(\mathbb{Q}, \mathrm{ad}(\rho))$ is spanned by

$$\Theta_{\lambda,\mu,\nu} = \begin{bmatrix} -\lambda\phi^{-1}\eta & \lambda\phi\eta^2 + \mu(1-\phi) + \nu\eta \\ \lambda(1-\phi^{-1}) & \lambda\phi^{-1}\eta \end{bmatrix} \tag{1.29}$$

for $\lambda, \mu, \nu \in \bar{\mathbb{Q}}_p$. There is a filtration of $\mathrm{ad}(\rho)$ given by $W_0 = \mathrm{ad}(\rho)$ and

$$W_1 = \ker\left(W_0 \to \phi^{-1}, \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right] \mapsto \gamma\right) \tag{1.30}$$

$$W_2 = \ker\left(W_1 \to \mathbb{Q}_p^2, \left[\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}\right] \mapsto (\alpha, \delta)\right) \tag{1.31}$$

$$W_3 = 0 \tag{1.32}$$

with graded pieces

$$W_0/W_1 \simeq \phi^{-1}, \quad W_1/W_2 \simeq \bar{\mathbb{Q}}_p^2, \quad W_2/W_3 \simeq \phi.$$

For every $0 \leq i \leq 3$ denote $W_i^0 = W_i \cap \mathrm{ad}^0(\rho)$.

1.5.1 The tangent space of the ordinary deformation ring

Exploiting the filtration defined above, we compute the dimensions of the Zariski tangent spaces for the functors $\mathcal{D}_\rho^{\mathrm{ord}}$ and $\mathcal{D}_\rho^{\mathrm{n.ord}}$ as in [BDP, Prop. 2.1]. Denote by $t_\rho^{\mathrm{n.ord}}$ (resp. $t_\rho^{\mathrm{n.ord},0}, t_\rho^{\mathrm{ord}}, t_\rho^{\mathrm{ord},0}$) the Zariski tangent space of the functor $\mathcal{D}_\rho^{\mathrm{n.ord}}$ (resp. $\mathcal{D}_\rho^{\mathrm{n.ord}} \cap \mathcal{D}_\rho^0$, $\mathcal{D}_\rho^{\mathrm{ord}}$, $\mathcal{D}_\rho^{\mathrm{ord}} \cap \mathcal{D}_\rho^0$). Denote by $(e_{1,\varepsilon}, e_{2,\varepsilon})$ the standard basis of $\bar{\mathbb{Q}}_p[\varepsilon]^2$.

Proposition 1.5.1. *(i) There are isomorphisms*

$$\mathrm{H}^1(\mathbb{Q}, W_1) \oplus \bar{\mathbb{Q}}_p \to t_\rho^{\mathrm{n.ord}} \qquad \textit{and} \qquad \mathrm{H}^1(\mathbb{Q}, W_1^0) \oplus \bar{\mathbb{Q}}_p \to t_\rho^{\mathrm{n.ord},0}$$

given by $(\Theta, \nu) \mapsto [(1 + \varepsilon\Theta)\rho, e_{1,\varepsilon} + \varepsilon\nu e_{2,\varepsilon}]$.

(ii) The forgetful functor $\mathcal{D}_\rho^{\mathrm{n.ord}} \to \mathcal{D}_\rho$ *induces isomorphisms*

$$t_\rho^{\mathrm{ord}} \simeq \mathrm{Im}(t_\rho^{\mathrm{n.ord}} \to t_\rho) \qquad \textit{and} \qquad t_\rho^{\mathrm{ord},0} \simeq \mathrm{Im}(t_\rho^{\mathrm{n.ord},0} \to t_\rho^0).$$

Proof. (i) As in the proof of Proposition 1.3.5, the nearly ordinary deformation ring $\mathcal{R}_\rho^{\mathrm{n.ord}}$ is representable by the quotient of $\mathcal{R}_\rho[\![Y]\!]$ by the ideal $I_\rho^{\mathrm{n.ord}}$, once we fix a representative of the equivalence class of the universal representation $\rho_{\mathcal{R}_\rho} \colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{R}_\rho)$. Thus, an element of the tangent space $t_\rho^{\mathrm{n.ord}}$ is the equivalence class of a pair $(\rho_\varepsilon, L_\varepsilon)$, where $\rho_\varepsilon \colon G_\mathbb{Q} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p[\varepsilon])$ is a lift of ρ and $L_\varepsilon = \langle e_{1,\varepsilon} + \varepsilon v e_{2,\varepsilon} \rangle$ in the basis $(e_{1,\varepsilon}, e_{2,\varepsilon})$ of $\bar{\mathbb{Q}}_p[\varepsilon]^2$. The representation ρ_ε can be written as $\rho_\varepsilon = (1 + \varepsilon\Theta)\rho$ where $\Theta \in \mathrm{Z}^1(\mathbb{Q}, \mathrm{ad}(\rho))$. If we write $\Theta = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]$, we obtain

$$\rho_\varepsilon = \begin{bmatrix} 1+\varepsilon\alpha & \varepsilon\beta \\ \varepsilon\gamma & 1+\varepsilon\delta \end{bmatrix} \begin{bmatrix} \phi & \eta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \phi(1+\varepsilon\alpha) & \eta+\varepsilon(\eta\alpha+\beta) \\ \varepsilon\phi\gamma & 1+\varepsilon(\eta\gamma+\delta) \end{bmatrix}.$$

The pair $(\rho_\varepsilon, L_\varepsilon)$ is nearly ordinary if and only if the corresponding morphism $\mathcal{R}_\rho[\![Y]\!] \to \bar{\mathbb{Q}}_p[\varepsilon]$ factors through $I_\rho^{\mathrm{n.ord}}$, *i.e.*

$$0 = \varepsilon\phi\gamma + [1+\varepsilon(\eta\gamma+\delta) - \phi(1+\varepsilon\alpha)]\varepsilon\upsilon - [\eta+\varepsilon(\eta\alpha+\beta)](\varepsilon\upsilon)^2 \tag{1.33}$$

$$= \varepsilon(\phi\gamma + (1-\phi)\upsilon) \tag{1.34}$$

when restricted to $G_{\mathbb{Q}_p}$. Since $\phi(\sigma) = 1$ for all $\sigma \in G_{\mathbb{Q}_p}$, this yields $\gamma(\sigma) = 0$ for all $\sigma \in G_{\mathbb{Q}_p}$ and the condition is independent of υ. From (1.30), the map $\mathrm{ad}(\rho) \to \phi^{-1}$ sending $\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right] \mapsto \gamma$ is $G_{\mathbb{Q}}$-equivariant. Thus,

$$\gamma \in \ker(\mathrm{Z}^1(\mathbb{Q}, \phi^{-1}) \to \mathrm{Z}^1(\mathbb{Q}_p, \bar{\mathbb{Q}}_p)) = \mathrm{B}^1(\mathbb{Q}, \phi^{-1}).$$

which is equivalent to saying that the class of γ vanishes in $\mathrm{H}^1(\mathbb{Q}, \phi^{-1})$. The exact sequence of $\bar{\mathbb{Q}}_p[G_{\mathbb{Q}}]$-modules $0 \to W_1 \to \mathrm{ad}(\rho) \to \phi^{-1} \to 0$ induces an exact sequence in cohomology

$$\mathrm{H}^0(\mathbb{Q}, \phi^{-1}) \to \mathrm{H}^1(\mathbb{Q}, W_1) \to \mathrm{H}^1(\mathbb{Q}, \mathrm{ad}(\rho)) \to \mathrm{H}^1(\mathbb{Q}, \phi^{-1}).$$

Since ϕ^{-1} is not trivial, the first term is zero, so $\mathrm{H}^1(\mathbb{Q}, \phi^{-1})$ is the kernel of $\mathrm{H}^1(\mathbb{Q}, \mathrm{ad}(\rho)) \to \mathrm{H}^1(\mathbb{Q}, \phi^{-1})$. For any $\Theta \in \mathrm{H}^1(\mathbb{Q}, W_1)$ the condition (1.33) is automatically satisfied for every υ, which yields $t_\rho^{\mathrm{n.ord}} \simeq \mathrm{H}^1(\mathbb{Q}, W_1) \oplus \bar{\mathbb{Q}}_p$.

(ii) By Proposition 1.3.5, the morphism $\mathcal{R}_\rho \to \mathcal{R}_\rho^{\mathrm{ord}}$ is surjective; thus, the map $t_\rho^{\mathrm{ord}} \to t_\rho$

is injective and its image is contained in the image of the morphism $t_\rho^{\mathrm{n.ord}} \to t_\rho$ by construction. Therefore, it suffices to show that for every nearly ordinary representation $\rho_\varepsilon\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p[\varepsilon])$ there is a unique $G_{\mathbb{Q}_p}$-stable line with respect to which the quotient is unramified. From the description of the nearly ordinary tangent space, we can write $\rho_\varepsilon = (1+\varepsilon\Theta)\rho$ as above with $\Theta = \left[\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right] \in \mathrm{Z}^1(\mathbb{Q}, W_1)$ where γ is a coboundary for ϕ^{-1}. Up to modifying Θ by a coboundary $\Theta_{\lambda,0,0}$ as in (1.29) (which amounts to changing the basis for ρ_ε), we can suppose $\gamma = 0$. Thus, assume, without loss of generality, that $\gamma = 0$ in the basis $(e_{1,\varepsilon}, e_{2,\varepsilon})$ of $\bar{\mathbb{Q}}_p[\varepsilon]^2$. Then, the pair $(\rho_\varepsilon, L_\varepsilon)$ with $L_\varepsilon = \langle e_{1,\varepsilon} + \varepsilon v e_{2,\varepsilon}\rangle$ is ordinary if and only if

$$
\begin{aligned}
0 &= (1+\varepsilon\delta(\sigma)) - 1 - [\eta(\sigma) + \varepsilon\alpha(\sigma)\eta(\sigma) + \varepsilon\beta(\sigma)]\varepsilon v && (1.35)\\
&= \varepsilon(\delta(\sigma) - \eta(\sigma)\nu) && (1.36)
\end{aligned}
$$

for all σ in $I_{\mathbb{Q}_p}$ by (1.21). From (1.31), $\delta \in \mathrm{Z}^1(\mathbb{Q}, \mathbb{Q}_p)$; since $\eta \in \mathrm{Z}^1(\mathbb{Q}, \phi)$ and

$$
\mathrm{Im}(\mathrm{Z}^1(\mathbb{Q}, \bar{\mathbb{Q}}_\mathrm{p}) \to \mathrm{Z}^1(\mathrm{I}_{\mathbb{Q}_\mathrm{p}}, \bar{\mathbb{Q}}_\mathrm{p})) = \mathrm{Im}(\mathrm{Z}^1(\mathbb{Q}, \phi) \to \mathrm{Z}^1(\mathrm{I}_{\mathbb{Q}_\mathrm{p}}, \bar{\mathbb{Q}}_\mathrm{p}))
$$

is a one-dimensional $\bar{\mathbb{Q}}_p$-vector space and $\eta|_{I_{\mathbb{Q}_p}} \neq 0$, this gives a non-trivial linear condition on v. The claim follows.

□

Lemma 1.5.2. $\mathrm{H}^2(\mathbb{Q}, \phi) = \mathrm{H}^2(\mathbb{Q}, \phi^{-1}) = \mathrm{H}^2(\mathbb{Q}, \bar{\mathbb{Q}}_p) = 0.$

Proof. By the global Euler characteristic formula we have

$$
\begin{aligned}
\dim_{\bar{\mathbb{Q}}_p} \mathrm{H}^2(\mathbb{Q}, \phi) &= \dim_{\bar{\mathbb{Q}}_p} \mathrm{H}^1(\mathbb{Q}, \phi) - \dim_{\bar{\mathbb{Q}}_p} \mathrm{H}^0(\mathbb{Q}, \phi) + \dim_{\bar{\mathbb{Q}}_p} \mathrm{H}^0(\mathbb{R}, \phi) - \dim_{\bar{\mathbb{Q}}_p}(\phi)\\
&= 1 - 0 + 0 - 1 = 0,
\end{aligned}
$$

and the same applies replacing ϕ by ϕ^{-1}. Similarly,

$$\begin{aligned}\dim_{\bar{\mathbb{Q}}_p} \mathrm{H}^2(\mathbb{Q}, \bar{\mathbb{Q}}_p) &= \dim_{\bar{\mathbb{Q}}_p} \mathrm{H}^1(\mathbb{Q}, \bar{\mathbb{Q}}_p) - \dim_{\bar{\mathbb{Q}}_p} \mathrm{H}^0(\mathbb{Q}, \bar{\mathbb{Q}}_p) + \dim_{\bar{\mathbb{Q}}_p} \mathrm{H}^0(\mathbb{R}, \bar{\mathbb{Q}}_p) - \dim_{\bar{\mathbb{Q}}_p}(\bar{\mathbb{Q}}_p) \\ &= 1 - 1 + 1 - 1 = 0.\end{aligned}$$

$\square$

Lemma 1.5.3. *The $\bar{\mathbb{Q}}_p[G_{\mathbb{Q}}]$-linear maps*

$$W_1 \to \bar{\mathbb{Q}}_p^2, \quad \left[\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}\right] \mapsto (\alpha, \delta) \qquad \textit{and} \qquad W_1^0 \to \bar{\mathbb{Q}}_p, \quad \left[\begin{smallmatrix} \alpha & \beta \\ 0 & -\alpha \end{smallmatrix}\right] \mapsto \alpha$$

induce isomorphisms $\mathrm{H}^1(\mathbb{Q}, W_1) \to \mathrm{H}^1(\mathbb{Q}, \bar{\mathbb{Q}}_p^2)$ *and* $\mathrm{H}^1(\mathbb{Q}, W_1^0) \to \mathrm{H}^1(\mathbb{Q}, \bar{\mathbb{Q}}_p)$.

Proof. Consider the exact sequence of $\bar{\mathbb{Q}}_p[G_{\mathbb{Q}}]$-modules $0 \to W_2 \to W_1 \to \bar{\mathbb{Q}}_p^2 \to 0$ defined in (1.31). This yields a long exact sequence in cohomology

$$0 \to \mathrm{H}^0(\mathbb{Q}, W_1) \to \mathrm{H}^0(\mathbb{Q}, \bar{\mathbb{Q}}_p^2) \to \mathrm{H}^1(\mathbb{Q}, W_2) \to \mathrm{H}^1(\mathbb{Q}, W_1) \to \mathrm{H}^1(\mathbb{Q}, \bar{\mathbb{Q}}_p^2) \to 0,$$

because W_2 is isomorphic to ϕ and, by Lemma 1.5.2, $\mathrm{H}^2(\mathbb{Q}, \phi)$ is trivial. In order to show that $\mathrm{H}^1(\mathbb{Q}, W_1) \to \mathrm{H}^1(\mathbb{Q}, \bar{\mathbb{Q}}_p^2)$ is an isomorphism, it suffices to check that the map $\mathrm{H}^0(\mathbb{Q}, \bar{\mathbb{Q}}_p^2) \to \mathrm{H}^1(\mathbb{Q}, W_2)$ is surjective. Recall that W_1 is the subspace of $M_2(\bar{\mathbb{Q}}_p)$ spanned by $E_{1,2}, E_{1,1}, E_{2,2}$. Comparing with the adjoint representation matrix (1.24), we see that the $G_{\mathbb{Q}}$-invariant subspace of W_1 is spanned by the line $\langle E_{1,1} + E_{2,2} \rangle$, in particular $\mathrm{H}^0(\mathbb{Q}, W_1) \simeq \bar{\mathbb{Q}}_p$. Since $\mathrm{H}^0(\mathbb{Q}, \bar{\mathbb{Q}}_p^2)$ and $\mathrm{H}^1(\mathbb{Q}, W_2)$ are two and one-dimensional respectively, it follows that the map $\mathrm{H}^0(\mathbb{Q}, \bar{\mathbb{Q}}_p^2) \to \mathrm{H}^1(\mathbb{Q}, W_2)$ is surjective.

Similarly, W_1^0 fits in an exact sequence $0 \to W_2 \to W_1^0 \to \bar{\mathbb{Q}}_p \to 0$ where the map $W_1^0 \to \bar{\mathbb{Q}}_p$ is given by $\left[\begin{smallmatrix} \alpha & \beta \\ 0 & -\alpha \end{smallmatrix}\right] \mapsto \alpha$. The long exact sequence in cohomology yields

$$0 \to \mathrm{H}^0(\mathbb{Q}, W_1^0) \to \mathrm{H}^0(\mathbb{Q}, \bar{\mathbb{Q}}_p) \to \mathrm{H}^1(\mathbb{Q}, W_2) \to \mathrm{H}^1(\mathbb{Q}, W_1^0) \to \mathrm{H}^1(\mathbb{Q}, \bar{\mathbb{Q}}_p) \to 0,$$

and $\mathrm{H}^0(\mathbb{Q}, W_1^0) = 0$. Since $\mathrm{H}^0(\mathbb{Q}, \bar{\mathbb{Q}}_p)$ and $\mathrm{H}^1(\mathbb{Q}, W_2)$ are both one-dimensional, the map $\mathrm{H}^0(\mathbb{Q}, \bar{\mathbb{Q}}_p) \to \mathrm{H}^1(\mathbb{Q}, W_2)$ is surjective. This implies that $\mathrm{H}^1(\mathbb{Q}, W_1^0) \to \mathrm{H}^1(\mathbb{Q}, \bar{\mathbb{Q}}_p)$ is an isomorphism. □

Proposition 1.5.4. *(i) There is a set of representatives of t_ρ^{ord} of the form $(\rho_\varepsilon, L_\varepsilon)$ where*

$$\rho_\varepsilon = \begin{bmatrix} \phi(1+\varepsilon\lambda\eta_{\mathbb{1}}) & \eta + \varepsilon\xi \\ 0 & 1+\varepsilon\mu\eta_{\mathbb{1}} \end{bmatrix}, \qquad L_\varepsilon = \langle e_{1,\varepsilon} + \varepsilon\mu e_{2,\varepsilon}\rangle \tag{1.37}$$

for $(\lambda, \mu) \in \bar{\mathbb{Q}}_p^2$; the equivalence class $[\rho_\varepsilon, L_\varepsilon]$ is independent of ξ.

(ii) The Zariski tangent spaces of the functors $\mathcal{D}_\rho^{\mathrm{ord}}$, $\mathcal{D}_\rho^{\mathrm{ord},0}$, $\mathcal{D}_\rho^{\mathrm{n.ord}}$ and $\mathcal{D}_\rho^{\mathrm{n.ord},0}$ satisfy

$$\dim_{\bar{\mathbb{Q}}_p} t_\rho^{\mathrm{ord}} = 2, \qquad \dim_{\bar{\mathbb{Q}}_p} t_\rho^{\mathrm{ord},0} = 1, \tag{1.38}$$

$$\dim_{\bar{\mathbb{Q}}_p} t_\rho^{\mathrm{n.ord}} = 3, \qquad \dim_{\bar{\mathbb{Q}}_p} t_\rho^{\mathrm{n.ord},0} = 2. \tag{1.39}$$

Proof. (i) By Proposition 1.5.1, an element of t_ρ^{ord} is the equivalence class of a pairs $(\rho_\varepsilon, L_\varepsilon)$ where $\rho_\varepsilon = \left(1 + \varepsilon \left[\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}\right]\right)\rho$ for some $\left[\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}\right] \in \mathrm{Z}^1(\mathbb{Q}, W_1)$ and the equivalence class of such pair is uniquely determined by the image of $\left[\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}\right]$ in $\mathrm{H}^1(\mathbb{Q}, W_1)$. Recall that $\eta_{\mathbb{1}}$ is the generator of $\mathrm{Z}^1(\mathbb{Q}, \bar{\mathbb{Q}}_p)$ such that $\eta_{\mathbb{1}} = \log_p \circ \chi$. Then $\alpha, \delta \in \mathrm{Z}^1(\mathbb{Q}, \bar{\mathbb{Q}}_p)$ can be written as $\alpha = \lambda\eta_{\mathbb{1}}$ and $\delta = \mu\eta_{\mathbb{1}}$. By Lemma 1.5.3, the class of $\left[\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}\right]$ in $\mathrm{H}^1(\mathbb{Q}, W_1)$ is independent of β. The line L_ε is the unique $G_{\mathbb{Q}_p}$-stable line yielding an unramified quotient; a direct calculation shows $L_\varepsilon = \langle e_{1,\varepsilon} + \mu e_{2,\varepsilon}\rangle$ since $\eta|_{I_{\mathbb{Q}_p}} = \eta_{\mathbb{1}}|_{I_{\mathbb{Q}_p}}$.

(ii) The claim about the dimensions of t_ρ^{ord} and $t_\rho^{\mathrm{ord},0}$ follows from (i). The dimensions of $t_\rho^{\mathrm{n.ord}}$ and $t_\rho^{\mathrm{n.ord},0}$ are obtained by combining (i) with Proposition 1.5.1.

□

Remark 1.5.5. Denote $t_\rho^{\mathrm{ord,red}}$ the tangent space of the reducible ordinary deformation ring $\mathcal{R}_\rho^{\mathrm{ord,red}}$. It follows from Proposition 1.5.4 that the natural inclusion

$$t_\rho^{\mathrm{ord,red}} \hookrightarrow t_\rho^{\mathrm{ord}}$$

is, in fact, an isomorphism.

1.5.2 The tangent space of the universal deformation ring

Lemma 1.5.6. *We have* $\mathrm{H}^2(\mathbb{Q}, W_i) = \mathrm{H}^2(\mathbb{Q}, W_i^0) = 0$ *for every* $0 \leq i \leq 3$*. In particular, the representation* ρ *is unobstructed.*

Proof. The claim is trivially true for $i = 3$ since $W_3 = 0$. For every $0 \leq i \leq 2$ we have an exact sequence of $\bar{\mathbb{Q}}_p[G_\mathbb{Q}]$-modules $0 \to W_{i+1} \to W_i \to W_i/W_{i+1}$ such that $\mathrm{H}^2(\mathbb{Q}, W_{i+1}/W_i) = 0$ by Lemma 1.5.2. Thus, $\mathrm{H}^2(\mathbb{Q}, W_i) = 0$ by induction. Similarly, $\mathrm{H}^2(\mathbb{Q}, W_i^0) = 0$ for every $0 \leq i \leq 3$. □

Proposition 1.5.7. *We have* $\dim_{\bar{\mathbb{Q}}_p} t_\rho = 3$ *and* $\dim_{\bar{\mathbb{Q}}_p} t_\rho^0 = 2$.

Proof. We have $t_\rho = \mathrm{H}^1(\mathbb{Q}, W_0)$ and $t_\rho^0 = \mathrm{H}^1(\mathbb{Q}, W_0^0)$. We have exact sequences

$$0 \to W_1 \to W_0 \to \phi^{-1} \to 0, \qquad 0 \to W_1 \to W_0 \to \phi^{-1} \to 0.$$

Since $\mathrm{H}^0(\mathbb{Q}, \phi^{-1}) = 0$ and $\mathrm{H}^2(\mathbb{Q}, W_1) = \mathrm{H}^2(\mathbb{Q}, W_1^0) = 0$ by the previous lemma, we obtain the exact sequences in cohomology $0 \to \mathrm{H}^1(\mathbb{Q}, W_1) \to \mathrm{H}^1(\mathbb{Q}, W_0) \to \mathrm{H}^1(\mathbb{Q}, \phi^{-1}) \to 0$ and $0 \to \mathrm{H}^1(\mathbb{Q}, W_1^0) \to \mathrm{H}^1(\mathbb{Q}, W_0^0) \to \mathrm{H}^1(\mathbb{Q}, \phi^{-1}) \to 0$. Since $\mathrm{H}^1(\mathbb{Q}, \phi^{-1})$ is one-dimensional, the claim follows from Lemma 1.5.3. □

Corollary 1.5.8. *The universal deformation ring* $\mathcal{R}_\rho$ *is isomorphic to the ring of power series in three variables over* $\bar{\mathbb{Q}}_p$.

Proof. This is follows from [Maz89, Proposition 2], given that ρ is unobstructed by Lemma 1.5.6 and that the tangent space t_ρ is three-dimensional by Proposition 1.5.7. □

1.5.3 The tangent space of the Eisenstein and cuspidal deformation rings

Denote by $t_\rho^{\mathrm{eis}}, t_\rho^{\mathrm{eis},0}, t^{\mathrm{cusp}}, t^{\mathrm{cusp},0}$ the Zariski tangent spaces of the functors $\mathcal{D}_\rho^{\mathrm{eis}}, \mathcal{D}_\rho^{\mathrm{eis}} \cap \mathcal{D}_\rho^0$, $\mathcal{D}^{\mathrm{cusp}}$, and $\mathcal{D}^{\mathrm{cusp}} \cap \mathcal{D}_\rho^0$ respectively.

Proposition 1.5.9. *There is a set of representatives of t_ρ^{eis} of the form $(\rho_\varepsilon, L_\varepsilon)$ where*

$$\rho_\varepsilon = \begin{bmatrix} \phi(1+\varepsilon\lambda\eta_{\mathbb{1}}) & \eta+\varepsilon\xi \\ 0 & 1 \end{bmatrix}, \qquad L_\varepsilon = \langle e_{1,\varepsilon} \rangle \tag{1.40}$$

for $\lambda \in \bar{\mathbb{Q}}_p$; the equivalence class is independent of ξ. In particular, $\dim_{\bar{\mathbb{Q}}_p} t_\rho^{\mathrm{eis}} = 1$ and $\dim_{\bar{\mathbb{Q}}_p} t_\rho^{\mathrm{eis},0} = 0$.

Proof. Let $(\rho_\varepsilon, L_\varepsilon)$ be a lift of ρ to $\bar{\mathbb{Q}}_p[\varepsilon]$ whose equivalence class is in t_ρ^{eis}. By Proposition 1.5.4, it can be written in the form (1.37). The unique $G_{\mathbb{Q}}$-stable line for ρ_ε is $\langle e_{1,\varepsilon} \rangle$; hence the equivalence class of $(\rho_\varepsilon, L_\varepsilon)$ is in t_ρ^{ord} if and only if the ordinary filtration coincides with the $G_{\mathbb{Q}}$-stable subspace, *i.e.* $\mu = 0$. The description of t_ρ^{eis} follows. For $t_\rho^{\mathrm{eis},0}$, it suffices to add the condition $\det \rho_\varepsilon = \phi$, which implies that $\lambda = 0$. □

Remark 1.5.10. Comparing Proposition 1.5.9 with Remark 1.5.5, we see that the inclusion $t_\rho^{\mathrm{eis}} \hookrightarrow t_\rho^{\mathrm{ord,red}}$ is not surjective. In particular, the kernel of the projection $\mathcal{R}_\rho^{\mathrm{ord,red}} \to \mathcal{R}^{\mathrm{eis}}$ is not trivial.

We compute the dimension of the cuspidal tangent space as in [BDP, Prop. 2.7].

Proposition 1.5.11. *(i) The tangent space t^{cusp} is the space of pairs $([\rho_\varepsilon, L_\varepsilon], [\rho'_\varepsilon, L'_\varepsilon])$ in $t_\rho^{\mathrm{ord}} \times t_{\rho'}^{\mathrm{ord}}$ where a set of representatives of the equivalence classes $[\rho_\varepsilon, L_\varepsilon]$ and $[\rho'_\varepsilon, L'_\varepsilon]$*

can be chosen as

$$\rho_\varepsilon = \begin{bmatrix} \phi(1+\varepsilon\lambda\eta_{\mathbb{1}}) + \varepsilon\mu\eta & (1+\varepsilon\lambda\eta_{\mathbb{1}})\eta + \varepsilon\beta \\ -\varepsilon\mu(1-\phi^{-1}) & 1+\varepsilon\mu(\eta_{\mathbb{1}}-\eta) \end{bmatrix}, \qquad L_\varepsilon = \langle e_{1,\varepsilon}\rangle \tag{1.41}$$

$$\rho'_\varepsilon = \begin{bmatrix} (1+\varepsilon\mu\eta_{\mathbb{1}}) + \varepsilon\lambda\eta' & \phi((1+\varepsilon\mu\eta_{\mathbb{1}})\eta' + \varepsilon\beta') \\ \varepsilon\lambda(1-\phi) & \phi(1+\varepsilon\lambda(\eta_{\mathbb{1}}-\eta')) \end{bmatrix}, \qquad L'_\varepsilon = \langle e_{1,\varepsilon}\rangle \tag{1.42}$$

with $\lambda\mathcal{L}(\phi) - \mu\mathcal{L}(\phi^{-1}) = 0$*; the equivalence classes are independent of* β *and* β' *respectively.*

(ii) We have $\dim_{\bar{\mathbb{Q}}_p} t^{\mathrm{cusp}} = 1$ *and* $\dim_{\bar{\mathbb{Q}}_p} t^{\mathrm{cusp},0} = 0$*.*

Proof. (i) By definition of the functor $\mathcal{D}^{\mathrm{cusp}}$, we can describe the tangent space as the set of pairs of equivalence classes $([\rho_\varepsilon, L_\varepsilon], [\rho'_\varepsilon, L'_\varepsilon]) \in t^{\mathrm{ord}}_\rho \times t^{\mathrm{ord}}_{\rho'}$ satisfying the conditions

$$\mathrm{Tr}(\rho_\varepsilon) = \mathrm{Tr}(\rho'_\varepsilon) \quad \text{ and } \vartheta_\varepsilon(\mathrm{Frob_p}) = \vartheta'_\varepsilon(\mathrm{Frob_p}).$$

By Proposition 1.5.4, a representative of the equivalence class of t^{ord}_ρ can be chosen of the form (1.37) in the basis $(e_{1,\varepsilon}, e_{2,\varepsilon})$ of $\bar{\mathbb{Q}}_p[\varepsilon]^2$. The matrix of ρ_ε in the basis $(e_{1,\varepsilon} + \mu e_{2,\varepsilon}, e_{2,\varepsilon})$ is

$$\begin{bmatrix} \phi(1+\varepsilon\lambda\eta_{\mathbb{1}}) + \varepsilon\mu\eta & (1+\varepsilon\lambda\eta_{\mathbb{1}})\eta + \varepsilon\beta \\ -\varepsilon\mu(1-\phi^{-1}) & 1+\varepsilon\mu(\eta_{\mathbb{1}}-\eta) \end{bmatrix},$$

hence

$$\mathrm{Tr}(\rho_\varepsilon) = 1 + \phi + \varepsilon(\mu + \phi\lambda)\eta_{\mathbb{1}} \tag{1.43}$$

and $\vartheta_\varepsilon \colon G_{\mathbb{Q}_p} \to \bar{\mathbb{Q}}_p[\varepsilon]^\times$ is the unramified character satisfying

$$\vartheta_\varepsilon(\mathrm{Frob}_p) = 1 + \varepsilon\mu(\eta_{\mathbb{1}} - \eta)(\mathrm{Frob}_p) = 1 - \varepsilon\mu\mathcal{L}(\phi^{-1}) \tag{1.44}$$

by Proposition 1.1.6. Similarly, since $\rho' = \left[\begin{smallmatrix} 1 & \phi\eta' \\ 0 & \phi \end{smallmatrix}\right] = \left[\begin{smallmatrix} \phi^{-1} & \eta' \\ 0 & 1 \end{smallmatrix}\right] \otimes \phi$, one can describe

$t^{\mathrm{ord}}_{\rho'} \simeq t^{\mathrm{ord}}_{\rho'\otimes\phi^{-1}}$ by simply replacing ϕ by ϕ^{-1} and η by η' in the above description of t^{ord}_{ρ}. Thus, in an appropriate basis of $\bar{\mathbb{Q}}_p[\varepsilon]^2$, the representation ρ'_ε is

$$\begin{bmatrix} (1+\varepsilon\lambda'\eta_{\mathbb{1}})+\varepsilon\mu'\eta' & \phi((1+\varepsilon\lambda'\eta_{\mathbb{1}})\eta'+\varepsilon\beta') \\ \varepsilon\mu'(1-\phi) & \phi(1+\varepsilon\mu'(\eta_{\mathbb{1}}-\eta')) \end{bmatrix}.$$

for some $\lambda', \mu' \in \bar{\mathbb{Q}}_p$. Thus, we have

$$\mathrm{Tr}(\rho'_\varepsilon) = 1+\phi+\varepsilon(\lambda'+\mu'\phi)\eta_{\mathbb{1}}, \tag{1.45}$$

$$\vartheta'_\varepsilon(\mathrm{Frob}_p) = 1+\varepsilon\mu'(\eta_{\mathbb{1}}-\eta')(\mathrm{Frob}_p) = 1-\mu'\mathcal{L}(\phi)\varepsilon. \tag{1.46}$$

From (1.43), (1.44), (1.45) and (1.46) one sees that

$$(\rho_\varepsilon, \rho'_\varepsilon) \in t^{\mathrm{cusp}} \iff \lambda = \mu',\ \mu = \lambda' \text{ and } \mu\mathcal{L}(\phi^{-1}) = \lambda\mathcal{L}(\phi). \tag{1.47}$$

(ii) By Proposition 1.1.8, $\mathcal{L}(\phi)$ and $\mathcal{L}(\phi^{-1})$ are non-zero, hence the dimension of t^{cusp} is one. To compute the relative tangent space $t^{\mathrm{cusp},0}$ it suffices to add to (1.47) the condition $\det\rho_\varepsilon = \phi$, which is equivalent to $\lambda+\mu = 0$. This equation is linearly independent from (1.47) provided that $\mathcal{L}(\phi)+\mathcal{L}(\phi^{-1}) \neq 0$. By Proposition 1.1.8, $\mathcal{L}(\phi)$ and $\mathcal{L}(\phi^{-1})$ are linearly independent over $\bar{\mathbb{Q}}$ if ϕ is not quadratic; instead, when ϕ is quadratic one has $\mathcal{L}(\phi) = \mathcal{L}(\phi^{-1}) \neq 0$. In either case, $\dim_{\bar{\mathbb{Q}}_p} t^{\mathrm{cusp},0} = 0$.

□

Corollary 1.5.12. *We have $t^{\mathrm{ord}}_\rho = t^{\mathrm{cusp}} \oplus t^{\mathrm{eis}}_\rho$.*

Proof. Let λ, μ be the coordinates of t^{ord}_ρ given in Proposition 1.5.4. On the one hand, by Proposition 1.5.11, the equation defining t^{cusp} is $\mu\mathcal{L}(\phi^{-1}) = \lambda\mathcal{L}(\phi)$, with $\mathcal{L}(\phi) \neq 0$. On the other hand, by Proposition 1.5.9 the equation defining t^{eis}_ρ is $\mu = 0$. □

The Eisenstein deformation ring

Let S be the set containing the places of $\mathbb{Q}$ dividing Np and and the place ∞. Denote by $G_{\mathbb{Q},S}$ the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside S.

We show the existence of a non-torsion cohomology class in the Iwasawa cohomology group $\mathrm{H}^1(G_{\mathbb{Q},S}, \kappa_\Lambda\phi^\pm)$. For $n \in \mathbb{Z}_{\geq 1}$, let $\kappa_n = \kappa_\Lambda \bmod X^n$; in particular, $\kappa_1 = \mathbb{1}$. We have the following [BDP, Prop. 2.9].

Proposition 1.5.13. *(i) For every $n \in \mathbb{Z}_{\geq 1}$, we have*

$$\mathrm{H}^1(G_{\mathbb{Q},S}, \kappa_n\phi^\pm) = \Lambda/(X^n) \qquad \textit{and} \qquad \mathrm{H}^2(G_{\mathbb{Q},S}, \kappa_n\phi^\pm) = 0;$$

(ii) For every $n \in \mathbb{Z}_{\geq 1}$ and $1 \leq i \leq 2$ we have

$$\mathrm{H}^i(G_{\mathbb{Q},S}, \kappa_\Lambda\phi^\pm) \otimes_\Lambda \Lambda/(X^n) \xrightarrow{\sim} \mathrm{H}^i(G_{\mathbb{Q},S}, \kappa_n\phi^\pm).$$

In particular, $\mathrm{H}^1(G_{\mathbb{Q},S}, \kappa_\Lambda\phi^\pm) = \Lambda$ *and* $\mathrm{H}^2(G_{\mathbb{Q},S}, \kappa_\Lambda\phi^\pm) = 0$.

Proof. (i) There is a short exact sequence of $\Lambda[G_{\mathbb{Q},S}]$-modules

$$0 \to \kappa_{n-1}\phi^\pm \xrightarrow{\cdot X} \kappa_n\phi^\pm \to \phi^\pm \to 0 \tag{1.48}$$

which yields the exact sequence in cohomology for $i = 1, 2$

$$\mathrm{H}^{i-1}(G_{\mathbb{Q},S}, \phi) \to \mathrm{H}^i(G_{\mathbb{Q},S}, \kappa_{n-1}\phi^\pm) \xrightarrow{\cdot X} \mathrm{H}^i(G_{\mathbb{Q},S}, \kappa_n\phi^\pm) \to \mathrm{H}^i(G_{\mathbb{Q},S}, \phi^\pm). \tag{1.49}$$

by [Nek06, Prop.3.5.1.3]. For $i = 2$ an induction on n shows that $\mathrm{H}^i(G_{\mathbb{Q},S}, \kappa_n\phi^\pm) = 0$ for every n, since $\mathrm{H}^2(G_{\mathbb{Q},S}, \phi) = 0$ by Lemma 1.5.2. For $i = 1$, it follows that there is an exact sequence of Λ-modules

$$0 \to \mathrm{H}^1(G_{\mathbb{Q},S}, \kappa_{n-1}\phi^{\pm}) \xrightarrow{\cdot X} \mathrm{H}^1(G_{\mathbb{Q},S}, \kappa_n\phi^{\pm}) \to \mathrm{H}^1(G_{\mathbb{Q},S}, \phi^{\pm}) \to 0.$$

Assume by induction that $\mathrm{H}^1(G_{\mathbb{Q},S}, \kappa_{n-1}\phi^{\pm}) \simeq \Lambda/(X^{n-1})$. Since the map $\mathrm{H}^1(G_{\mathbb{Q},S}, \kappa_{n-1}\phi^{\pm}) \to \mathrm{H}^1(G_{\mathbb{Q},S}, \kappa_n\phi^{\pm})$ is given by multiplication by X, it follows that $\mathrm{H}^1(G_{\mathbb{Q},S}, \kappa_n\phi^{\pm}) \otimes \Lambda/(X) \simeq \mathrm{H}^1(G_{\mathbb{Q},S}, \phi^{\pm}) \simeq \Lambda/(X)$. In particular, by Nakayama's lemma, $\mathrm{H}^1(G_{\mathbb{Q},S}, \kappa_n\phi^{\pm})$ is generated by one element, and therefore, isomorphic to $\Lambda/(X^n)$.

(ii) There is an exact sequence of $\Lambda[G_{\mathbb{Q},S}]$-modules

$$0 \to \kappa_\Lambda\phi^{\pm} \xrightarrow{\cdot X^n} \kappa_\Lambda\phi^{\pm} \to \kappa_n\phi^{\pm} \to 0.$$

Using the result (i), we obtain an exact sequence in cohomology

$$\begin{aligned} 0 \to \mathrm{H}^1(G_{\mathbb{Q},S}, \kappa_\Lambda\phi^{\pm}) &\xrightarrow{\cdot X^n} \mathrm{H}^1(G_{\mathbb{Q},S}, \kappa_\Lambda\phi^{\pm}) \to \mathrm{H}^1(G_{\mathbb{Q},S}, \kappa_n\phi^{\pm}) \\ \to \ \mathrm{H}^2(G_{\mathbb{Q},S}, \kappa_\Lambda\phi^{\pm}) &\xrightarrow{\cdot X^n} \mathrm{H}^2(G_{\mathbb{Q},S}, \kappa_\Lambda\phi^{\pm}) \to 0. \end{aligned}$$

Applying Nakayama's Lemma in the case $n = 1$, we deduce that $\mathrm{H}^2(G_{\mathbb{Q},S}, \kappa_\Lambda\phi^{\pm}) = 0$ and $\mathrm{H}^1(G_{\mathbb{Q},S}, \kappa_\Lambda\phi^{\pm}) = 0$ is generated by one element. Since $\mathrm{H}^1(G_{\mathbb{Q},S}, \kappa_\Lambda\phi^{\pm}) \otimes \Lambda/(X^n) \simeq \mathrm{H}^1(G_{\mathbb{Q},S}, \kappa_n\phi^{\pm}) \simeq \Lambda/(X^n)$ for every n, it follows that $\mathrm{H}^1(G_{\mathbb{Q},S}, \kappa_\Lambda\phi^{\pm}) = 0$ is isomorphic to Λ.

□

The existence of this cohomology class determines the structure of the deformation ring $\mathcal{R}_\rho^{\mathrm{eis}}$ [BDP, Lemma 3.5].

Corollary 1.5.14. *The morphism $\Lambda \to \mathcal{R}_\rho^{\mathrm{eis}}$ and $\Lambda \to \mathcal{R}_{\rho'}^{\mathrm{eis}}$ are isomorphisms.*

Proof. By Proposition 1.5.13, there exists a cocycle $\eta_\Lambda \in \mathrm{Z}^1(G_{\mathbb{Q},S}, \kappa_\Lambda\phi)$ such that $[\eta_\Lambda \otimes$

$1_{\bar{\mathbb{Q}}_p}] = [\eta] \in \mathrm{H}^1(\mathbb{Q}, \phi)$. Up to modify η_Λ by a coboundary, we can assume $\eta_\Lambda \otimes 1_{\bar{\mathbb{Q}}_p} = \eta$. Then $\rho_\Lambda = \left[\begin{smallmatrix} \kappa_\Lambda \phi & \eta_\Lambda \\ 0 & 1 \end{smallmatrix}\right]$ in the basis $e_{1,\Lambda}, e_{2.\Lambda}$ of Λ^2 is a lift of ρ to Λ. The pair $(\rho_\Lambda, \langle e_{1,\Lambda} \rangle)$ induces a morphism $\mathcal{R}_\rho^{\mathrm{eis}} \to \Lambda$. Since $\det \rho_\Lambda = \kappa_\Lambda \phi$, the morphism $\mathcal{R}_\rho^{\mathrm{eis}} \to \Lambda$ is Λ-linear. In particular, the structural morphism $\Lambda \to \mathcal{R}_\rho^{\mathrm{eis}}$ is injective. To conclude that it is surjective, it suffices to notice that by Proposition 1.5.9 the induced morphism on the tangent spaces is injective. □

Remark 1.5.15. The proof that $\mathcal{R}_\rho^{\mathrm{eis}}$ is isomorphic to Λ via the structural morphism is purely Galois theoretic. We will see that the same is true for the cuspidal deformation ring $\mathcal{R}^{\mathrm{cusp}}$. However, in order to show this, we will use some modular input. Indeed, our computation on the tangent space implies that the Krull dimension of $\mathcal{R}^{\mathrm{cusp}}$ is at most one, but it is not a priori clear that $\mathcal{R}^{\mathrm{cusp}}$ is not an artinian ring. This is because any infinitesimal pair of lifts of Λ to the tangent space does not a priori lift to Λ. A Λ-adic lift of the pair (ρ, ρ') will arise from the representation attached to a Λ-adic family of cuspforms.

Chapter 2

The Modularity Theorems

The aim of this chapter is proving a modularity theorem for a family of Galois representations arising from the deformation ring $\mathcal{R}^{\mathrm{cusp}}$. More precisely, we prove the existence of an isomorphism between $\mathcal{R}^{\mathrm{cusp}}$ and the Hecke algebra given by the completed local ring of the eigencurve at the point corresponding to an irregular weight one Eisenstein series, denoted by $\mathcal{T}^{\mathrm{cusp}}$. This type of result is an example of an $R = T$ theorem, an isomorphism between a deformation ring R and a Hecke algebra T. Results of this type have been key steps towards the proof of many groundbreaking theorems, including Fermat's Last Theorem.

In this chapter, we follow the approach of [BD16]. A significant difference is that in our case the existence of a map $\mathcal{R}^{\mathrm{cusp}} \to \mathcal{T}^{\mathrm{cusp}}$ is unclear due to the reducibility of the representation attached to the Eisenstein series. To construct a map, we use a method developed by Mazur and Wiles [MW84] to construct two representations $G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{T}_{\mathrm{cusp}})$ with residual representations ρ and ρ'. Via the isomorphism between $\mathcal{R}^{\mathrm{cusp}}$ and $\mathcal{T}^{\mathrm{cusp}}$ we show that the cuspidal eigencurve is etale over the weight space.

We then proceed to study the local ring of the eigencurve. In addition to belonging to a cuspidal Hida family, irregular weight one Eisenstein points are the weight one specialization of two Eisenstein families. Using a powerful commutative algebra, Wiles' Numerical Criterion, we refine the result $\mathcal{R}^{\mathrm{cusp}} = \mathcal{T}^{\mathrm{cusp}}$ and obtain an isomorphisms between the ordinary deformations rings $\mathcal{R}_{\rho}^{\mathrm{ord}}$ and $\mathcal{R}_{\rho'}^{\mathrm{ord}}$, and the local rings of the eigencurve corresponding to the systems of eigenvalues of the Hecke module generated by the cuspidal family and one Eisenstein family.

Finally, we describe the structure of the completed local ring of the eigencurve, denoted by $\mathcal{T}$. We show that $\mathcal{T}$ is not Gorenstein; thus, not of complete intersection. In particular, this provides an example of a height one prime ideal of the ordinary Hida Hecke algebra such that the localization is not Gorenstein.

2.1 Generalities on the eigencurve

2.1.1 The modular curve and its canonical locus

Let $X = X(\Gamma_1(N))$ be the proper smooth modular curve over $\mathbb{Z}_p$ for a prime $p \nmid N$, and let $E^{\mathrm{univ}} \to X$ be the universal generalized elliptic curve over X with identity section $e\colon X \to E^{\mathrm{univ}}$. Let $D \subset X$ be the divisor corresponding to the cusps of X. The open modular curve $X \setminus D$ is a moduli space for elliptic curves E together with an embedding $\mu_N \hookrightarrow E[N]$. Denote by ω the invertible sheaf $e^*(\Omega^1_{E^{\mathrm{univ}}/X})$ on X.

Let $\Gamma = \Gamma_0(p) \cap \Gamma_1(N)$ and let $X_\Gamma = X(\Gamma_0(p) \cap \Gamma_1(N))$ over $\mathbb{Z}_p$ be the proper semistable model for the modular curve of level $\Gamma_1(N) \cap \Gamma_0(p)$. Denote $D_\Gamma \subset X_\Gamma$ the divisor of the cusps. Then $X_\Gamma \setminus D_\Gamma$ parametrizes triples given by an elliptic curve E, with an embedding $\mu_N \hookrightarrow E[N]$ and a finite flat subroup $H \subset E[p]$ of rank p.

Denote $\mathcal{X} = X^{\mathrm{an}}_{\mathbb{Q}_p}$ and $\mathcal{X}_\Gamma = X^{\mathrm{an}}_{\Gamma,\mathbb{Q}_p}$ the analytification of the generic fibers of X and X_Γ respectively. Let ω^{an} be the analytification of the invertible sheaf ω. Forgetting the

p-level structure yields a natural projection of rigid analytic spaces

$$\pi\colon \mathcal{X}_\Gamma \to \mathcal{X}.$$

Denote $\mathcal{X}^{\mathrm{ord}}$, $\mathcal{X}_\Gamma^{\mathrm{ord}}$ the ordinary loci of $\mathcal{X}$ and $\mathcal{X}^{\mathrm{ord}}$. While the ordinary locus $\mathcal{X}^{\mathrm{ord}}$ is connected, the ordinary locus $\mathcal{X}_\Gamma^{\mathrm{ord}}$ has two connected component, the multiplicative and etale locus, denoted by $\mathcal{X}^{\mathrm{mult}}$ and $\mathcal{X}^{\mathrm{et}}$. The multiplicative locus $\mathcal{X}_\Gamma^{\mathrm{mult}}$ parametrizes elliptic curves with good ordinary reduction, level $\Gamma_1(N)$-structure and a subgroup of order p lifting the kernel of the Frobenius morphism. For $v \geq 0$, denote by $\mathcal{X}(v)$ (resp. $\mathcal{X}_\Gamma^{\mathrm{mult}}(v)$) the v-overconvergent neighborhood of $\mathcal{X}^{\mathrm{ord}}$ (resp. $\mathcal{X}_\Gamma^{\mathrm{mult}}(v)$). When v is sufficiently small, the construction of the canonical subgroup by Katz and Lubin [Kat73] yields to a section of the projection π

$$s^{\mathrm{can}}\colon \mathcal{X}(v) \to \mathcal{X}_\Gamma^{\mathrm{mult}}(v). \tag{2.1}$$

Denote by $\mathcal{D}_\Gamma^{\mathrm{can}} \subset \mathcal{D}_\Gamma$ the image of the divisor $\mathcal{D} \subset \mathcal{X}^{\mathrm{ord}}$ under the canonical section s^{can}.

The weight space

Let $\mathbf{q} = p$ if p is odd, and let $\mathbf{q} = 4$ if $p = 2$. Let F be a finite extension of $\mathbb{Q}_p$ containing the values of all characters of $(\mathbb{Z}/N\mathbb{Z})^\times$; let $\mathcal{O}_F$ be the ring of integers of F. Denote $\mathbb{Z}_{p,N}^\times = (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times$ and consider the completed group ring

$$\mathcal{O}_F[\![\mathbb{Z}_{p,N}^\times]\!] = \varprojlim_n \mathcal{O}_F[(\mathbb{Z}/p^nN\mathbb{Z})^\times].$$

It comes equipped with a natural universal character $\mathbb{Z}_{p,N}^\times \to \mathcal{O}_F[\![\mathbb{Z}_{p,N}^\times]\!]^\times$ sending x to $[x]$. By a construction due to Berthelot, one can associate to $\mathcal{O}_F[\![\mathbb{Z}_{p,N}^\times]\!][\frac{1}{p}]$ a rigid analytic space $\mathcal{W}$ over F. The $\mathbb{C}_p$-valued points of the weight space $\mathcal{W}$ are in bijection with the continuous homomorphisms $\mathrm{Hom}(\mathbb{Z}_{p,N}^\times, \mathbb{C}_p^\times)$. We refer to these points as weight-characters. A weight-character is called *classical* if it corresponds to a morphism $\psi = (\varphi{\cdot}\chi_k)\colon \mathbb{Z}_{p,N}^\times \to \mathbb{C}_p^\times$ where φ is a finite order character and χ_k is defined by $\chi_k(x) = x^k$ for $k \in \mathbb{Z}_{>0}$; we say that ψ is of weight k and character φ. Classical weights are Zariski-dense in the weight space $\mathcal{W}$.

From the split exact sequence (1.13), there is a canonical isomorphism $\mathcal{O}_F[\![\mathbb{Z}_{p,N}^\times]\!] = \mathcal{O}_F[(\mathbb{Z}/\mathbf{q}N\mathbb{Z})^\times] \otimes_{\mathcal{O}_F} \mathcal{O}_F[\![1+\mathbf{q}\mathbb{Z}_p]\!]$. Since $1+\mathbf{q}\mathbb{Z}_p$ is topologically generated by $(1+\mathbf{q})$ there is an isomorphism

$$\mathcal{O}_F[\![1+\mathbf{q}\mathbb{Z}_p]\!] \to \mathcal{O}_F[\![X]\!], \qquad\qquad [1+\mathbf{q}] \mapsto 1+X.$$

Thus, there is an isomorphism of rigid analytic spaces

$$\mathcal{W}_F = \bigsqcup_{\psi \in \mathrm{Hom}((\mathbb{Z}/\mathbf{q}N\mathbb{Z})^\times, F^\times)} \mathcal{W}_\psi$$

where $\mathcal{W}_\psi$ is the rigid analytic space associated to $\mathcal{O}_F[\![X]\!][\frac{1}{p}]$; in other words, $\mathcal{W}$ is the disjoint union of open unit discs indexed by the characters of $(\mathbb{Z}/\mathbf{q}N\mathbb{Z})^\times$.

The eigencurve and the full eigencurve

For the tame level $N = 1$, and a prime $p > 2$, Coleman and Mazur constructed a rigid analytic space, the eigencurve, whose $\mathbb{C}_p$-valued points are in bijection with overconvergent p-adic eigenforms of tame level 1 and finite slope [CM96]. This construction has since been extended by Buzzard [Buz07] to a much broader class of settings and applies in particular to arbitrary prime p and tame level N with $(N, p) = 1$.

Let $\mathcal{C}$ and $\mathcal{C}_{\mathrm{full}}$ be the reduced eigencurve and the full eigencurve of level N respectively; they are rigid analytic spaces over $\mathbb{Q}_p$, equipped with locally-in-the-domain-finite-flat maps

$$w\colon \mathcal{C} \to \mathcal{W} \qquad \text{and} \qquad w_{\mathrm{full}}\colon \mathcal{C}_{\mathrm{full}} \to \mathcal{W}.$$

The reduced eigencurve is constructed via the spectral theory of the Hecke algebra generated by the diamond operators, U_p and T_ℓ for primes $\ell \nmid Np$. In addition to these operators, $\mathcal{C}_{\text{full}}$ encodes Hecke eigenvalues for operators U_ℓ with $\ell \mid N$. Note that the eigencurve $\mathcal{C}$ is a reduced rigid analytic space (see, for example [Che05, Prop. 3.9]), while the full eigencurve $\mathcal{C}_{\text{full}}$ is not reduced in general.

Let $\mathcal{H}$ be the ring of polynomials over $\mathcal{O}_F[\![\mathbb{Z}_{p,N}^\times]\!]$ generated by formal variables corresponding to the diamond operators $\langle d \rangle$ for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, the Hecke operators U_p and T_ℓ for ℓ prime not dividing Np; let $\mathcal{H}_{\text{full}} = \mathcal{H}[\{U_\ell \mid \ell \text{ prime } | N\}]$. By construction of the eigencurve, there are $\mathcal{O}_F[\![\mathbb{Z}_{p,N}^\times]\!]$-linear homomorphisms

$$\mathcal{H} \to \mathcal{O}_{\mathcal{C}}(\mathcal{C}) \qquad \text{and} \qquad \mathcal{H}_{\text{full}} \to \mathcal{O}_{\mathcal{C}_{\text{full}}}(\mathcal{C}_{\text{full}}),$$

and the image of the T_ℓ and U_q under these maps are rigid analytic functions bounded by 1. There is a commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{\text{full}} \arrow[rr, "\pi"] \arrow[dr, "w_{\text{full}}"'] & & \mathcal{C} \arrow[dl, "w"] \\
& \mathcal{W} &
\end{tikzcd}$$

that we now describe on $\mathbb{C}_p$-valued points. The $\mathbb{C}_p$-valued points of $\mathcal{C}$ and $\mathcal{C}_{\text{full}}$ are in one-to-one correspondence with systems of Hecke eigenvalues of overconvergent modular forms with finite slope. For $\kappa \in \mathcal{W}(\mathbb{C}_p)$, the points of the fiber $w^{-1}(\kappa)$ (resp. $w_{\text{full}}^{-1}(\kappa)$) arise as follows. Let $g \in M_\kappa^\dagger(N, \mathbb{C}_p)$ be a normalized overconvergent eigenform with q-expansion $g(q) = \sum_{n=0}^\infty a_n(g) q^n$ with finite slope, *i.e.* $a_p(g) \neq 0$. Then g determines a point $y_g \in \mathcal{C}$ (resp. $x_g \in \mathcal{C}_{\text{full}}$), corresponding to the ring homomorphism $\lambda_g \colon \mathcal{H} \to \mathbb{C}_p$ (respectively $\lambda_{g,\text{full}} \colon \mathcal{H}_{\text{full}} \to \mathbb{C}_p$) sending

$$T_\ell \mapsto a_\ell(g), \ell \nmid Np \qquad \text{and} \qquad U_q \mapsto a_q(g), q = p \text{ (resp. } q | Np)$$

such that the kernel of $\mathcal{O}_F[\![\mathbb{Z}_{p,N}^\times]\!] \to \mathcal{H} \xrightarrow{\lambda_g} \mathbb{C}_p$ (resp. $\mathcal{O}_F[\![\mathbb{Z}_{p,N}^\times]\!] \to \mathcal{H} \xrightarrow{\lambda_{g,\mathrm{full}}} \mathbb{C}_p$) is the ideal corresponding to the weight character κ. Note that the eigenform g is completely determined by the corresponding point $x_g \in \mathcal{C}_{\mathrm{full}}$, but its image in $y_g \in \mathcal{C}$ only accounts for the Hecke eigenvalues for the good Hecke operators and U_p. The map π is locally finite and surjective.

The *ordinary locus* $\mathcal{C}^{\mathrm{ord}}$ of $\mathcal{C}$ (resp. $\mathcal{C}^{\mathrm{ord}}_{\mathrm{full}}$ of $\mathcal{C}_{\mathrm{full}}$) is the admissible open and closed subspace characterized by the condition $|U_p| = 1$. By construction of the eigencurve, the ordinary locus of the eigencurve is isomorphic to the rigid analytic space attached to the maximal spectrum of the generic fiber of the p-ordinary Hecke algebra of tame level N.

The *cuspidal eigencurve* $\mathcal{C}^{\mathrm{cusp}} \hookrightarrow \mathcal{C}$ (resp. $\mathcal{C}^{\mathrm{cusp}}_{\mathrm{full}} \hookrightarrow \mathcal{C}_{\mathrm{full}}$) is a Zariski closed subspace of $\mathcal{C}$; it is also equidimensional of dimension one [Buz07, Lemma 5.8]. We have

$$\mathcal{C}^{\mathrm{ord}}_{\mathrm{full}} = \pi^{-1}(\mathcal{C}^{\mathrm{ord}}) \qquad \text{and} \qquad \mathcal{C}^{\mathrm{cusp}}_{\mathrm{full}} = \pi^{-1}(\mathcal{C}^{\mathrm{cusp}}).$$

Remark 2.1.1. The comparison between the Hida Hecke algebra and the ring of analytic functions over $\mathcal{C}^{\mathrm{ord}}$ relies on the fact that ordinary modular forms are automatically overconvergent (see [Pil13, Prop. 6.2]). In particular, this applies to Eisenstein families.

A family of pseudorepresentations

For any classical point $y_g \in \mathcal{C}^{\mathrm{cusp}}(\bar{\mathbb{Q}}_p)$ corresponding to a normalized eigenform g, there exists a continuous absolutely irreducible odd representation $\rho_g \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Q}_p)$ unramified outside Np such that $\mathrm{Tr}(\rho_g(\mathrm{Frob}_\ell))$ is equal to the T_ℓ-eigenvalue of g for every $\ell \nmid Np$. A priori there is no global representation with values in $\mathrm{GL}_2(\mathcal{O}_{\mathcal{C}}(\mathcal{C}))$ that p-adically interpolates the representations ρ_g (although one can be constructed over the rigid analytic functions on the normalization of $\mathcal{C}$ (see [CM96, Theorem 5.1.2]).

However, there is an odd two-dimensional pseudorepresentation

$$\mathbf{T}_{\mathcal{O}_{\mathcal{C}}(\mathcal{C})}\colon G_{\mathbb{Q}} \to \mathcal{O}_{\mathcal{C}}(\mathcal{C})$$

sending Frob_ℓ to T_ℓ for every $\ell \nmid Np$ [Che14a]. By Chebotarev density theorem, it follows that $\mathbf{T}_{\mathcal{O}_{\mathcal{C}}(\mathcal{C})} \otimes k(x_g) = \mathrm{Tr}(\rho_g)$. For every open affinoid subdomain $\mathrm{Spm}(A) \subset \mathcal{C}$, we obtain a pseudorepresentations $\mathbf{T}_A\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(A)$

2.2 p-adic families of Eisenstein series

Let ψ, φ be two Dirichlet characters of coprime conductors N_ψ and N_φ and let $k \in \mathbb{Z}_{\geq 3}$ such that $\psi(-1)\phi(-1) = (-1)^k$. Consider $E_k(\psi, \varphi)$ the Eisenstein series of weight k, level $N = N_\varphi N_\psi$ and character $\varphi\psi$ with q-expansion

$$E_k(\psi,\varphi)(q) = c(\psi)\frac{L(\varphi, 1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k,n}(\psi,\varphi)q^n,$$

where $\sigma_{k,n}(\psi, \varphi) = \sum_{d|n} \psi\left(\frac{n}{d}\right)\varphi(d)d^{k-1}$ and

$$c(\psi) = \begin{cases} 1, \text{if } \psi = \mathbb{1} \\ 0, \ \text{otherwise} \end{cases}.$$

Fix a prime $p \nmid N$. The p-th Hecke polynomial factors as

$$X^2 - a_{k,p}(\varphi,\psi)X + \varphi(p)\psi(p)p^{k-1} = (X - \psi(p))(X - \varphi(p)p^{k-1}).$$

Denote

$$\alpha_p = \psi(p) \qquad \text{and} \qquad \beta_p = \varphi(p)p^{k-1}.$$

The Eisenstein series $E_k(\varphi,\psi)$ is an eigenform for the (classical) Hecke algebra of level N, in particular for T_p. The p-stabilizations

$$E_k^{(p)}(\psi,\varphi)(q) = E_k(\psi,\varphi)(q) - \beta_p E_k(\psi,\varphi)(q^p) \tag{2.2}$$

$$E_k^{\mathrm{crit}}(\psi,\varphi)(q) = E_k(\psi,\varphi)(q) - \alpha_p E_k(\psi,\varphi)(q^p) \tag{2.3}$$

are eigenforms of level $\Gamma_1(N)\cap\Gamma_0(p)$ with U_p eigenvalues α_p and β_p respectively.

The stabilization $E_k^{\mathrm{crit}}(\psi,\varphi)$ is *critical*, in the sense that its U_p-eigenvalue has p-adic valuation $k-1$. By the work of Coleman [Col96], the eigencurve is known to be etale over the weight space for points corresponding to cuspforms of weight k and slope smaller than $k-1$ regular at p; moreover, those cuspforms are necessarily classical. On the other hand, many questions related to the smoothness of the eigencurve at points of critical slope remain open even for Eisenstein critical points. For $k \geq 2$ and Eisenstein series of character $\varphi = \psi = \mathbb{1}$, this question is addressed in [BC06].

On the other hand, the ordinary p-stabilizations $E_k^{(p)}(\psi,\varphi)$ can be p-adically interpolated as follows. Much like in the work of Serre [Ser73] for $\psi = \varphi = \mathbb{1}$, we have that for $k = 1 \bmod p-1$, the n-th coefficient of the q-expansion of $E_k^{(p)}(\psi,\varphi)$, given by

$$\sigma_{k,n}^{(p)}(\psi,\varphi) = \sum_{d|n, p\nmid d} \psi\left(\frac{n}{d}\right)\varphi(d)d^{k-1} = \sum_{d|n, p\nmid d} \psi\left(\frac{n}{d}\right)\varphi(d)\langle\!\langle d\rangle\!\rangle^{k-1}$$

extends to an analytic functions of $k \in \mathbb{Z}_p$. Moreover, the arguments of Serre in *loc.cit.* show that for $\psi = \mathbb{1}$, the constant term inherits the same analyticity property, giving rise to the Kubota-Leopoldt p-adic L-function $L_p(\varphi\omega_p, s)$ for $s \in \mathbb{Z}_p$ satisfying the relation

$$L_p(\varphi\omega_p, 1-k) = (1-\varphi(p)p^{k-1})L(\varphi, 1-k) \tag{2.4}$$

for every $k = 1 \bmod p-1$. There is a corresponding element $\zeta_\varphi \in \mathcal{O}_F[\![X]\!]$ satisfying

$$\zeta_\varphi((1+p)^{k-1}-1) = L_p(\varphi\omega_p, 1-k). \tag{2.5}$$

We denote $\mathcal{E}_{\psi,\varphi}$ the family of Eisenstein series with q-expansion

$$\mathcal{E}_{\psi,\varphi} = c(\psi)\frac{\zeta_\varphi}{2} + \sum_{d|n, p\nmid d} \psi\left(\frac{n}{d}\right)\varphi(d)[\langle\!\langle d\rangle\!\rangle] \in \mathcal{O}_F[\![X]\!][\![q]\!]$$

interpolating the eigenvalues of $E_k^{(p)}(\psi,\varphi)$ for $k = 1 \bmod (p-1)$. Let $\mathcal{W}_{\varphi\psi\omega_p}$ be the connected component of the weight space $\mathcal{W}$ for tame level N corresponding to the character $\varphi\psi\omega_p$. Then the systems of Hecke eigenvalues of $\mathcal{E}_{\psi,\varphi}$ define sections

$$\lambda_{\psi,\varphi} \colon \mathcal{W}_{\varphi\psi\omega_p} \to \mathcal{C} \qquad \text{and} \qquad \lambda_{\psi,\varphi,\mathrm{full}} \colon \mathcal{W}_{\varphi\psi\omega_p} \to \mathcal{C}_{\mathrm{full}}$$

of the projections w and w_{full}. Note that since the $\mathcal{C}$ and $\mathcal{C}^{\mathrm{full}}$ are the eigencurves of tame level $N = N_\psi N_\varphi$, π defines an isomorphism between $\lambda_{\psi,\varphi,\mathrm{full}}(\mathcal{W}_{\varphi\psi\omega_p})$ and $\lambda_{\psi,\varphi}(\mathcal{W}_{\varphi\psi\omega_p})$. With an abuse of notation, we will also refer to the irreducible components of the eigencurves $\mathcal{C}$ and $\mathcal{C}_{\mathrm{full}}$ given by the images of $\lambda_{\psi,\varphi}$ and $\lambda_{\psi,\varphi,\mathrm{full}}$ as the Eisenstein components $\mathcal{E}_{\psi,\varphi}$.

2.2.1 Cuspidality and overconvergence

Consider the classical weight 1 Eisenstein series of level N with q-expansion

$$E_1(\varphi,\psi)(q) = E_1(\psi,\varphi)(q) = c(\psi)\frac{L(\varphi,0)}{2} + c(\varphi)\frac{L(\psi,0)}{2} + \sum_{n=1}^{\infty}\sigma_{k,n}(\psi,\varphi)q^n.$$

The p-stabilizations

$$E_1^{(p)}(\psi,\varphi)(q) = E_1(\psi,\varphi)(q) - \varphi(p)E_1(\psi,\varphi)(q^p), \tag{2.6}$$

$$E_1^{(p)}(\varphi,\psi)(q) = E_1(\varphi,\psi)(q) - \psi(p)E_1(\varphi,\psi)(q^p), \tag{2.7}$$

of U_p-eigenvalue $\psi(p)$ and $\varphi(p)$ are the weight one specializations of the $\mathcal{O}_F[\![X]\!]$-adic families $\mathcal{E}_{\psi,\varphi}$ and $\mathcal{E}_{\varphi,\psi}$ respectively. In particular, the irreducible components of the eigencurve $\mathcal{C}$ (resp. $\mathcal{C}_{\text{full}}$) corresponding to $\mathcal{E}_{\psi,\varphi}$ and $\mathcal{E}_{\varphi,\psi}$ intersect at weight 1 precisely when $\psi(p) = \varphi(p)$.

Denote $f = E_1^{(p)}(\psi,\varphi)$ and let $x_f \in \mathcal{C}_{\text{full}}(\bar{\mathbb{Q}}_p)$ and $y_f = \pi(x_f) \in \mathcal{C}(\bar{\mathbb{Q}}_p)$ be the points corresponding to its systems of Hecke eigenvalues. The constant term of the q-expansion of the $\mathcal{O}_F[\![X]\!]$-adic family $\mathcal{E}_{\psi,\varphi}$ at the cusp ∞ is identically zero if $\psi \neq \mathbb{1}$. If ψ is trivial, by the interpolation property (2.4), the constant term of the q-expansion at infinity is $L_p(\varphi\omega_p, 0)$, which has a trivial zero whenever $\varphi(p) = 1$. As a classical form of level $\Gamma_1(N) \cap \Gamma_0(p)$, f is not cuspidal, because the constant term of its q-expansion does not vanish at all cusps. We have, however, the following result [DLR15b, Prop. 1.3].

Proposition 2.2.1. *The constant term of the q-expansion vanishes at all cusps in the $\mathcal{D}_\Gamma^{\text{can}} = \Gamma_0(p)\infty$ if and only if $\varphi(p) = \psi(p)$.*

Following [CM96, Sec. 3.6], we say that a classical form $g \in M_k(\Gamma, \mathbb{C}_p)$ is *cuspidal-overconvergent* if the constant term of its q-expansion vanishes at all cusps in $\mathcal{D}_\Gamma^{\text{can}}$. Given g a cuspidal-overconvergent form, the pullback of g through s^{can} is cuspidal when viewed as a p-adic modular form of tame level N. Moreover, if g is a classical eigenform of level Γ, the pullback of g to $\mathcal{X}(v)$ for some $v > 0$ is also an eigenform as an overconvergent p-adic modular form.

Corollary 2.2.2. *The following are equivalent:*

(i) $\varphi(p) = \psi(p)$*;*

(ii) f is cuspidal-overconvergent;

(iii) The point $x_f \in \mathcal{C}_{\mathrm{full}}$ (resp. $y_f \in \mathcal{C}$) belongs to the intersection of the Eisenstein components $\mathcal{E}_{\psi,\varphi}$ and $\mathcal{E}_{\varphi,\psi}$.

We say that f is *irregular* if the assumptions of the previous corollary are satisfied, regular otherwise. If f is regular, it is not cuspidal and belongs to a unique Eisenstein component $\mathcal{E}_{\psi,\varphi}$ of $\mathcal{C}$ (resp. $\mathcal{C}_{\mathrm{full}}$). Since the component is the image of $\lambda_{\psi,\varphi}$ (resp. $\lambda_{\psi,\varphi,\mathrm{full}}$), the projection w (resp. w_{full}) is etale at y_f (resp. x_f).

Thus, we focus our attention on the irregular case. For simplicity of notation, assume that $\psi = \mathbb{1}$ and $\varphi = \phi$, and suppose $\phi(p) = 1$. We denote

$$\mathcal{T} = (\mathcal{O}_{\mathcal{C},y_f} \otimes_F \bar{\mathbb{Q}}_p)^{\wedge} \qquad \mathcal{T}^{\mathrm{cusp}} = (\mathcal{O}_{\mathcal{C}^{\mathrm{cusp}},y_f} \otimes_F \bar{\mathbb{Q}}_p)^{\wedge}$$

$$\mathcal{T}_{\mathrm{full}} = (\mathcal{O}_{\mathcal{C}_{\mathrm{full}},x_f} \otimes_F \bar{\mathbb{Q}}_p)^{\wedge} \qquad \mathcal{T}^{\mathrm{cusp}}_{\mathrm{full}} = (\mathcal{O}_{\mathcal{C}^{\mathrm{cusp}}_{\mathrm{full}},x_f} \otimes_F \bar{\mathbb{Q}}_p)^{\wedge}$$

where $(-)^{\wedge}$ denotes the completion with respect to ideal corresponding to the system of eigenvalues of f. Note that all these rings belong to the category $\mathfrak{C}_{\bar{\mathbb{Q}}_p}$. Let $w_f = w(y_f) \in \mathcal{W}(\bar{\mathbb{Q}}_p)$ be the weight-character of weight one and nebentypus ϕ, belonging to the component $\mathcal{W}_{\omega_p\phi}$ of the weight space. The completion of $(\mathcal{O}_{\mathcal{W},w_f} \otimes_F \bar{\mathbb{Q}}_p)^{\wedge}$ with respect to its maximal ideal is canonically isomorphic to Λ and the rings $\mathcal{T}$, $\mathcal{T}^{\mathrm{cusp}}$, $\mathcal{T}_{\mathrm{full}}$, $\mathcal{T}^{\mathrm{cusp}}_{\mathrm{full}}$ are thus endowed with a Λ-algebra structure.

Remark 2.2.3. One can assume without loss of generality that ψ is trivial because, for every representation ρ whose deformation functor is representable, the deformation rings for ρ and $\rho \otimes \psi$ are canonically isomorphic [Maz89, Sec.1.3], so all the Galois theoretic arguments in the previous chapter carry over unchanged.

2.3 Cuspidal Modularity Theorem

2.3.1 Construction of a Galois representation over the cuspidal Hecke algebra

The aim of this section is to construct a surjective homomorphism $\mathcal{R}^{\mathrm{cusp}} \to \mathfrak{T}^{\mathrm{cusp}}$. We will later show that this map is an isomorphism. Following the work of Hida, we can construct an ordinary Galois representation V_K over the total fraction field K of $\mathfrak{T}^{\mathrm{cusp}}$. We want to show that there exist two $G_{\mathbb{Q}}$-stable $\mathfrak{T}^{\mathrm{cusp}}$-lattices in V_K with residual representations isomorphic to ρ and ρ'. The existence of such lattices is related to the fact that V_K is an irreducible but residually reducible representation. This result can be viewed as a generalization of Ribet's lemma. As in [BDP, Prop.3.3], we follow the version [BC06] of a method due to Mazur and Wiles [MW84].

Proposition 2.3.1. *There exists a pair of odd absolutely irreducible Galois representations* $\rho_{\mathfrak{T}^{\mathrm{cusp}}}, \rho'_{\mathfrak{T}^{\mathrm{cusp}}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathfrak{T}^{\mathrm{cusp}})$ *such that*

(i) $\rho_{\mathfrak{T}^{\mathrm{cusp}}} \bmod \mathfrak{m}_{\mathfrak{T}^{\mathrm{cusp}}} = \rho$ *and* $\rho'_{\mathfrak{T}^{\mathrm{cusp}}} \bmod \mathfrak{m}_{\mathfrak{T}^{\mathrm{cusp}}} = \rho'$*;*

(ii) $\det \rho_{\mathfrak{T}^{\mathrm{cusp}}} = \det \rho'_{\mathfrak{T}^{\mathrm{cusp}}} = \phi\kappa_{\Lambda}$*;*

(iii) For every prime $\ell \nmid Np$*, the representations* $\rho_{\mathfrak{T}^{\mathrm{cusp}}}$ *and* $\rho'_{\mathfrak{T}^{\mathrm{cusp}}}$ *are unramified at* ℓ *and*

$$\mathrm{Tr}(\rho_{\mathfrak{T}^{\mathrm{cusp}}}(\mathrm{Frob}_\ell)) = \mathrm{Tr}(\rho'_{\mathfrak{T}^{\mathrm{cusp}}}(\mathrm{Frob}_\ell)) = T_\ell;$$

(iv) The representations $\rho_{\mathfrak{T}^{\mathrm{cusp}}}, \rho'_{\mathfrak{T}^{\mathrm{cusp}}}$ *are ordinary. The characters* $\vartheta_{\mathfrak{T}^{\mathrm{cusp}}}, \vartheta'_{\mathfrak{T}^{\mathrm{cusp}}}$ *acting on the unramified quotients satisfy* $\vartheta_{\mathfrak{T}^{\mathrm{cusp}}}(\mathrm{Frob}_p) = \vartheta'_{\mathfrak{T}^{\mathrm{cusp}}}(\mathrm{Frob}_p) = U_p$*.*

Let $K = \mathrm{Frac}(\mathfrak{T}^{\mathrm{cusp}})$ be the total fraction field of $\mathfrak{T}^{\mathrm{cusp}}$. Denote by $\mathfrak{p}_i$ the minimal prime ideals of $\mathfrak{T}^{\mathrm{cusp}}$ and let K_i be the fraction field of $\mathfrak{T}^{\mathrm{cusp}}/\mathfrak{p}_i$. We have $K = \bigoplus_i K_i$. Since $\mathfrak{T}^{\mathrm{cusp}}$ is reduced, the map $\mathfrak{T}^{\mathrm{cusp}} \to K$ is injective. It is well-known that a representation satisfying the above properties can be constructed over K [Hid93, Sec. 7.5].

Lemma 2.3.2. *There exists an odd absolutely irreducible two-dimensional representation $\rho_K = \bigoplus \rho_{K_i} \colon G_{\mathbb{Q}} \to \bigoplus \mathrm{GL}_2(K_i)$ over K satisfying properties (ii), (iii), (iv) of Proposition 2.3.1.*

We denote such representation by V_K. Let $\mathbf{T}_{\mathcal{T}^{\mathrm{cusp}}}$ be the trace of ρ_K; it coincides with the pseudorepresentation induced by the global pseudorepresentation over $\mathcal{C}$ composed with the morphism $\mathcal{O}_{\mathcal{C}}(\mathcal{C}) \to \mathcal{T}^{\mathrm{cusp}}$ (in fact, any odd two-dimensional absolutely irreducible pseudorepresentation over a field is the trace of a representation).

We fix the basis (v_-, v_+) of V_K in which the complex conjugation τ acts as $\left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Denote by $\rho = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ the matrix of of the representation in such basis. Then

$$a(\sigma) = \frac{1}{2}(\mathbf{T}_{\mathcal{T}^{\mathrm{cusp}}}(\sigma) - \mathbf{T}_{\mathcal{T}^{\mathrm{cusp}}}(\tau\sigma)) \qquad d(\sigma) = \frac{1}{2}(\mathbf{T}_{\mathcal{T}^{\mathrm{cusp}}}(\sigma) + \mathbf{T}_{\mathcal{T}^{\mathrm{cusp}}}(\tau\sigma)), \tag{2.8}$$

so the functions a, d are valued in $\mathcal{T}^{\mathrm{cusp}}$. Similarly, the function $X(\sigma, \sigma') = b(\sigma)c(\sigma')$ is also valued in $\mathcal{T}^{\mathrm{cusp}}$, since $X(\sigma, \sigma') = a(\sigma\sigma') - a(\sigma)a(\sigma')$. Denote by B and C the $\mathcal{T}^{\mathrm{cusp}}$-submodules of K generated by $b(\sigma)$ and $c(\sigma)$ for $\sigma \in G_{\mathbb{Q}}$ respectively.

Let $J \subset \mathfrak{m}_{\mathcal{T}^{\mathrm{cusp}}}$ be an ideal of $\mathcal{T}^{\mathrm{cusp}}$ such that the pseudorepresentation $\mathbf{T}_{\mathcal{T}^{\mathrm{cusp}}} \bmod J$ is the sum of two characters. Since $\mathbf{T}_{\mathcal{T}^{\mathrm{cusp}}}$ is odd, the two characters must be of different signature, denoted by $\psi_\pm$ such that $\psi_\pm(\tau) = \pm 1$. Then $\mathbf{T}_{\mathcal{T}^{\mathrm{cusp}}} \bmod J$ satisfies the relations

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \psi_+(\sigma) \\ \psi_-(\sigma) \end{bmatrix} = \begin{bmatrix} \mathbf{T}_{\mathcal{T}^{\mathrm{cusp}}}(\sigma) \\ \mathbf{T}_{\mathcal{T}^{\mathrm{cusp}}}(\tau\sigma) \end{bmatrix}.$$

Since $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is invertible in $\mathrm{GL}_2(\mathcal{T}^{\mathrm{cusp}}/J)$, the characters $\psi_\pm$ are uniquely determined by these relations, hence by (2.8) that

$$\psi_- = a \bmod J \quad \text{and} \quad \psi_+ = d \bmod J$$

Moreover, since a mod J is a character, it follows that $X(\sigma,\sigma')\in J$ for every $\sigma,\sigma'\in G_{\mathbb{Q}}$, so $BC\subset J$. Let R be the $\mathfrak{T}^{\mathrm{cusp}}$-subalgebra of $M_2(K)$ generated of elements $\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]$ such that $a,d\in\mathfrak{T}^{\mathrm{cusp}}$, $b\in B$ and $c\in C$. For any $\mathfrak{T}^{\mathrm{cusp}}$-module M and $m\in M$, denote by $\overline{m}$ the image of m in M/JM. It follows from the above discussion that the maps $r_B,r_C\colon G_{\mathbb{Q}}\to(R/J)^{\times}$ defined as

$$r_B(\sigma)=\begin{bmatrix}\bar a(\sigma) & \bar b(\sigma)\\ 0 & \bar d(\sigma)\end{bmatrix}\qquad\qquad r_C(\sigma)=\begin{bmatrix}\bar a(\sigma) & 0\\ \bar c(\sigma) & \bar d(\sigma)\end{bmatrix}\tag{2.9}$$

are group homomorphisms. This allows us to define maps

$$j_B\colon \mathrm{Hom}_{\mathfrak{T}^{\mathrm{cusp}}}(B/JB,\mathfrak{T}^{\mathrm{cusp}}/J)\to \mathrm{Z}^1(\mathbb{Q},\psi_-\psi_+^{-1}),\quad j_B(f)=\psi_+^{-1}\cdot(f\circ\bar b),$$
$$j_C\colon \mathrm{Hom}_{\mathfrak{T}^{\mathrm{cusp}}}(C/JC,\mathfrak{T}^{\mathrm{cusp}}/J)\to \mathrm{Z}^1(\mathbb{Q},\psi_+\psi_-^{-1}),\quad j_C(f)=\psi_-^{-1}\cdot(f\circ\bar c).$$

The maps j_B,j_C are injective because by definition B/JB and C/JC are generated by the image of $G_{\mathbb{Q}}$ over $\mathfrak{T}^{\mathrm{cusp}}/J$. In fact a stronger statement is true. Let $[j_B]$ and $[j_C]$ images of j_B and j_C in $\mathrm{H}^1(\mathbb{Q},\psi_-\psi_+^{-1})$ and $\mathrm{H}^1(\mathbb{Q},\psi_+\psi_-^{-1})$ respectively.

Lemma 2.3.3. *The maps $[j_B]$ and $[j_C]$ are injective.*

Proof. Let f be in the kernel of $[j_B]$. This means that $j_B(f)$ is a coboundary for the character $\psi_-\psi_+^{-1}$, *i.e.* $j_B(f)=\lambda(1-\psi_-\psi_+^{-1})$ for some $\lambda\in\mathfrak{T}^{\mathrm{cusp}}/J$.

In particular, the restriction of $j_B(f)$ to the kernel of $\psi_-\psi_+^{-1}$ is zero. The kernel of $j_B(f)$ contains the commutator of $G_{\mathbb{Q}}$, and in particular $[\sigma,\tau]$ for every $\sigma\in G_{\mathbb{Q}}$. A direct calculation shows that

$$b([\sigma,\tau])=2\det\rho(\sigma)^{-1}a(\sigma)b(\sigma).$$

Since $2\det\rho(\sigma)^{-1}a(\sigma)\in(\mathfrak{T}^{\mathrm{cusp}})^{\times}$, and $f(\bar b([\sigma,\tau]))=0$, it follows that f is identically zero. □

Lemma 2.3.4. *The $\mathfrak{T}^{\mathrm{cusp}}$-modules B and C are free of rank one.*

Proof. Let $J = \mathfrak{m}_{\mathcal{T}^{\text{cusp}}}$. By Lemma [BC06, Sec. 2, Lemma 4] B and C are $\mathcal{T}^{\text{cusp}}$-modules of finite type, because the representations ρ_j are irreducible. By Lemma 2.3.3 there are

$$\mathrm{Hom}_{\mathcal{T}^{\text{cusp}}}(B/\mathfrak{m}_{\mathcal{T}^{\text{cusp}}}B, \bar{\mathbb{Q}}_p) \hookrightarrow \mathrm{H}^1(\mathbb{Q}, \phi)$$

Since the latter module is one-dimensional over $\bar{\mathbb{Q}}_p$, it follows that $B/\mathfrak{m}_{\mathcal{T}^{\text{cusp}}}B \simeq \bar{\mathbb{Q}}_p$. Thus, by Nakayama's Lemma, B is generated by one element. Denote by $b_0 = b(\sigma_0)$ the generator of B over $\mathcal{T}^{\text{cusp}}$ and consider the morphism of $\mathcal{T}^{\text{cusp}}$-modules $\mathcal{T}^{\text{cusp}} \to B$ sending 1 to b_0. The kernel of this map is the annihilator of B. By definition B is a submodule of $K = \bigoplus_i K_i$; denote by B_i the image of B in K_i. Let $t \in \mathrm{Ann}(B)$ be such that its image in K_i is non-zero. This implies that $B_i = 0$, which contradicts the fact that ρ_i is irreducible. Thus $\mathrm{Ann}(B) = 0$ and B is free of rank one over $\mathcal{T}^{\text{cusp}}$. The same argument applies to C. □

Proof of Proposition 2.3.1. Denote by $b_0 = b(\sigma_0')$ and $c(\sigma_0')$ the generators of B and C respectively. Consider the bases of K^2 given by $\mathcal{V}_B = (v_-, b_0 v_+)$ and $\mathcal{V}_C = (c_0 v_-, v_+)$. The representation ρ_K in these bases is given by

$$\begin{bmatrix} a & bb_0^{-1} \\ cb_0 & d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & bc_0 \\ cc_0^{-1} & d \end{bmatrix} \tag{2.10}$$

respectively. These representations take value in $\mathrm{GL}_2(\mathcal{T}^{\text{cusp}})$ because a, d, X are valued in $\mathcal{T}^{\text{cusp}}$ and b_0, c_0 are the generators of B and C respectively. Thus, the $\mathcal{T}^{\text{cusp}}$-lattices $M = \langle v_-, b_0 v_+\rangle$ and $M' = \langle c_0, v_+\rangle$ are $G_{\mathbb{Q}}$-stable. By construction the reduction modulo $\mathfrak{m}_{\mathcal{T}^{\text{cusp}}}$ of the representation $\left[\begin{smallmatrix} a & bb_0^{-1} \\ cb_0 & d \end{smallmatrix}\right]$ is upper triangular with non-zero upper right entry. In fact, by Lemma 2.3.3, $\overline{b}\overline{b}_0^{-1}$ defines a non-trivial extension of $\mathbb{1}$ by ϕ. Since $\mathrm{H}^1(\mathbb{Q}, \phi)$ is one-dimensional, $[\overline{b}\overline{b}_0^{-1}]$ is a non-scalar multiple of $[\eta]$. Both η and $\overline{b}\overline{b}_0^{-1}$ vanish when evaluated at τ, so $\overline{b}\overline{b}_0^{-1}$ is a scalar multiple of τ. Thus, up to multiplying $b_0 v_+$ by a unit of $\mathfrak{T}^{\mathrm{cusp}}$, we can assume that the reduction of the representation in the basis (v_-, b_0v_+) is equal to ρ. We denote such representation by $\rho_{\mathfrak{T}^{\mathrm{cusp}}}$. Similarly, up to replacing c_0v_- by a multiple in $\mathfrak{T}^{\mathrm{cusp}}$, we obtain a basis (c_0v_-, v_+) of M' assume that the reduction of $\begin{bmatrix} a & cc_0 \\ bc_0^{-1} & d \end{bmatrix}$ is ρ'. We denote this representation by $\rho'_{\mathfrak{T}^{\mathrm{cusp}}}$. This completes the proof of the first two claims.

It remains to show that $\rho_{\mathfrak{T}^{\mathrm{cusp}}}$ and $\rho'_{\mathfrak{T}^{\mathrm{cusp}}}$ are ordinary. We will show this for $\rho_{\mathfrak{T}^{\mathrm{cusp}}}$, adapting an argument in [BD16]. The representation V_K is ordinary, *i.e.* it admits a filtration of $K[G_{\mathbb{Q}_p}]$-modules

$$0 \to V_K^{\mathrm{sub}} \to V_K \to V_K^{\mathrm{quo}} \to 0,$$

where V_K^{sub} and V_K^{quo} are free rank one K-modules and the action of $G_{\mathbb{Q}_p}$ on the V_K^{quo} is given by the unramified character $\vartheta_{\mathfrak{T}^{\mathrm{cusp}}}$ by Lemma 2.3.2.

Denote by

$$M^{\mathrm{sub}} = V_K^{\mathrm{sub}} \cap M \qquad \text{and} \qquad M^{\mathrm{quo}} = \mathrm{Im}(\mathrm{M} \to \mathrm{V_K^{quo}}).$$

There is an exact sequence of $\mathfrak{T}^{\mathrm{cusp}}[G_{\mathbb{Q}_p}]$-modules

$$0 \to M^{\mathrm{sub}} \to M \to M^{\mathrm{quo}} \to 0 \tag{2.11}$$

and $G_{\mathbb{Q}_p}$ acts on M^{quo} through $\vartheta_{\mathfrak{T}^{\mathrm{cusp}}}$. Since $M^{\mathrm{sub}} \otimes K = V_K^{\mathrm{sub}}$ and $M^{\mathrm{quo}} \otimes K = V_K^{\mathrm{quo}}$, the modules M^{sub} and M^{quo} are generically of rank one; it suffices to show that they are free over $\mathfrak{T}^{\mathrm{cusp}}$. Tensoring with $\mathfrak{T}^{\mathrm{cusp}}/\mathfrak{m}_{\mathfrak{T}^{\mathrm{cusp}}}$ yields a surjective map $M/\mathfrak{m}_{\mathfrak{T}^{\mathrm{cusp}}}M \to M^{\mathrm{quo}}/\mathfrak{m}_{\mathfrak{T}^{\mathrm{cusp}}}M^{\mathrm{quo}}$. If $M^{\mathrm{quo}}/\mathfrak{m}_{\mathfrak{T}^{\mathrm{cusp}}}M^{\mathrm{quo}}$ is two-dimensional over $\bar{\mathbb{Q}}_p$, the above map is an isomorphism of $\bar{\mathbb{Q}}_p[G_{\mathbb{Q}_p}]$-module, contradicting the fact that the residual representation is isomorphic to ρ. Hence, by Nakayama's Lemma, the module M^{quo} is generated by one element, and since it's generically of rank one, it is free of rank one. Since the sequence of 2.11 is split as a sequence of $\mathcal{T}^{\mathrm{cusp}}$-modules, M^{sub} is also projective, hence free because $\mathcal{T}^{\mathrm{cusp}}$ is local. $\square$

We record for later use a refinement of the previous result. The representations $\rho_{\mathcal{T}^{\mathrm{cusp}}}$ and $\rho'_{\mathcal{T}^{\mathrm{cusp}}}$ can be extended to an affinoid neighborhood of f on the cuspidal eigencurve.

Proposition 2.3.5. *There exists an affinoid neighborhood* $\mathrm{Spm}A$ *of* f *in* $\mathcal{C}^{\mathrm{cusp}}$ *and a representation* $\rho_A\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(A)$ *such that*

$$\mathrm{Tr}(\rho_A) = \mathbf{T}_A.$$

Proof. Fix an affinoid neighborhood $\mathrm{Spm}(A)$ of the cuspidal eigencurve $\mathcal{C}^{\mathrm{cusp}}$ containing f and let L be the ring of fractions of A. Denote by $\mathfrak{q}_j$ the minimal prime ideals of A, and L_j be the fraction field of $A/\mathfrak{q}_j$, so that $L = \bigoplus_j L_j$ and $A \hookrightarrow L$ because A is reduced. For every j we obtain by composition with $A \to L_j$ a pseudorepresentation $\mathbf{T}_{L_j}$ which is odd and absolutely irreducible because the specialization of $\mathbf{T}_A$ (the pseudorepresentation obtained by restricting $\mathbf{T}_{\mathcal{O}_{\mathcal{C}}(\mathcal{C})}$ to $\mathrm{Spm}(A)$) at classical weights on $\mathrm{Spm}(A)$ is odd and absolutely irreducible. Since L_j is a field, there exists a representation $\rho_{L_j}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(L_j)$ such that $\mathrm{Tr}(\rho_{L_j}) = \mathbf{T}_{L_j}$. Denote $V_L = L^2$ with the action of $G_{\mathbb{Q}}$ given by ρ_L. Since ρ_L is odd, there exists a basis (v^-, v^+) of V_L given by eigenvectors for the action of τ. Let $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] : G_{\mathbb{Q}} \to \mathrm{GL}_2(L)$ be the matrix of the representation in the basis (v^-, v^+) and let B (resp. C) be the A-submodules of L generated by the $b(g)$ (resp. $c(g)$) for $g \in G_{\mathbb{Q}}$. The same argument as in the proof of Proposition 2.3.1 then shows that the localization of B (resp. C) at the maximal ideal corresponding to f) is free of rank one. Thus, again by the same argument, we can conclude that there exists a free submodule M_A in V_L of rank two stable under the action of $G_{\mathbb{Q}}$. $\square$

2.3.2 Modularity Theorem for the cuspidal Hecke algebra

From the previous theorem, we obtain a pair of ordinary lifts $(\rho_{\mathcal{T}^{\mathrm{cusp}}}, \rho'_{\mathcal{T}^{\mathrm{cusp}}})$ of (ρ, ρ') to $\mathcal{T}^{\mathrm{cusp}}$ that by construction have the same determinant, trace and unramified quotient, because they are given by different choices of $\mathcal{T}^{\mathrm{cusp}}$-lattices in the same V_K. The pair corresponds to a unique morphism

$$\varphi^{\mathrm{cusp}} \colon \mathcal{R}^{\mathrm{cusp}} \to \mathcal{T}^{\mathrm{cusp}}.$$

We show that φ^{cusp} is an isomorphism [BDP, Thm 3.4].

Theorem 2.3.6. *(i) The map φ^{cusp} is a Λ-linear isomorphism.*

(ii) $\mathcal{T}^{\mathrm{cusp}}$ is etale over Λ.

Proof. The map φ^{cusp} is Λ-linear because $\det \rho_{\mathcal{T}^{\mathrm{cusp}}} = \det \rho'_{\mathcal{T}^{\mathrm{cusp}}} = \phi\kappa_\Lambda$ is the character induced by the universal character of Λ through the structural morphism $\Lambda \to \mathcal{T}^{\mathrm{cusp}}$ by Proposition 2.3.1. Moreover,

$$\varphi^{\mathrm{cusp}}(\mathrm{Tr}\rho_{\mathcal{R}^{\mathrm{ord}}}(\mathrm{Frob}_\ell) \otimes 1) = T_\ell, \;\; \ell \nmid Np \qquad \text{and} \qquad \varphi^{\mathrm{cusp}}(\vartheta_{\mathcal{R}^{\mathrm{ord}}}(\mathrm{Frob}_p) \otimes 1) = U_p,$$

which shows that φ^{cusp} is surjective. Since $\mathcal{T}^{\mathrm{cusp}}$ has Krull dimension one, this implies that $\mathcal{R}^{\mathrm{cusp}}$ has Krull dimension greater than or equal to one. By Proposition 1.5.11, the tangent space of $\mathcal{R}^{\mathrm{cusp}}$ is one-dimensional, which implies that $\mathcal{R}^{\mathrm{cusp}}$ is isomorphic to the ring of power series in one variable over $\bar{\mathbb{Q}}_p$. Since φ^{cusp} is a closed immersion and $\mathcal{T}^{\mathrm{cusp}}$ has Krull dimension one, it follows that φ^{cusp} is an isomorphism. This concludes the proof of (i). By Proposition 1.5.11, the map $\Lambda \to \mathcal{T}^{\mathrm{cusp}}$ induces an isomorphism on the tangent spaces, which implies that it is an isomorphism since Λ and $\mathcal{T}^{\mathrm{cusp}}$ are both rings of power series in one variable over $\bar{\mathbb{Q}}_p$. Since φ^{cusp} is an isomorphism, this shows (ii). □

Corollary 2.3.7. *The eigencurve $\mathcal{C}$ has a unique cuspidal component passing through f. The projection of this component to the weight space is etale.*

Denote $\pi^{\mathrm{cusp}}\colon \mathcal{T}^{\mathrm{cusp}} \to \Lambda$ the inverse of the structural morphism $\Lambda \to \mathcal{T}^{\mathrm{cusp}}$.

Corollary 2.3.8. *The morphism π^{cusp} satisfies the relations*

$$\pi^{\mathrm{cusp}}(T_\ell) = 1 + \phi(\ell) + \tfrac{\log_p(\ell)}{\log_p(1+\mathbf{q})}\left(\tfrac{\mathcal{L}(\phi^{-1})}{\mathcal{L}(\phi^{-1})+\mathcal{L}(\phi)}\phi(\ell) + \tfrac{\mathcal{L}(\phi)}{\mathcal{L}(\phi^{-1})+\mathcal{L}(\phi)}\right) X \bmod X^2, \tag{2.12}$$

$$\pi^{\mathrm{cusp}}(U_p) = 1 - \tfrac{\mathcal{L}(\phi)\mathcal{L}(\phi^{-1})}{\log_p(1+\mathbf{q})(\mathcal{L}(\phi^{-1})+\mathcal{L}(\phi))} X \bmod X^2, \tag{2.13}$$

for every prime $\ell \nmid Np$.

Proof. The composition $\pi^{\mathrm{cusp}} \circ \varphi^{\mathrm{cusp}}\colon \mathcal{R}^{\mathrm{cusp}} \to \Lambda$ corresponds to a pair of representations $(\rho_\Lambda, \rho'_\Lambda)\colon G_{\mathbb{Q}} \to \Lambda$ with determinant $\det \rho_\Lambda = \det \rho'_\Lambda = \phi\kappa_\Lambda$, because $\pi^{\mathrm{cusp}} \circ \varphi^{\mathrm{cusp}}$ is Λ-linear. Let $(\rho_{\Lambda/(X^2)}, \rho'_{\Lambda/(X^2)})$ be the reduction of $(\rho_\Lambda, \rho'_\Lambda)$ modulo the ideal (X^2); after identifying $\Lambda/(X^2)$ with $\bar{\mathbb{Q}}_p[\varepsilon]$ via $X \mapsto \varepsilon$, this yields an element of the tangent space t^{cusp}. Since $\kappa_\Lambda = 1 + \frac{X}{\log_p(1+\mathbf{q})}\eta_{\mathbb{1}} \bmod X^2$ by Lemma 1.2.2, the pair $(\rho_{\Lambda/(X^2)}, \rho'_{\Lambda/(X^2)})$ can be described as in Proposition 1.5.11 with

$$\lambda = \tfrac{\mathcal{L}(\phi^{-1})}{(\mathcal{L}(\phi)+\mathcal{L}(\phi^{-1}))\log_p(1+\mathbf{q})} \qquad \text{and} \qquad \mu = \tfrac{\mathcal{L}(\phi)}{(\mathcal{L}(\phi)+\mathcal{L}(\phi^{-1}))\log_p(1+\mathbf{q})}. \tag{2.14}$$

It follows that

$$\mathrm{Tr}\rho_\Lambda(\mathrm{Frob}_\ell) = 1 + \phi(\ell) + \tfrac{\log_{\mathrm{p}}(\ell)}{\log_{\mathrm{p}}(1+\mathbf{q})}\left(\tfrac{\mathcal{L}(\phi^{-1})}{\mathcal{L}(\phi^{-1})+\mathcal{L}(\phi)}\phi(\ell) + \tfrac{\mathcal{L}(\phi)}{\mathcal{L}(\phi^{-1})+\mathcal{L}(\phi)}\right) \mathrm{X} \bmod X^2,$$

$$\vartheta_\Lambda(\mathrm{Frob}_p) = 1 - \tfrac{\mathcal{L}(\phi)\mathcal{L}(\phi^{-1})}{\log_p(1+\mathbf{q})(\mathcal{L}(\phi^{-1})+\mathcal{L}(\phi))} X \bmod X^2,$$

which proves the desired relations. □

Remark 2.3.9 (A residually semisimple lattice for the cuspidal representation)**.** The cuspidal deformation ring $\mathcal{R}^{\mathrm{cusp}}$ is a discrete valuation ring. It follows from Proposition

1.5.11 that the image of the reducibility ideal I_ρ^{red} in $\mathcal{R}^{\mathrm{cusp}}$ is contained in the square of the maximal ideal of $\mathcal{R}^{\mathrm{cusp}}$. Adapting the arguments in Proposition 2.3.1, one can choose a lattice for the cuspidal representation such that $B, C \subset (X)$. In particular, the corresponding residual representation is semisimple and the decomposition group at p acts trivially. Pick any line $L \subset \bar{\mathbb{Q}}_p^2$ which is *not* stable under the action of $G_{\mathbb{Q}}$ as in [CE05]. An alternative approach to constructing a deformation ring corresponding to $\mathcal{T}^{\mathrm{cusp}}$ would be to consider the functor classifying equivalence classes of pairs of (ρ_A, L_A) were ρ_A is an ordinary lift of $\mathbb{1} \oplus \phi$ and L_A is line lifting L such that L_A is the ordinary filtration. One can show that deformation functor is representable and its universal ring is isomorphic to $\mathcal{R}^{\mathrm{cusp}}$ by computing its tangent space.

2.4 Ordinary Modularity Theorem

Recall that $\mathcal{T}$ is the completed local ring of the eigencurve at the point corresponding to f. There is a unique cuspidal component of the eigencurve passing through f, corresponding to a minimal prime ideal $\mathfrak{p}^{\mathrm{cusp}}$. In addition to it, there are two minimal prime ideals corresponding to the Eisenstein components of the eigencurve passing through f, given by the system of eigenvalues of the Λ-adic forms $\mathcal{E}_{\mathbb{1},\phi}$ and $\mathcal{E}_{\phi,\mathbb{1}}$. Denote $\mathfrak{p}_{\mathbb{1},\phi}^{\mathrm{eis}}$ and $\mathfrak{p}_{\phi,\mathbb{1}}^{\mathrm{eis}}$ the kernels of morphisms $\pi_{\mathbb{1},\phi}^{\mathrm{eis}}, \pi_{\phi,\mathbb{1}}^{\mathrm{eis}} : \mathcal{T} \to \Lambda$ defined by

$$\pi_{\mathbb{1},\phi}^{\mathrm{eis}}(T_\ell) = 1 + \phi(\ell)[\langle\!\langle \ell \rangle\!\rangle], \quad \ell \nmid Np, \qquad \pi_{\mathbb{1},\phi}^{\mathrm{eis}}(U_p) = 1 \tag{2.15}$$

$$\pi_{\phi,\mathbb{1}}^{\mathrm{eis}}(T_\ell) = \phi(\ell) + [\langle\!\langle \ell \rangle\!\rangle], \quad \ell \nmid Np, \qquad \pi_{\phi,\mathbb{1}}^{\mathrm{eis}}(U_p) = 1. \tag{2.16}$$

Lemma 2.4.1. *The morphisms $\pi^{\mathrm{eis}}_{\mathbb{1},\phi}, \pi^{\mathrm{eis}}_{\phi,\mathbb{1}} : \mathfrak{T} \to \Lambda$ satisfy the relations*

$$\pi^{\mathrm{eis}}_{\mathbb{1},\phi}(T_\ell) = 1 + \phi(\ell) + \frac{\log_p(\ell)}{\log_p(1+\mathbf{q})}\phi(\ell)X \mod X^2,$$
$$\pi^{\mathrm{eis}}_{\phi,\mathbb{1}}(T_\ell) = 1 + \phi(\ell) + \frac{\log_p(\ell)}{\log_p(1+\mathbf{q})}X \mod X^2$$

for every prime $\ell \nmid Np$ and $\pi^{\mathrm{eis}}_{\mathbb{1},\phi}(U_p) = \pi^{\mathrm{eis}}_{\phi,\mathbb{1}}(U_p) = 1$.

Proof. This follows by formulas (2.15) and the fact that $\langle\!\langle \ell \rangle\!\rangle \in 1 + \mathbf{q}\mathbb{Z}_p$ is viewed as an element of $\Lambda^\times$ via the character $1 + \mathbf{q}\mathbb{Z}_p \to \Lambda^\times$ determined by $(1+\mathbf{q}) \mapsto (1+X)$. $\square$

Denote $\mathfrak{T}^{\mathrm{eis}}_{\mathbb{1},\phi} = \mathfrak{T}/\mathfrak{p}^{\mathrm{eis}}_{\mathbb{1},\phi}$ and $\mathfrak{T}^{\mathrm{eis}}_{\phi,\mathbb{1}} = \mathfrak{T}/\mathfrak{p}^{\mathrm{eis}}_{\phi,\mathbb{1}}$. There is a Λ-algebra morphism

$$(\pi^{\mathrm{eis}}_{\mathbb{1},\phi}, \pi^{\mathrm{eis}}_{\phi,\mathbb{1}}, \pi^{\mathrm{cusp}}) \colon \mathfrak{T} \to \Lambda \times \Lambda \times \Lambda, \tag{2.17}$$

which is injective because $\mathfrak{T}$ is reduced, since $\mathfrak{C}$ is reduced. Thus, determining the structure of $\mathfrak{T}$ as a Λ-module amounts to computing the image of this morphism.

Lemma 2.4.2. *There exists an indecomposable reducible representation $\rho_{\mathfrak{T}^{\mathrm{eis}}_{\mathbb{1},\phi}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathfrak{T}^{\mathrm{eis}}_{\mathbb{1},\phi})$ (resp. $\rho'_{\mathfrak{T}^{\mathrm{eis}}_{\phi,\mathbb{1}}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathfrak{T}^{\mathrm{eis}}_{\phi,\mathbb{1}})$) unramified outside Np such that $\rho_{\mathfrak{T}^{\mathrm{eis}}_{\mathbb{1},\phi}} \bmod \mathfrak{m}_{\mathfrak{T}^{\mathrm{eis}}_{\mathbb{1},\phi}} = \rho$ (resp. ρ') and $\mathrm{Tr}\rho_{\mathfrak{T}^{\mathrm{eis}}_{\mathbb{1},\phi}}(\mathrm{Frob}_\ell) = T_\ell$ (resp. $\mathrm{Tr}\rho'_{\mathfrak{T}^{\mathrm{eis}}_{\phi,\mathbb{1}}}(\mathrm{Frob}_\ell) = T_\ell$) for every $\ell \nmid Np$. Moreover, the quotient is a one-dimensional representation of $G_{\mathbb{Q}}$ unramified at p, on which $G_{\mathbb{Q}}$ acts through a character satisfying $\vartheta_{\mathfrak{T}^{\mathrm{eis}}_{\mathbb{1},\phi}}(\mathrm{Frob}_p) = U_p$ (resp. $\vartheta'_{\mathfrak{T}^{\mathrm{eis}}_{\phi,\mathbb{1}}}(\mathrm{Frob}_p) = U_p$).*

Proof. By Corollary 1.5.14, there is an isomorphism $\mathcal{R}^{\mathrm{eis}}_\rho \to \Lambda$, yielding a deformation of $\rho_\Lambda \colon G_{\mathbb{Q}} \to \mathcal{R}^{\mathrm{eis}}_\rho$ such that the semisimplification is $\rho^{\mathrm{ss}}_\Lambda = \mathbb{1} + \phi\kappa_\Lambda$. Composing this map with the structural morphism $\Lambda \to \mathfrak{T}^{\mathrm{eis}}_{\mathbb{1},\phi}$ and comparing with (2.15) we obtain the desired result. Similarly, we obtain a representation valued in $\mathfrak{T}^{\mathrm{eis}}_{\phi,\mathbb{1}}$ by composing the map $\mathcal{R}^{\mathrm{eis}}_{\rho'} \to \Lambda$ with the structural morphism $\Lambda \to \mathfrak{T}^{\mathrm{eis}}_{\phi,\mathbb{1}}$. $\square$

The lemma yields Λ-linear isomorphisms $\varphi_\rho^{\mathrm{eis}} : \mathcal{R}_\rho^{\mathrm{eis}} \to \mathcal{T}_{\mathbb{1},\phi}^{\mathrm{eis}}$ and $\varphi_{\rho'}^{\mathrm{eis}} : \mathcal{R}_{\rho'}^{\mathrm{eis}} \to \mathcal{T}_{\phi,\mathbb{1}}^{\mathrm{eis}}$. Denote $\mathcal{T}_\rho^{\mathrm{ord}}$ and $\mathcal{T}_{\rho'}^{\mathrm{ord}}$ the images of $\mathcal{T}$ in $\Lambda \times \Lambda$ through the maps $(\pi_{\mathbb{1},\phi}^{\mathrm{eis}}, \pi^{\mathrm{cusp}})$ and $(\pi_{\phi,\mathbb{1}}^{\mathrm{eis}}, \pi^{\mathrm{cusp}})$ respectively. We have the following result [BDP, Prop. 4.2]

Lemma 2.4.3. *(i) There are isomorphisms*

$$\mathcal{T}_\rho^{\mathrm{ord}} \simeq \mathcal{T}_{\mathbb{1},\phi}^{\mathrm{eis}} \times_{\bar{\mathbb{Q}}_p} \mathcal{T}^{\mathrm{cusp}} \simeq \Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda \qquad \textit{and} \qquad \mathcal{T}_{\rho'}^{\mathrm{ord}} \simeq \mathcal{T}_{\phi,\mathbb{1}}^{\mathrm{eis}} \times_{\bar{\mathbb{Q}}_p} \mathcal{T}^{\mathrm{cusp}} \simeq \Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda.$$

(ii) There are surjective Λ-algebras morphisms $\varphi_\rho^{\mathrm{ord}} : \mathcal{R}_\rho^{\mathrm{ord}} \to \mathcal{T}_\rho^{\mathrm{ord}}$ and $\varphi_{\rho'}^{\mathrm{ord}} : \mathcal{R}_{\rho'}^{\mathrm{ord}} \to \mathcal{T}_{\rho'}^{\mathrm{ord}}$.

Proof. (i) Since $\mathcal{T}$ is a local ring with maximal ideal defined as the kernel of the morphism $\mathcal{T} \to \bar{\mathbb{Q}}_p$ sending T_ℓ to $a_\ell(f)$, every prime ideal is contained in the kernel of this map, thus $\mathcal{T}_\rho^{\mathrm{ord}} \subset \Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda$. By Corollary 2.3.8 and Lemma 2.4.1, we can find a pair (a,b) in the image of $\mathcal{T}_\rho^{\mathrm{ord}}$ in $\Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda$ such that $a \neq b \bmod (X^2)$. Since $a - b \in (X) \setminus (X^2)$, it is easy to see that $(0, X) = \left(\frac{b-a}{X}\right)^{-1}((a,b) - (a,a))$ is in the image of $\mathcal{T}_\rho^{\mathrm{ord}}$ and thus to conclude that the image is isomorphic to $\Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda$.

(ii) We have isomorphisms $\mathcal{R}_\rho^{\mathrm{eis}} \to \mathcal{T}_{\mathbb{1},\phi}^{\mathrm{eis}}$ and $\mathcal{R}^{\mathrm{cusp}} \to \mathcal{T}^{\mathrm{cusp}}$. Composing with the projections of $\mathcal{R}_\rho^{\mathrm{ord}}$ onto $\mathcal{R}_\rho^{\mathrm{eis}}$ and $\mathcal{R}^{\mathrm{cusp}}$, we obtain a map $\mathcal{R}_\rho^{\mathrm{ord}} \to \mathcal{T}_{\mathbb{1},\phi}^{\mathrm{eis}} \times_{\bar{\mathbb{Q}}_p} \mathcal{T}^{\mathrm{cusp}}$, which is isomorphic to $\mathcal{T}_\rho^{\mathrm{ord}}$. Denote $\varphi_\rho^{\mathrm{ord}} : \mathcal{R}_\rho^{\mathrm{ord}} \to \mathcal{T}_\rho^{\mathrm{ord}}$. It is Λ-linear because all the morphisms above are and surjective because $\varphi_\rho^{\mathrm{ord}}(\mathrm{Frob}_\ell) = T_\ell$ and $\vartheta_\rho^{\mathrm{ord}}(\mathrm{Frob}_p) = U_p$.

□

Remark 2.4.4. It follows that the Λ-algebra $\mathcal{T}_\rho^{\mathrm{ord}}$ is isomorphic to $\Lambda[\![T]\!]/(T(X - T))$. In particular, it is a quotient of a ring of polynomials in one variable over Λ modulo the ideal generated by one element, so it is of complete intersection.

Remark 2.4.5. Although $\mathcal{R}_\rho^{\mathrm{ord}}$ admits quotients $\mathcal{R}_\rho^{\mathrm{eis}}$ and $\mathcal{R}^{\mathrm{cusp}}$, it is not *a priori* clear that the map $\mathcal{R}_\rho^{\mathrm{ord}} \to \mathcal{R}_\rho^{\mathrm{eis}} \times_{\bar{\mathbb{Q}}_p} \mathcal{R}^{\mathrm{cusp}}$ is injective, because the Krull dimension of the local ring $\mathcal{R}_\rho^{\mathrm{ord}}$ might be two. Even if the Krull dimension is one, $\mathcal{R}_\rho^{\mathrm{ord}}$ might have more than one irreducible component. We show that this situation does not arise and that $\mathcal{R}_\rho^{\mathrm{ord}} \to \mathcal{R}_\rho^{\mathrm{eis}} \times_{\bar{\mathbb{Q}}_p} \mathcal{R}^{\mathrm{cusp}}$ is an isomorphism in Corollary 2.4.10.

We wish to prove that the map $\varphi_\rho^{\mathrm{ord}}$ is an isomorphism of complete intersection Λ-algebras. We invoke a version of Wiles' Numerical Criterion due to Lenstra, which states the following.

Theorem 2.4.6 (Wiles' Numerical Criterion)**.** *Let $\varphi \colon R \to T$ be a surjective homomorphism of local Λ-algebras. Suppose that T is finite and flat as Λ-module and let $\pi_T : T \to \Lambda$ be a Λ-algebra homomorphism. Let $\mathfrak{p}_T = \ker \pi_T$, $\mathfrak{p}_R = \ker(\pi_T \circ \varphi)$ and assume that $\eta_T = \pi_T(\mathrm{Ann}_{\mathrm{T}}(\mathfrak{p}_T)) \neq 0$. Then*

$$\mathrm{length}_\Lambda(\mathfrak{p}_R/\mathfrak{p}_R^2) \geqslant \mathrm{length}_\Lambda(\Lambda/\eta_T)$$

and the equality holds if and only if φ is an isomorphism of relative complete intersection rings over Λ.

We wish to apply the above criterion to the following setup. Let $R = \mathcal{R}_\rho^{\mathrm{ord}}$, $T = \mathcal{T}_\rho^{\mathrm{ord}}$ and $\varphi = \varphi_\rho^{\mathrm{ord}}$. Consider the diagram

$$\begin{tikzcd} \mathcal{R}_\rho^{\mathrm{ord}} \arrow[rr, "\varphi_\rho^{\mathrm{ord}}"] \arrow[dr, "\pi_{\mathbb{1},\phi}^{\mathrm{eis}} \circ \varphi_\rho^{\mathrm{ord}}"'] & & \mathcal{T}_\rho^{\mathrm{ord}} \arrow[dl, "\pi_{\mathbb{1},\phi}^{\mathrm{eis}}"] \\ & \Lambda & \end{tikzcd}$$

and let $\pi_T = \pi_{\mathbb{1},\phi}^{\mathrm{eis}}$. By Lemma 2.4.3, there is an isomorphism $(\pi_{\mathbb{1},\phi}^{\mathrm{eis}}, \pi^{\mathrm{cusp}}) \colon \mathcal{T}_\rho^{\mathrm{ord}} \to \Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda$, so $\mathrm{Ann}_{\mathcal{T}_\rho^{\mathrm{ord}}}(\ker \pi_{\mathbb{1},\phi}^{\mathrm{eis}}) = \ker \pi^{\mathrm{cusp}} = ((\mathrm{X}, 0))$ and $\eta_T = \pi_{\mathbb{1},\phi}^{\mathrm{eis}}(\ker \pi^{\mathrm{cusp}}) = (X)$. Thus, by Theorem 2.4.6, in order to show that $\varphi_\rho^{\mathrm{ord}}$ is an isomorphism, it suffices to show that $\mathrm{length}_\Lambda(\mathfrak{p}_{\mathrm{R}}/\mathfrak{p}_{\mathrm{R}}^2) = 1$. The morphism $\pi_{\mathbb{1},\phi}^{\mathrm{eis}} \circ \varphi_\rho^{\mathrm{ord}}$ factors through the quotient $\mathcal{R}_\rho^{\mathrm{eis}}$ of $\mathcal{R}_\rho^{\mathrm{ord}}$. Since $\varphi_\rho^{\mathrm{eis}} \colon \mathcal{R}_\rho^{\mathrm{eis}} \to \mathcal{T}_{\mathbb{1},\phi}^{\mathrm{eis}}$ is an isomorphisms, the ideal $\mathfrak{p}_R$ is equal to the kernel of the projection $\mathcal{R}_\rho^{\mathrm{ord}} \to \mathcal{R}_\rho^{\mathrm{eis}}$, which is by construction the ideal of $\mathcal{R}_\rho^{\mathrm{ord}}$

$$C_\rho^{\mathrm{ord}} = \langle c(\sigma) \mid \sigma \in G_{\mathbb{Q}} \rangle$$

where $\rho_{\mathcal{R}_\rho^{\mathrm{ord}}} = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is the universal ordinary representation in a basis $e_{1,\mathcal{R}_\rho^{\mathrm{ord}}}$, $e_{2,\mathcal{R}_\rho^{\mathrm{ord}}}$ such that $\langle e_{1,\mathcal{R}_\rho^{\mathrm{ord}}} \rangle$ is the ordinary filtration. We have the following result [BDP, Prop. 4.9].

Proposition 2.4.7. *$C_\rho^{\mathrm{ord}}/C_\rho^{\mathrm{ord},2}$ is a torsion Λ-module of length one.*

Proof. Since C_ρ^{ord} is a $\mathcal{R}_\rho^{\mathrm{ord}}$-module of finite type, $C_\rho^{\mathrm{ord}}/C_\rho^{\mathrm{ord},2}$ is a $\mathcal{R}_\rho^{\mathrm{ord}}/C_\rho^{\mathrm{ord}} \simeq \Lambda$-module of finite type as well. Thus, it suffices to show that for every $n \geq 1$

$$\dim_{\bar{\mathbb{Q}}_p} \mathrm{Hom}_\Lambda(C_\rho^{\mathrm{ord}}/C_\rho^{\mathrm{ord},2}, \Lambda/(X^n)) = 1.$$

Indeed, for $n = 1$, the statement implies that $\dim_{\bar{\mathbb{Q}}_p} C_\rho^{\mathrm{ord}} \otimes \mathcal{R}_\rho^{\mathrm{ord}}/\mathfrak{m}_{\mathcal{R}_\rho^{\mathrm{ord}}} = 1$ and hence by Nakayama's Lemma $C_\rho^{\mathrm{ord}}/C_\rho^{\mathrm{ord},2}$ is generated by one element over Λ. Then

$$\mathrm{Hom}_\Lambda(C_\rho^{\mathrm{ord}}/C_\rho^{\mathrm{ord},2}, \Lambda/(X^n)) \simeq \Lambda/(X^n, \mathrm{Ann}_\Lambda(\mathrm{C}_\rho^{\mathrm{ord}}/\mathrm{C}_\rho^{\mathrm{ord},2}))$$

and the claim follows. In the above basis, the representation induced by $\rho_{\mathcal{R}_\rho^{\mathrm{ord}}}$ on $\mathcal{R}_\rho^{\mathrm{ord}}/C_\rho^{\mathrm{ord}} \simeq \Lambda$ is $\rho_\Lambda = \left[\begin{smallmatrix} \phi\kappa_\Lambda & \eta_\Lambda \\ 0 & 1 \end{smallmatrix}\right]$ and the restriction of c to $G_{\mathbb{Q}_p}$ is identically zero. Thus, we obtain a Λ-linear morphism

$$j_{C_\rho^{\mathrm{ord}}} \colon \mathrm{Hom}_\Lambda(C_\rho^{\mathrm{ord}}/C_\rho^{\mathrm{ord},2}, \Lambda/(X^n)) \to \ker(\mathrm{Z}^1(\mathbb{Q}, (\phi\kappa_n)^{-1}) \to \mathrm{Z}^1(\mathbb{Q}_p, \kappa_n^{-1})).$$

The morphism $j_{C_\rho^{\mathrm{ord}}}$ is injective. Thus, the dimension over $\bar{\mathbb{Q}}_p$ of the right hand side gives an upper bound for the dimension of $\mathrm{Hom}_\Lambda(C_\rho^{\mathrm{ord}}/C_\rho^{\mathrm{ord},2}, \Lambda/(X^n))$.

Lemma 2.4.8. *For every $n \in \mathbb{Z}_{\geq 1}$, we have*

(i) $\mathrm{H}^1(G_{\mathbb{Q},S}, \phi^{\pm}\kappa_n^{-1}) = \Lambda/(X^n)$ *and* $\mathrm{H}^2(G_{\mathbb{Q},S}, \phi^{\pm}\kappa_n^{-1}) = 0$;

(ii) $\mathrm{H}^1(\mathbb{Q}_p, \kappa_n^{-1}) = \Lambda/(X^n) \oplus \bar{\mathbb{Q}}_p$ *and* $\mathrm{H}^2(\mathbb{Q}_p, \kappa_n^{-1}) = 0$;

(iii) the map $\mathrm{H}^1(G_{\mathbb{Q},S}, \phi^{\pm}\kappa_n^{-1}) \to \mathrm{H}^1(\mathbb{Q}_p, \kappa_n^{-1})$ *is injective.*

Proof. The proof of (i) is identical to the proof of Proposition 1.5.13.

For (ii), by local Tate duality, for every $m \in \mathbb{Z}_{\geq 1}$, $\mathrm{H}^2(\mathbb{Q}_p, \mathbb{Z}/p^m\mathbb{Z})$ is the $\mathbb{Q}/\mathbb{Z}$-dual to $\mathrm{H}^0(\mathbb{Q}_p, \mu_{p^m}) = 0$. Thus, $\mathrm{H}^2(\mathbb{Q}_p, \bar{\mathbb{Q}}_p) = 0$. There is an exact sequence of $\Lambda[G_{\mathbb{Q}_p}]$-modules

$$0 \to \kappa_{n-1}^{-1} \xrightarrow{\cdot X} \kappa_n^{-1} \to \bar{\mathbb{Q}}_p \to 0 \tag{2.18}$$

inducing a long exact sequence in cohomology from which one easily deduces by induction on n that $\mathrm{H}^2(\mathbb{Q}, \kappa_n^{-1}) = 0$ for every $n \in \mathbb{Z}_{\geq 1}$. From the exact sequence of $\Lambda[G_{\mathbb{Q}_p}]$-modules

$$0 \to \kappa_\Lambda^{-1} \xrightarrow{\cdot X^n} \kappa_\Lambda^{-1} \to \kappa_n^{-1} \to 0, \tag{2.19}$$

we obtain a long exact sequence in cohomology

$$\mathrm{H}^{i-1}(\mathbb{Q}_p, \kappa_n^{-1}) \to \mathrm{H}^i(\mathbb{Q}_p, \kappa_\Lambda^{-1}) \xrightarrow{\cdot X^n} \mathrm{H}^i(\mathbb{Q}_p, \kappa_\Lambda^{-1}) \to \mathrm{H}^i(\mathbb{Q}_p, \kappa_n^{-1}) \to \mathrm{H}^{i+1}(\mathbb{Q}_p, \kappa_\Lambda^{-1}) \tag{2.20}$$

for $i = 1, 2$. Since $\mathrm{H}^2(\mathbb{Q}_p, \bar{\mathbb{Q}}_p) = 0$, from the exact sequence above for $n = 0$ and $i = 2$ we get that $\mathrm{H}^2(\mathbb{Q}_p, \kappa_\Lambda^{-1}) \otimes_\Lambda \bar{\mathbb{Q}}_p \simeq \mathrm{H}^2(\mathbb{Q}_p, \bar{\mathbb{Q}}_p) = 0$. As a consequence, the exact sequence above for $i = 1$ implies that $\mathrm{H}^1(\mathbb{Q}_p, \kappa_\Lambda^{-1}) \otimes \Lambda/(X^n) \simeq \mathrm{H}^1(\mathbb{Q}_p, \kappa_n^{-1})$ for every $n \in \mathbb{Z}_{\geq 1}$. In particular, $\mathrm{H}^1(\mathbb{Q}_p, \kappa_\Lambda^{-1}) \otimes \bar{\mathbb{Q}}_p \simeq \mathrm{H}^1(\mathbb{Q}_p, \bar{\mathbb{Q}}_p)$, which has dimension two over $\bar{\mathbb{Q}}_p$, so $\mathrm{H}^1(\mathbb{Q}_p, \kappa_n^{-1})$ is generated by two elements over Λ for every n. From the short exact sequence (2.18), we obtain an exact sequence

$$0 \to \mathrm{H}^0(\mathbb{Q}_p, \bar{\mathbb{Q}}_p) \to \mathrm{H}^1(\mathbb{Q}_p, \kappa_{n-1}^{-1}) \xrightarrow{\cdot X} \mathrm{H}^1(\mathbb{Q}_p, \kappa_n^{-1}) \to \mathrm{H}^1(\mathbb{Q}_p, \bar{\mathbb{Q}}_p) \to 0.$$

for every $n \in \mathbb{Z}_{\geq 1}$, which implies that $\dim_{\bar{\mathbb{Q}}_p} \mathrm{H}^1(\mathbb{Q}_p, \kappa_{n-1}^{-1}) = n + 1$ for every n. Hence

$\mathrm{H}^1(\mathbb{Q}_p, \kappa_\Lambda^{-1}) \simeq \Lambda \oplus \bar{\mathbb{Q}}_p$ and the second statement follows.

It remains to show (iii). Denote by $\mathrm{res}_{p,n}\colon \mathrm{H}^1(G_{\mathbb{Q},S}, \phi^{\pm}\kappa_n^{-1}) \to \mathrm{H}^1(\mathbb{Q}_p, \kappa_n)$ the restriction map, for $n \in \mathbb{Z}_{\geq 1}$. The map $\mathrm{res}_{\mathrm{p},1}$ is injective. For $n \in \mathbb{Z}_{>1}$, the exact sequence (2.18) gives rise to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1(G_{\mathbb{Q},S}, \phi^{\pm}\kappa_{n-1}^{-1}) \arrow[r, "\cdot X"] \arrow[d, "\mathrm{res}_{\mathrm{p,n-1}}"] & \mathrm{H}^1(G_{\mathbb{Q},S}, \phi^{\pm}\kappa_n^{-1}) \arrow[r] \arrow[d, "\mathrm{res}_{\mathrm{p,n}}"] & \mathrm{H}^1(G_{\mathbb{Q},S}, \phi^{\pm}) \arrow[r] \arrow[d, "\mathrm{res}_{\mathrm{p},1}"] & 0 \\
\mathrm{H}^0(\mathbb{Q}_p, \bar{\mathbb{Q}}_p) \arrow[r, "\delta^0"] & \mathrm{H}^1(\mathbb{Q}_p, \kappa_{n-1}^{-1}) \arrow[r, "\cdot X"] & \mathrm{H}^1(\mathbb{Q}_p, \kappa_n^{-1}) \arrow[r] & \mathrm{H}^1(\mathbb{Q}_p, \bar{\mathbb{Q}}_p) \arrow[r] & 0
\end{tikzcd} \tag{2.21}$$

where the horizontal arrows are exact. For $n = 2$, the diagram above shows that it suffices to verify $\mathrm{Im}(\mathrm{res}_{\mathrm{p},1}) \cap \mathrm{Im}(\delta^0) = 0$ because $\mathrm{res}_{\mathrm{p},1}$ is injective. The image of $\mathrm{res}_{\mathrm{p},1}$ is the restriction of $\eta_{\phi^{\pm}}$ to $G_{\mathbb{Q}_p}$. For $\lambda \in \bar{\mathbb{Q}}_p = \mathrm{H}^0(\mathbb{Q}_p, \bar{\mathbb{Q}}_p)$ the map δ^0 is given by $\delta^0(\lambda) = \frac{\lambda}{X}(1-\kappa_\Lambda) = \frac{\lambda}{\log_p(1+\mathbf{q})}\eta_{\mathbb{1}}$ by Lemma 1.2.2; thus the image of δ^0 is the restriction of $\eta_{\mathbb{1}}$ to $G_{\mathbb{Q}_p}$. Since $(\eta_{\phi^{\pm}} - \eta_{\mathbb{1}})(\mathrm{Frob}_p) = \mathcal{L}(\phi^{\mp}) \neq 0$ by Proposition 1.1.8, it follows that $\mathrm{res}_{\mathrm{p},2}$ is injective. For $n \geq 2$ the claim follows from the diagram above by induction on n, since the injectivity of $\mathrm{res}_{\mathrm{p,n-1}}$ for $n > 2$ automatically implies that $\mathrm{Im}(\delta^0) \cap \mathrm{Im}(\mathrm{res}_{\mathrm{p,n-1}}) = 0$ because $\mathrm{H}^1(G_{\mathbb{Q},S}, \phi^{\pm}\kappa_{n-1}^{-1}) \simeq \Lambda/(X^{n-1})$ and $\mathrm{H}^1(\mathbb{Q}_p, \kappa_{n-1}^{-1}) \simeq \Lambda/(X^{n-1}) \oplus \bar{\mathbb{Q}}_p$. $\square$

From the Lemma, it follows that for every $n \in \mathbb{Z}_{\geq 1}$,

$$\ker(\mathrm{Z}^1(G_{\mathbb{Q},S}, (\phi\kappa_n)^{-1}) \to \mathrm{Z}^1(\mathbb{Q}_p, \kappa_n^{-1})) = \ker(\mathrm{B}^1(G_{\mathbb{Q},S}, (\phi\kappa_n)^{-1}) \to \mathrm{B}^1(\mathbb{Q}_p, \kappa_n^{-1}))$$

Since $\mathrm{B}^1(G_{\mathbb{Q},S}, (\phi\kappa_n)^{-1}) = (1 - \phi^{-1}\kappa_n^{-1})\Lambda/(X^n)$ and $\mathrm{B}^1(\mathbb{Q}_p, \kappa_n^{-1}) \simeq \Lambda/(X^{n-1})$ and the map sends generator to generator, the kernel is isomorphic to $\bar{\mathbb{Q}}_p$. It follows that the dimension over $\bar{\mathbb{Q}}_p$ of $\mathrm{Hom}_\Lambda(C_\rho^{\mathrm{ord}}/C_\rho^{\mathrm{ord},2}, \Lambda/(X^n))$ is at most one for every $n \in \mathbb{Z}_{\geq 1}$, and in fact the equality holds because C_ρ^{ord} is non zero. $\square$

Applying Wiles' Numerical Criterion in our setting yields the following result.

Theorem 2.4.9. *The map* $\varphi_\rho^{\mathrm{ord}}\colon \mathcal{R}_\rho^{\mathrm{ord}} \to \mathcal{T}_\rho^{\mathrm{ord}}$ *(resp.* $\varphi_{\rho'}^{\mathrm{ord}}\colon \mathcal{R}_{\rho'}^{\mathrm{ord}} \to \mathcal{T}_{\rho'}^{\mathrm{ord}}$*) is a* Λ*-algebra isomorphism.*

From this we can deduce some consequences about the ring $\mathcal{R}_\rho^{\mathrm{ord}}$.

Corollary 2.4.10. *The ring* $\mathcal{R}_\rho^{\mathrm{ord}}$ *(resp.* $\mathcal{R}_{\rho'}^{\mathrm{ord}}$*) is a* Λ*-algebra of Krull dimension one and of complete intersection. Moreover,* $\mathcal{R}_\rho^{\mathrm{ord}} = \mathcal{R}_\rho^{\mathrm{eis}} \times_{\bar{\mathbb{Q}}_p} \mathcal{R}^{\mathrm{cusp}}$ *(resp.* $\mathcal{R}_{\rho'}^{\mathrm{ord}} = \mathcal{R}_{\rho'}^{\mathrm{eis}} \times_{\bar{\mathbb{Q}}_p} \mathcal{R}^{\mathrm{cusp}}$*).*

2.5 Structure of the completed local ring of the eigencurve

We wish to determine the structure of the Hecke algebra $\mathcal{T}$ over Λ. Denote by $\mathcal{T}^{(p)}$ the Λ-subalgebra of $\mathcal{T}$ generated by the Hecke operators T_ℓ for $\ell \nmid Np$; clearly $\mathcal{T} = \mathcal{T}^{(p)}[U_p]$. It suffices to determine the image of $\mathcal{T}$ and $\mathcal{T}^{(p)}$ under the (injective) morphism (2.24). Denote

$$S = \Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda = \{(a,b,c) \in \Lambda \times \Lambda \times \Lambda \mid a(0) = b(0) = c(0)\}$$
$$S^{(p)} = \{(a,b,c) \in S \mid \mathcal{L}(\phi^{-1})\frac{\partial a}{\partial X}(0) + \mathcal{L}(\phi)\frac{\partial b}{\partial X}(0) = (\mathcal{L}(\phi) + \mathcal{L}(\phi^{-1}))\frac{\partial c}{\partial X}(0)\}.$$

We determine the structure of $\mathcal{T}$ and $\mathcal{T}^{(p)}$ as Λ-algebras [BDP, Prop. 5.1].

Theorem 2.5.1. *There are isomorphisms of* Λ*-algebras*

$$\mathcal{T}^{(p)} \simeq S^{(p)} \qquad \textit{and} \qquad \mathcal{T} \simeq S.$$

Proof. The image of the map $(\pi_{\mathbb{1},\phi}^{\mathrm{eis}}, \pi_{\phi,\mathbb{1}}^{\mathrm{eis}}, \pi^{\mathrm{cusp}})$ is contained in S, because $\mathcal{T}$ has a unique maximal ideal. For every prime $\ell \nmid Np$, Corollary 2.3.8 and Lemma 2.4.1 yield the congruence

$$(\pi^{\mathrm{eis}}_{\mathbb{1},\phi},\pi^{\mathrm{eis}}_{\phi,\mathbb{1}},\pi^{\mathrm{cusp}})(T_\ell)-1-\phi(\ell)=\frac{\log_p(\ell)}{\log_p(1+\mathbf{q})}\left(\phi(\ell)X,X,\frac{(\phi(\ell)\mathcal{L}(\phi^{-1})+\mathcal{L}(\phi))X}{\mathcal{L}(\phi^{-1})+\mathcal{L}(\phi)}\right) \bmod X^2$$

The image of $\mathcal{T}^{(p)}$ is contained in $S^{(p)}$. Since X and $(\pi^{\mathrm{eis}}_{\mathbb{1},\phi},\pi^{\mathrm{eis}}_{\phi,\mathbb{1}},\pi^{\mathrm{cusp}})(T_\ell)-1-\phi(\ell)$ generate $\mathfrak{m}_{S^{(p)}}/\mathfrak{m}^2_{S^{(p)}}$ over $\bar{\mathbb{Q}}_p$, this shows that the image of $\mathcal{T}^{(p)}$ is equal to $S^{(p)}$. Since the ring $\mathcal{T}$ is $\mathcal{T}^{(p)}[U_p]$, it remains to compute the image of U_p. We have

$$(\pi^{\mathrm{eis}}_{\mathbb{1},\phi},\pi^{\mathrm{eis}}_{\phi,\mathbb{1}},\pi^{\mathrm{cusp}})(U_p)-1=(0,0,-\frac{\mathcal{L}(\phi)\mathcal{L}(\phi^{-1})}{\log_p(1+\mathbf{q})(\mathcal{L}(\phi^{-1})+\mathcal{L}(\phi))}X) \bmod X^2$$

Thus, we see that $X,(\pi^{\mathrm{eis}}_{\mathbb{1},\phi},\pi^{\mathrm{eis}}_{\phi,\mathbb{1}},\pi^{\mathrm{cusp}})(T_\ell)-1-\phi(\ell)$ and $(\pi^{\mathrm{eis}}_{\mathbb{1},\phi},\pi^{\mathrm{eis}}_{\phi,\mathbb{1}},\pi^{\mathrm{cusp}})(U_p)-1$ span $\mathfrak{m}_S/\mathfrak{m}_S^2$, so the image of $\mathcal{T}$ is equal to S. □

Remark 2.5.2 (U_p operator and pseudorepresentations)**.** The existence of an isomorphism $\mathcal{R}^{\mathrm{ord}}_\rho\simeq\mathcal{T}^{\mathrm{ord}}_\rho$ (resp. $\mathcal{R}^{\mathrm{ord}}_{\rho'}\simeq\mathcal{T}^{\mathrm{ord}}_{\rho'}$) shows, in particular, that the map $\mathcal{R}^{\mathrm{ps}}_{1+\phi}\to\mathcal{T}^{\mathrm{ord}}_\rho$ is surjective, *i.e.* that the Hecke algebras $\mathcal{T}^{\mathrm{ord}}_\rho$ and $\mathcal{T}^{\mathrm{ord}}_{\rho'}$ are generated by traces of representations. The pseudorepresentation $\mathbf{T}_{\mathcal{T}^{\mathrm{cusp}}}$ induces a Λ-equivariant morphism $\mathcal{R}^{\mathrm{ps}}_{1+\phi}\to\mathcal{T}$. By Chebotarev density Theorem, the image of $\mathcal{R}^{\mathrm{ps}}_{1+\phi}$ in $\mathcal{T}$ is generated by the T_ℓ operators for infinitely many primes ℓ, so it is equal to $\mathcal{T}^{(p)}$. As a consequence of Theorem 2.5.3 , we see that the Hecke algebra $\mathcal{T}$ is *not* generated by traces of representations. In geometric terms, this means that the map from the eigencurve to the pseudodeformation space is not (locally) a closed immersion, despite the fact that for each irreducible component it is. This feature is specific to the irregular weight one setting; it was observed by Calegari and Specter [CS] and motivated the definition of ordinary determinants in *loc.cit.*

2.5.1 Ring-theoretic properties of the Hecke algebra

We now examine in more detail the algebraic properties of the rings $\mathcal{T}^{(p)}$ and $\mathcal{T}$, describing them as quotients of rings of formal power series. These characterizations are not at all canonical but useful in order to determine regularity properties of the rings [BDP, Cor.5.3].

Theorem 2.5.3. *(i) The ring $\mathcal{T}^{(p)}$ is isomorphic to $\Lambda[\![T]\!]/(T(T-X)(T-\frac{\mathcal{L}(\phi^{-1})}{\mathcal{L}(\phi^{-1})+\mathcal{L}(\phi)}X))$ as a Λ-algebra. In particular, $\mathcal{T}^{(p)}$ is of complete intersection but not regular.*

(ii) The ring $\mathcal{T}$ is isomorphic to $\Lambda[\![T_1,T_2]\!]/(T_1T_2,T_1(X-T_1),T_2(X-T_2))$. In particular, $\mathcal{T}$ is Cohen-Macaulay, but not Gorenstein.

Proof. (i) By Theorem 2.5.3, $\mathcal{T}^{(p)}$ is isomorphic to $S^{(p)}$ as a Λ-algebra. Since the tangent space of $S^{(p)}$ is generated by (X,X,X) and $(X,0,\frac{\mathcal{L}(\phi^{-1})}{\mathcal{L}(\phi^{-1})+\mathcal{L}(\phi)}X)$ there is a surjective map $\Lambda[\![T]\!]\to S^{(p)}$ sending T to $(X,0,\frac{\mathcal{L}(\phi^{-1})}{\mathcal{L}(\phi^{-1})+\mathcal{L}(\phi)}X)$ with kernel generated by $T(T-X)(T-\frac{\mathcal{L}(\phi^{-1})}{\mathcal{L}(\phi^{-1})+\mathcal{L}(\phi)}X)$. In particular, $\mathcal{T}^{(p)}$ is the quotient of a regular ring by the ideal generated by a non-zero divisor, so $\mathcal{T}^{(p)}$ is of complete intersection. It is not regular because it is not a domain.

(ii) By Theorem 2.5.3, the ring $\mathcal{T}$ is isomophic to S. The maximal ideal of S is generated by $(X,X,X),(X,0,0)$, and $(0,0,X)$, thus there is a surjective Λ-algebra morphism $\Lambda[\![T_1,T_2]\!]\to S$ sending $T_1\mapsto(X,0,0)$ and $T_2\mapsto(0,X,0)$ with kernel $((T_1T_2,T_1(X-T_1),T_2(X-T_2))$. In particular, S has codimension 2 in the regular ring $\Lambda[\![T_1,T_2]\!]$, hence by a theorem of Serre [Eis04, Corollary 21.20], it is Gorenstein if and only if it is of complete intersection. Thus, it suffices to show that S is not of complete intersection. Consider the triple $(x_1,x_2,x_3)=((X,0,0),(0,X,0),(0,0,X))$ of generators of $\mathfrak{m}_S$. Denote by $K_\bullet(x_1,x_2,x_3)$ the Koszul complex of the free S-module $M=Su_1\oplus Su_2\oplus Su_3$, so that $K_r(x_1,x_2,x_3)=\bigwedge^r M$ with differential maps $d_i\colon K_r(x_1,x_2,x_3)\to K_{r-1}(x_1,x_2,x_3)$

satisfying

$$d_r(u_{i_1} \wedge u_{i_2} \cdots \wedge u_{i_r}) = \sum_{k=1}^{r} (-1)^k x_{i_k} u_{i_1} \wedge u_{i_2} \wedge \ldots \widehat{u}_{i_k} \cdots \wedge u_{i_r}$$

The homology groups $\mathrm{H}_i(K_\bullet(x_1, x_2, x_3))$ are $S/\mathfrak{m}_S = \bar{\mathbb{Q}}_p$-modules. The homology group $\mathrm{H}_1(K_\bullet(x_1, x_2, x_3)) = \ker d_1/\mathrm{Im}(d_2)$ can be computed as follows. The kernel of d_1 is given by the space of $\sum_i (a_i, b_i, c_i)u_i \in M$ with $(a_i, b_i, c_i) \in S$ satisfying $a_1 = b_2 = c_3 = 0$. The image of d_2 is the subspace of $\ker(d_1)$ defined by the equations

$$\frac{\partial b_1}{\partial X}(0) = \frac{\partial a_2}{\partial X}(0), \quad \frac{\partial c_1}{\partial X}(0) = \frac{\partial a_3}{\partial X}(0), \quad \frac{\partial c_2}{\partial X}(0) = \frac{\partial b_3}{\partial X}(0).$$

Thus, the standard deviation $\varepsilon_1(R) = \dim_{\bar{\mathbb{Q}}_p} \mathrm{H}_1(K_\bullet(x_1, x_2, x_3))$ is 3. Since

$$\dim_{\bar{\mathbb{Q}}_p} \mathfrak{m}_S/\mathfrak{m}_S^2 = 3 < 3 + 1 = \varepsilon_1(S) + \text{Krull}\dim(S),$$

the ring S is neither of complete intersection nor Gorenstein. An analogous computation shows that $\mathrm{H}_3(K_\bullet(x_1, x_2, x_3))$ vanishes and $\dim_{\bar{\mathbb{Q}}_p} \mathrm{H}_2(K_\bullet(x_1, x_2, x_3)) = 1$, so S has depth one. Since the depth is equal to the Krull dimension, the ring S is Cohen-Macaulay.

□

2.6 The full cuspidal Hecke algebra

In this section, we show an isomorphism between the completed local rings of the eigencurve $\mathcal{C}$ (resp. $\mathcal{C}^{\mathrm{cusp}}$) and the full eigencurve $\mathcal{C}^{\mathrm{full}}$ (resp. $\mathcal{C}^{\mathrm{cusp}}_{\mathrm{full}}$), denoted by $\mathcal{T}$ (resp. $\mathcal{T}^{\mathrm{cusp}}$) and $\mathcal{T}^{\mathrm{full}}$ (resp. $\mathcal{T}^{\mathrm{cusp}}_{\mathrm{full}}$) respectively [BDP, Prop.4.4]. This argument is a variation of the proof of the analogous statement in [BD16, Prop. 7.1]. The main differences are that in our case ρ is indecomposable and has infinite image, causing some issues when analyzing the action of the inertia groups for primes dividing the level.

The morphism $\iota_{\mathrm{full}}^{\mathrm{cusp}}\colon \mathfrak{T}^{\mathrm{cusp}} \hookrightarrow \mathfrak{T}_{\mathrm{full}}^{\mathrm{cusp}}$ induces by composition a representation $\rho_{\mathfrak{T}_{\mathrm{full}}^{\mathrm{cusp}}} = \iota_{\mathrm{full}}^{\mathrm{cusp}} \circ \rho_{\mathfrak{T}^{\mathrm{cusp}}}$. Denote by M (resp. M_{full}) the standard free $\mathfrak{T}^{\mathrm{cusp}}$(resp. $\mathfrak{T}_{\mathrm{full}}^{\mathrm{cusp}}$)-module of rank 2 with the action of $G_{\mathbb{Q}}$ given by $\rho_{\mathfrak{T}^{\mathrm{cusp}}}$ (resp. $\rho_{\mathfrak{T}_{\mathrm{full}}^{\mathrm{cusp}}}$).

Theorem 2.6.1. *Let ℓ be a prime dividing N. Then:*

(i) $\rho_{\mathfrak{T}^{\mathrm{cusp}}}(I_{\mathbb{Q}_\ell})$ is finite;

(ii) The module $M^{I_{\mathbb{Q}_\ell}}$ is a free direct summand of M of rank one;

(iii) The module $M_{\mathrm{full}}^{I_{\mathbb{Q}_\ell}}$ is isomorphic to $M^{I_{\mathbb{Q}_\ell}} \otimes_{\mathfrak{T}^{\mathrm{cusp}}} \mathfrak{T}_{\mathrm{full}}^{\mathrm{cusp}}$ and the action of Frob_ℓ *on $M_{\mathrm{full}}^{I_{\mathbb{Q}_\ell}}$ is given by multiplication by U_ℓ.*

Proof. (i) Let $t_p\colon I_{\mathbb{Q}_\ell} \to \mathbb{Z}_p$ be a non-trivial continuous homomorphism. Denote $H = \rho_{\mathfrak{T}^{\mathrm{cusp}}}(I_{\mathbb{Q}_\ell}) \simeq \rho_{\mathfrak{T}_{\mathrm{full}}^{\mathrm{cusp}}}(I_{\mathbb{Q}_\ell})$. By the Grothendieck Monodromy Theorem in families [BC09, Lemma 7.8.14], there exists a nilpotent matrix $N \in M_2(\mathfrak{T}^{\mathrm{cusp}})$ and a finite index subgroup of $I_{\mathbb{Q}_\ell}$ such that the restriction of $\rho_{\mathfrak{T}^{\mathrm{cusp}}}$ to the subgroup is $\exp(t_p(\cdot)N)$. Thus, to show that the image of $I_{\mathbb{Q}_\ell}$ is finite, it suffices to prove that $N = 0$. If $N \neq 0$, the representation $\rho_{\mathfrak{T}_{\mathrm{full}}^{\mathrm{cusp}}}$ to $G_{\mathbb{Q}_\ell}$ is of Steinberg type, *i.e.*

$$\rho_{\mathfrak{T}_{\mathrm{full}}^{\mathrm{cusp}}}|_{G_{\mathbb{Q}_\ell}} \simeq \begin{bmatrix} \chi & * \\ 0 & 1 \end{bmatrix} \otimes \psi$$

for some unramified character $\psi\colon G_{\mathbb{Q}_\ell} \to \bar{\mathbb{Q}}_p^\times$. In particular, $\mathrm{Tr}(\rho_{\mathfrak{T}_{\mathrm{full}}^{\mathrm{cusp}}})|_{I_{\mathbb{Q}_\ell}} = \mathrm{Tr}(\rho)|_{I_{\mathbb{Q}_\ell}} = \psi + \psi\chi \neq 1 + \phi$, by [BC09, Lemma 7.8.17]. Thus N must be zero and the image of H is finite.

(ii) Since the image of $I_{\mathbb{Q}_\ell}$ is finite, the inclusion $M^{I_{\mathbb{Q}_\ell}} \hookrightarrow M$ admits a retraction $s = \frac{1}{|H|}\sum_{h \in H} h$. Thus, $M^{I_{\mathbb{Q}_\ell}}$ is a direct summand of M; in particular the map $M^{I_{\mathbb{Q}_\ell}} \otimes \bar{\mathbb{Q}}_p \to (M \otimes \bar{\mathbb{Q}}_p)^{I_{\mathbb{Q}_\ell}}$ is injective. Since $(M \otimes \bar{\mathbb{Q}}_p)^{I_{\mathbb{Q}_\ell}}$ is the image of s as endomorphism of $M \otimes \bar{\mathbb{Q}}_p$, the map is also surjective. It follows that $M^{I_{\mathbb{Q}_\ell}} \otimes \bar{\mathbb{Q}}_p$ has dimension one over $\bar{\mathbb{Q}}_p$. Since $M^{I_{\mathbb{Q}_\ell}}$ is a direct summand of M, it is free of rank one.

(iii) Since $M_{\mathrm{full}}^{I_{\mathbb{Q}_\ell}}$ is the image of $s \otimes 1_{\mathcal{T}_{\mathrm{full}}^{\mathrm{cusp}}}$ on M_{full}, we have that $M^{I_{\mathbb{Q}_\ell}} \otimes_{\mathcal{T}^{\mathrm{cusp}}} \mathcal{T}_{\mathrm{full}}^{\mathrm{cusp}} \simeq M_{\mathrm{full}}^{I_{\mathbb{Q}_\ell}}$. It remains to show that the action of a Frobenius element at ℓ is given by multiplication by U_ℓ. By Proposition 2.3.5, there exists an affinoid neighborhood $\mathrm{Spm}(A)$ of f in $\mathcal{C}^{\mathrm{cusp}}$ and a representation $\rho_A \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(A)$ such that $\rho_A \otimes \mathcal{T}^{\mathrm{cusp}} = \rho_{\mathcal{T}^{\mathrm{cusp}}}$. Denote $\mathrm{Spm}(B)$ an affinoid neighborhood of f in $\mathcal{C}_{\mathrm{full}}^{\mathrm{cusp}}$ such that the image of $\mathrm{Spm}(B)$ is contained in $\mathrm{Spm}(A)$. By composition with $A \to B$, we get a lift ρ_B of $\rho_{\mathcal{T}_{\mathrm{full}}^{\mathrm{cusp}}}$; denote by M_B the corresponding $B[G_{\mathbb{Q}}]$-module. Up to shrinking $\mathrm{Spm}(B)$, we can assume that $M_B^{I_{\mathbb{Q}_\ell}}$ is free of rank one over B and $M_B^{I_{\mathbb{Q}_\ell}} \otimes k(x) \to (M_B \otimes k(x))^{I_{\mathbb{Q}_\ell}}$ is an isomorphism for every point $x \in \mathrm{Spm}(B)$, where $k(x)$ is the residue field of x. Denote by $\rho_B^{I_{\mathbb{Q}_\ell}} \colon G_{\mathbb{Q}_\ell} \to B$ the unramified character given by the action of $G_{\mathbb{Q}_\ell}$ on $M_B^{I_{\mathbb{Q}_\ell}}$. The function $\rho_B^{I_{\mathbb{Q}_\ell}}(\mathrm{Frob}_\ell) - U_\ell$ vanishes at all classical points of $\mathrm{Spm}(B)$. Since classical points are Zariski-dense in $\mathcal{C}_{\mathrm{full}}^{\mathrm{cusp}}$, it follows that $\rho_B^{I_{\mathbb{Q}_\ell}}(\mathrm{Frob}_\ell) = U_\ell$ in B. Thus, the equality holds in $\mathcal{T}_{\mathrm{full}}^{\mathrm{cusp}}$.

□

Corollary 2.6.2. *(i) The map $\iota_{\mathrm{full}}^{\mathrm{cusp}} \colon \mathcal{T}^{\mathrm{cusp}} \hookrightarrow \mathcal{T}_{\mathrm{full}}^{\mathrm{cusp}}$ is an isomorphism.*

(ii) Denote $\tilde{\pi}_{\mathbb{1},\phi}^{\mathrm{cusp}} = \pi_{\mathbb{1},\phi}^{\mathrm{cusp}} \circ \iota_{\mathrm{full}}^{\mathrm{cusp}-1} \colon \mathcal{T}_{\mathrm{full}}^{\mathrm{cusp}} \to \Lambda$. Then $\tilde{\pi}^{\mathrm{cusp}}$ satisfies the relations

$$\tilde{\pi}^{\mathrm{cusp}}(U_\ell) = 1 + \tfrac{\log_p(\ell)}{\log_p(1+\mathbf{q})} \left(\tfrac{\mathcal{L}(\phi)}{\mathcal{L}(\phi^{-1}) + \mathcal{L}(\phi)} \right) X \bmod X^2$$

for every prime $\ell \mid N$.

Proof. (i) By Theorem 2.6.1, $M^{I_{\mathbb{Q}_\ell}}$ is a direct summand of M. Hence, there exists a matrix $g \in \mathrm{GL}_2(\mathcal{T}^{\mathrm{cusp}})$ such that $(g \rho_{\mathcal{T}^{\mathrm{cusp}}} g^{-1})(\mathrm{Frob}_\ell) = \left[\begin{smallmatrix} u & * \\ 0 & * \end{smallmatrix}\right]$. Then $\iota_{\mathrm{full}}^{\mathrm{cusp}}(u) = U_\ell$.

(ii) Via the isomorphism $\mathcal{R}^{\mathrm{cusp}} \simeq \Lambda$, we obtain a deformation $\rho_\Lambda \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\Lambda)$ of ρ

such that

$$\rho_\Lambda = \begin{bmatrix} \phi(1+X\lambda\eta_{\mathbb{1}}) & \eta + X\xi \\ 0 & 1+X\mu\eta_{\mathbb{1}} \end{bmatrix} \bmod X^2 \tag{2.22}$$

in some basis $e_{1,\Lambda}, e_{2,\Lambda}$ of $M_\Lambda = \Lambda^2$ by Proposition 1.5.4 and the parameters satisfy

$$\lambda = \tfrac{\mathcal{L}(\phi^{-1})}{(\mathcal{L}(\phi)+\mathcal{L}(\phi^{-1}))\log_p(1+\mathbf{q})} \qquad \text{and} \qquad \mu = \tfrac{\mathcal{L}(\phi)}{(\mathcal{L}(\phi)+\mathcal{L}(\phi^{-1}))\log_p(1+\mathbf{q})} \tag{2.23}$$

by Proposition 1.5.11. The U_ℓ-eigenvalue is then given by the action of Frob_ℓ on the $I_{\mathbb{Q}_\ell}$-invariant subspace. By comparing with the inertia-invariant subspace of the residual representation, it follows that $M_\Lambda^{I_{\mathbb{Q}_\ell}} \otimes_\Lambda \bar{\mathbb{Q}}_p = \langle e_2 \rangle$, so (2.22) shows that

$$\tilde{\pi}^{\mathrm{cusp}}(\mathrm{Frob}_\ell) = \mu\eta_{\mathbb{1}}(\mathrm{Frob}_\ell) = \frac{\mathcal{L}(\phi)\log_p(\ell)}{(\mathcal{L}(\phi)+\mathcal{L}(\phi^{-1})).\log_p(1+\mathbf{q})} \bmod X^2$$

□

Theorem 2.6.3. *The inclusion $\iota_{\mathrm{full}} \colon \mathcal{T} \hookrightarrow \mathcal{T}_{\mathrm{full}}$ is an isomorphism.*

Proof. The irreducible components of the eigencurve $\mathcal{C}$ corresponding to the Eisenstein families $\mathcal{E}_{\mathbb{1},\phi}$ and $\mathcal{E}_{\phi,\mathbb{1}}$ are isomorphic to their preimages in $\mathcal{C}^{\mathrm{full}}$. The morphism (2.24) extends to an injective morphism

$$(\tilde{\pi}^{\mathrm{eis}}_{\mathbb{1},\phi}, \tilde{\pi}^{\mathrm{eis}}_{\phi,\mathbb{1}}, \tilde{\pi}^{\mathrm{cusp}}) \colon \mathcal{T}_{\mathrm{full}} \to \Lambda \times \Lambda \times \Lambda, \tag{2.24}$$

such that

$$\tilde{\pi}^{\mathrm{eis}}_{\mathbb{1},\phi}(U_\ell) = 1 \qquad \text{and} \qquad \tilde{\pi}^{\mathrm{eis}}_{\phi,\mathbb{1}}(U_\ell) = [\langle\!\langle \ell \rangle\!\rangle],$$

for every $\ell | N$. The image of $(\tilde{\pi}^{\mathrm{eis}}_{\mathbb{1},\phi}, \tilde{\pi}^{\mathrm{eis}}_{\phi,\mathbb{1}}, \tilde{\pi}^{\mathrm{cusp}})$ is contained in $\Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda$, because $\mathcal{T}_{\mathrm{full}}$ is local. But since the image of $\mathcal{T}$ is isomorphic to $\Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda$, it follows that ι_{full} is surjective. □

Chapter 3

Arithmetic applications

In this chapter, we explore some arithmetic applications of the theorems proved in the previous chapters. One theme is the relation between the Kubota-Leopoldt p-adic L-function $L_p(\phi\omega_p, s)$ and congruences between cuspidal and Eisenstein families. This connection lies at the core of the proof of many deep results, including the original proof of the Iwasawa Main Conjecture by Mazur and Wiles [MW84]. Using the results of the previous chapters, we give an independent proof of the Ferrero-Greenberg Theorem, stating that the derivative of the Kubota Leopoldt p-adic L-function is non-zero. Moreover, we recover the precise formula for the latter proved by Gross [Gro82] which constitutes an instance of the celebrated Gross-Stark Conjecture.

Another theme is the relation between generalized weight one eigenforms and units of number fields. Since three irreducible components of the eigencurve meet at points corresponding to irregular weight one Eisenstein series, the generalized eigenspace corresponding to their systems of eigenvalues is non trivial. In the spirit of the recent works of Darmon, Rotger and Lauder [DLR15a], the coefficients of the q-expansion of overconvergent forms in the generalized eigenspace can be written in terms of p-adic logarithms of p-units in the splitting field of the charater ϕ.

3.1 The Ferrero-Greenberg Theorem

The aim of this section is showing that the $L_p(\phi\omega_p, s)$ has a simple zero at $s = 0$, recovering an instance of the famous theorem of Greenberg and Ferrero. Intuitively, since the Kubota-Leopoldt p-adic L-function is the leading term of the q-expansion of $\mathcal{E}_{\mathbb{1},\phi}$ at the cusp ∞, its order of vanishing is linked to congruences between $\mathcal{E}_{\mathbb{1},\phi}$ and a Λ-adic cuspform. However, a more precise argument should be made that involves the order of vanishing of the constant term of the q-expansion at *all cusps*.

3.1.1 Evaluation at the cusps

Evaluation at the cusps for ordinary forms

Let $\mathcal{U} = \mathrm{Spm}(A) \subset \mathcal{W}$ be an admissible open affinoid. By a construction of Andreatta, Iovita and Stevens [AIS14] (see also [Pil13]), there exists an invertible sheaf $\omega_{\mathcal{U}}$ over the rigid space $\mathcal{X}(0) \times \mathcal{U}$ satisfying the property that, for every classical weight k and trivial character in $\mathcal{U}$, the weight k-specialization of $\omega_{\mathcal{U}}$ is isomorphic to $\omega^{\otimes k}$. Moreover, the sheaves $\omega_{\mathcal{U}}$ are in fact overconvergent for sufficiently small $\mathcal{U}$; *i.e.* they extend to line bundles over $\mathcal{X}(v) \times \mathcal{U}$ for some $v > 0$. We denote D the divisor of cusps of $\mathcal{X}(0) \times \mathcal{U}$ and $\omega_{\mathcal{U}}(-D)$ the line bundle $\omega_{\mathcal{U}} \otimes \mathcal{O}(-D)$. Let

$$S_{\mathcal{U}} = \mathrm{H}^0(\mathcal{X}(0) \times \mathcal{U}, \omega_{\mathcal{U}}(-D)) \subset M_{\mathcal{U}} = \mathrm{H}^0(\mathcal{X}(0) \times \mathcal{U}, \omega_{\mathcal{U}})$$

be the $\mathcal{O}(\mathcal{U})$-modules of modular forms and cuspforms respectively. We denote

$$S_{\mathcal{U}}^{\dagger,0} = e^{\mathrm{ord}}(S_{\mathcal{U}}) \qquad \text{and} \qquad M_{\mathcal{U}}^{\dagger,0} = e^{\mathrm{ord}}(M_{\mathcal{U}}).$$

the images of the modules $S_{\mathcal{U}}$ and $M_{\mathcal{U}}$ under the Hida's ordinary projector $e^{\mathrm{ord}} = \lim_{n\to\infty} U_p^{n!}$. The notation here is motivated by the fact that slope 0-eigenforms are automatically overconvergent in light of Remark 2.1.1.

By Hida theory, the morphism $w_{|\mathcal{C}^{\mathrm{ord}}}\colon \mathcal{C}^{\mathrm{ord}} \to \mathcal{W}$ is finite [Hid93, Sec. 7.3]. The preimage of $\mathcal{U}$ is an affinoid of $\mathcal{C}^{\mathrm{ord}}$, denoted by $\mathrm{Spm}(B)$, endowed with a finite flat morphism $A \to B$. By construction of the eigencurve, the modules $S_{\mathcal{U}}^{\dagger,0}$ and $M_{\mathcal{U}}^{\dagger,0}$ come equipped with an action of the Hecke algebra B. There is an exact sequence

$$0 \to S_{\mathcal{U}} \to M_{\mathcal{U}} \xrightarrow{\mathrm{Res}_{\mathcal{U}}} \bigoplus_{\delta\in D} \mathcal{O}(\mathcal{U}) \tag{3.1}$$

where the map $\mathrm{Res}_{\mathcal{U}}$ sends a modular form $\mathcal{G}$ over $\mathcal{O}(\mathcal{U})$ to $\mathrm{Res}_{\mathcal{U}}(\mathcal{G}) = (A_\delta(\mathcal{G}))_\delta$ and A_δ is the constant term of $\mathcal{G}$ at the cusp δ. We have the following result about the image of the residue map $\mathrm{Res}_{\mathcal{U}}$ [BDP, Prop.3.1].

Lemma 3.1.1. *(i) The map* $\mathrm{Res}_{\mathcal{U}}$ *is surjective.*

(ii) Let $\mathscr{C}_{\mathcal{U}}$ *the image of* $M_{\mathcal{U}}^{\dagger,0}$ *under* $\mathrm{Res}_{\mathcal{U}}$*. Then* $\mathscr{C}_{\mathcal{U}}$ *is a direct summand of* $\bigoplus_{\delta\in D}\mathcal{O}(\mathcal{U})$*.*

Proof. There is an exact sequence of sheaves on $\mathcal{X}(0)\times\mathcal{U}$

$$0 \to \omega_{\mathcal{U}} \to \omega_{\mathcal{U}}(-D) \to \bigoplus_{\delta\in D} \mathcal{O}_{D\times U} \to 0.$$

where the sheaf $\mathcal{O}_{D\times\mathcal{U}}$ is supported over $D\times\mathcal{U}$. Applying the functor of global section, we obtain an exact sequence

$$\begin{aligned}0 &\to \mathrm{H}^0(\mathcal{X}(0)\times\mathcal{U}, \omega_{\mathcal{U}}(-D)) \to \mathrm{H}^0(\mathcal{X}(0)\times\mathcal{U}, \omega_{\mathcal{U}}) \xrightarrow{\mathrm{Res}_{\mathcal{U}}} \bigoplus_{\delta\in D}\mathcal{O}(\mathcal{U}) \\ &\to \mathrm{H}^1(\mathcal{X}(0)\times\mathcal{U}, \omega_{\mathcal{U}}(-D)).\end{aligned}$$

Since $\mathcal{X}(0)\times\mathcal{U}$ is an affinoid and $\omega_{\mathcal{U}}(-D)$ is a line bundle, the latter term of the sequence vanishes. This proves (i).

The inclusion $S_{\mathcal{U}} \to M_{\mathcal{U}}$ is compatible with the action of e^{ord}, so the quotient

$\bigoplus_{\delta\in D}\mathcal{O}(\mathcal{U})$ inherits an action of e^{ord}. Since e^{ord} is an idempotent, the exact sequence yields an exact sequence

$$0\to S_{\mathcal{U}}^{\dagger,0}\to M_{\mathcal{U}}^{\dagger,0}\xrightarrow{\mathrm{Res}_{\mathcal{U}}}\mathscr{C}_{\mathcal{U}}\to 0,$$

and $\mathscr{C}_{\mathcal{U}}$ is a direct summand of $\bigoplus_{\delta\in D}\mathcal{O}(\mathcal{U})$. □

Fix now $\mathcal{U}=\mathrm{Spm}(A)$ an affinoid subdomain containing the point w_f of weight one and character ϕ. Recall that Λ is the completion of the local ring of the weight space at w_f. Let S_Λ and M_Λ be the completions of the localizations of $S_{\mathcal{U}}$ and $M_{\mathcal{U}}$ at the maximal ideal of A corresponding to w_f and let

$$S_\Lambda^{\dagger,0}=e^{\mathrm{ord}}(S_\Lambda)\qquad\text{and}\qquad M_\Lambda^{\dagger,0}=e^{\mathrm{ord}}(M_\Lambda).$$

Corollary 3.1.2. *There is an exact sequence*

$$0\to S_\Lambda^{\dagger,0}\to M_\Lambda^{\dagger,0}\xrightarrow{\mathrm{Res}_\Lambda}\mathscr{C}_\Lambda\to 0,\tag{3.2}$$

where $\mathscr{C}_\Lambda$ is a direct sum of $\bigoplus_{\delta\in D}\Lambda$.

Denote by $\mathfrak{m}_f$ the maximal ideal of B corresponding to the system of eigenvalues of f, so that the completion of $B\otimes\bar{\mathbb{Q}}_p$ with respect to $\mathfrak{m}_f$ is isomorphic to $\mathcal{T}$. Then $(B\otimes\bar{\mathbb{Q}}_p)\otimes_{A\otimes\bar{\mathbb{Q}}_p}\Lambda$ is a finite Λ-algebra; thus it is a semilocal ring with maximal ideals in bijection with the systems of eigenvalues of overconvergent weight one ordinary eigenforms. In particular, $\mathcal{T}$ is a direct summand of $(B\otimes\bar{\mathbb{Q}}_p)\otimes_{A\otimes\bar{\mathbb{Q}}_p}\Lambda$.

The modules $S_\Lambda^{\dagger,0}$ and $M_\Lambda^{\dagger,0}$ have an action of the Hecke algebra of $\mathcal{T}$, so the quotient $\mathscr{C}_\Lambda$ inherits one as well. Thus, taking the localization at $\mathfrak{m}_f$ yields an exact sequence

$$0\to S_{\Lambda,\mathfrak{m}_f}^{\dagger,0}\to M_{\Lambda,\mathfrak{m}_f}^{\dagger,0}\xrightarrow{\mathrm{Res}_\Lambda}\mathscr{C}_{\Lambda,\mathfrak{m}_f}\to 0.\tag{3.3}$$

and $\mathscr{C}_{\Lambda,\mathfrak{m}_f}$ is a direct summand of $\mathscr{C}_\Lambda$ and, as a consequence, a direct summand of $\bigoplus_{\delta\in D}\Lambda$.

Remark 3.1.3. A direct description of the Hecke action on $\mathscr{C}_\Lambda$ can be given as in [Oht03, Sec. 2].

3.1.2 Duality for the cuspidal Hecke algebra

We now show that $S^{\dagger,0}_{\Lambda,\mathfrak{m}_f}$ is a free Λ-algebra of rank one. As an application of the modularity results for the cuspidal deformation ring $\mathcal{R}^{\mathrm{cusp}}$, we obtain an explicit formula for the derivatives of the q-expansion coefficients of a generator of $S^{\dagger,0}_{\Lambda,\mathfrak{m}_f}$.

Proposition 3.1.4. *The pairing*

$$(-,-)\colon S^{\dagger,0}_{\Lambda,\mathfrak{m}_f}\times\mathcal{T}^{\mathrm{cusp}}\to\Lambda,\qquad(\mathcal{G},T)=a_1(T\mathcal{G})\tag{3.4}$$

is perfect. In particular, $S^{\dagger,0}_{\Lambda,\mathfrak{m}_f}$ is a free Λ-module of rank one.

Proof. Consider the map $S^{\dagger,0}_{\Lambda,\mathfrak{m}_f}\to\mathrm{Hom}_\Lambda(\mathcal{T}^{\mathrm{cusp}},\Lambda)$ induced by the pairing above; the isomorphism $\mathcal{T}^{\mathrm{cusp}}\to\mathcal{T}^{\mathrm{cusp}}_{\mathrm{full}}$ established in Corollary 2.6.2, combined with the q-expansion principle, implies that the map is injective. Similarly, the q-expansion principle implies that $\mathcal{T}\to\mathrm{Hom}_\Lambda(S^{\dagger,0}_{\Lambda,\mathfrak{m}_f}\to\Lambda)$ is injective. Therefore, the pairing is perfect. By Theorem 2.3.6, it follows that $S^{\dagger,0}_{\Lambda,\mathfrak{m}_f}$ is isomorphic to Λ. $\square$

Proposition 3.1.5. *Let $\mathcal{F}=\sum_{n=1}^{\infty}a_n(\mathcal{F})q^n\in\Lambda[\![q]\!]$ be the q-expansion of the unique normalized Λ-adic cuspform specializing to f. Then the derivatives of the coefficients of $\mathcal{F}$ satisfy*

$$\frac{\partial}{\partial X}\Big|_{X=0}a_p(\mathcal{F})=-\frac{\mathcal{L}(\phi)\mathcal{L}(\phi^{-1})}{(\mathcal{L}(\phi)+\mathcal{L}(\phi^{-1}))\log_p(1+\mathbf{q})},\ \textit{and}\tag{3.5}$$

$$\frac{\partial}{\partial X}\Big|_{X=0}a_\ell(\mathcal{F})=\frac{(\phi(\ell)\mathcal{L}(\phi^{-1})+\mathcal{L}(\phi))\log_p(\ell)}{(\mathcal{L}(\phi)+\mathcal{L}(\phi^{-1}))\log_p(1+\mathbf{q})},\ \textit{for every prime}\ \ell\neq p.\tag{3.6}$$

Proof. Let $\mathcal{F}$ be the generator of the Λ-module $S^{\dagger,0}_{\Lambda,\mathfrak{m}_f}$ normalized so that $a_1(\mathcal{F}) = 1$. Then, $a_n(\mathcal{F}) = \pi^{\mathrm{cusp}}(T_n)$ for every n. The formulas above follow from Corollary 2.3.8 for primes $\ell \nmid N$ and Corollary 2.6.2 for primes dividing the N. $\square$

Remark 3.1.6. By choosing an appropriate infinitesimal parameter (*i.e.* fixing a homomorphism $\Lambda \to \bar{\mathbb{Q}}_p[\varepsilon]$ sending $X \mapsto \log_p(1+\mathbf{q})\varepsilon$, compare with Lemma 1.2.2), we obtain the q-expansion of a modular form of weight $1+\varepsilon$. Namely

$$\mathcal{F}_\varepsilon = \sum_{n>0} \left(a_n(f) + \frac{\varepsilon}{\log_p(1+\mathbf{q})} \frac{\partial}{\partial X}\Big|_{X=0} a_n(\mathcal{F}) \right) q^n \in \bar{\mathbb{Q}}_p[\varepsilon][\![q]\!]$$

can be viewed as the q-expansion of a cuspform of weight $1+\varepsilon$. The existence of this infinitesimal cuspform was observed in [DDP11], where its q-expansion was obtained via different methods based on calculations with families of Eisenstein series.

3.1.3 Evaluation of Eisenstein series at the cusps

We compute the constant term of the q-expansion of the Eisenstein families $\mathcal{E}_{\mathbb{1},\phi}$ and $\mathcal{E}_{\phi,\mathbb{1}}$ at all cusps [BDP, Prop.4.6]. In order to do so, we use the computation of the constant term at all cusps [Oza17] for classical Hilbert Eisenstein series, although these can be easily calculated directly for modular forms.

Proposition 3.1.7. *Let $A_\delta(\mathcal{E}_{\mathbb{1},\phi})$ and $A_\delta(\mathcal{E}_{\phi,\mathbb{1}})$ be the constant terms of $\mathcal{E}_{\mathbb{1},\phi}$ and $\mathcal{E}_{\phi,\mathbb{1}}$ at $\delta \in D$.*

(i) One has $A_\infty(\mathcal{E}_{\mathbb{1},\phi}) = \frac{1}{2}\zeta_\phi$, $A_0(\mathcal{E}_{\mathbb{1},\phi}) = 0$ and $A_\delta(\mathcal{E}_{\mathbb{1},\phi}) \in \Lambda \cdot \zeta_\phi$ for all $\delta \in D$.

(ii) One has $A_\infty(\mathcal{E}_{\phi,\mathbb{1}}) = 0$, $A_0(\mathcal{E}_{\phi,\mathbb{1}}) \in \Lambda^\times \zeta_{\phi^{-1}}$ and $A_\delta(\mathcal{E}_{\phi,\mathbb{1}}) \in \Lambda \cdot \zeta_{\phi^{-1}}$ for all $\delta \in D$.

Proof. (i) We will establish the lemma via a computation of the constant term of the specializations of the Eisenstein families $\mathcal{E}_{\mathbb{1},\phi}$ and $\mathcal{E}_{\phi,\mathbb{1}}$ at all classical weights $k \geq 3$ such that $\omega_p^{k-1} = \mathbb{1}$. Recall that D_Γ the cuspidal divisor of X_Γ and that D_Γ^{can} is the image of D under the canonical section (2.1). We have bijections

$$D = \Gamma_1(N)\backslash\mathbb{P}^1(\mathbb{Q}), \qquad D_\Gamma = \Gamma\backslash\mathbb{P}^1(\mathbb{Q}) \qquad \text{and} \qquad D_\Gamma^{\mathrm{can}} = \Gamma\backslash(\Gamma_0(p)\infty).$$

In particular, via this identification, an element of $\delta \in D_\Gamma^{\mathrm{can}}$ corresponds to an element $\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]$ where a, c are coprime integers and $p|c$. Denote $\iota_p = \left[\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right]$. For every $g \in M_k(\Gamma_1(N), \phi)$, we define

$$g^{(p)}(z) = (g - g_{|_k}\iota_p)(z) = g(z) - p^{k-1}g(pz) \in M_k(\Gamma, \phi).$$

We wish to relate the q-expansion of $g^{(p)}$ at all cusps in D_Γ^{can} to the q-expansion of g at cusps in D. For $\delta = \left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right] \in D_\Gamma^{\mathrm{can}}$, choose a matrix $\gamma_\delta = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$. The q-expansion of $g^{(p)}$ at the cusp δ is then given by

$$\begin{aligned}(g^{(p)}_{|_k}\gamma_\delta)(z) &= g_{|_k}\gamma_\delta(z) - g_{|_k}(\iota_p\gamma_\delta)(z) = g_{|_k}\gamma_\delta(z) - g_{|_k}(\gamma_{p\delta}\iota_p)(z) \\ &= g_{|_k}\gamma_\delta(z) - p^{k-1}g_{|_k}\gamma_{p\delta}(pz)\end{aligned}$$

where $\gamma_{p\delta} = \left[\begin{smallmatrix} a & bp \\ cp^{-1} & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$. Letting A_δ (resp. $A_\delta^{(p)}$) denote the constant term of the q-expansion of g (resp. $g^{(p)}$) at the cusp $\delta \in D$ (resp. $\delta \in D_\Gamma^{\mathrm{can}}$) we have $A_\delta^{(p)} = A_\delta - p^{k-1}A_{p\delta}$.

Let g be the weight k Eisenstein series $E_k(\mathbb{1}, \phi) \in M_k(\Gamma_1(N), \phi)$. Then $g^{(p)}$ in the above notation is the ordinary p-stabilization of g, which is the weight k-specialization of the Eisenstein family $\mathcal{E}_{\mathbb{1},\phi}$. The constant term of the q-expansion of f vanishes at all cusps outside $\Gamma_1(N)\backslash\Gamma_0(N)\infty$. More precisely, by [Oza17, Prop.1.1], for $\delta = \left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]$ we have

$$A_\delta = 0 \text{ , if } N \nmid c, \qquad\qquad A_\delta = \frac{\phi^{-1}(|a|)}{2}L(\phi, 1-k) \text{ , if } N \mid c.$$

Thus, if $N \nmid c$ it follows that $A_\delta^{(p)} = 0$; if $N \mid c$, by (2.4) and $\phi(p) = 1$, we have

$$A_\delta^{(p)} = A_\delta - p^{k-1} A_{p\delta} = (1 - p^{k-1}) \frac{\phi^{-1}(|a|)}{2} L(\phi, 1-k) = \frac{\phi^{-1}(|a|)}{2} L_p(\phi\omega_p, 1-k).$$

The points of weight k satisfying the assumption above are Zariski dense in the connected component of $\mathcal{W}$ containing w_f. Therefore, the constant term of the q-expansion of $\mathcal{E}_{\mathbb{1},\phi}$ at the cusp $\delta \in D_\Gamma^{\mathrm{can}}$ vanishes if $N \nmid c$ and is equal to $\frac{\phi^{-1}(|a|)}{2} \zeta_\phi(X)$ otherwise. In particular, the constant term of $\mathcal{E}_{\mathbb{1},\phi}$ at ∞ is $\frac{1}{2}\zeta_\phi(X)$ and vanishes at the cusp 0.

(ii) Let g be the weight k Eisenstein series $E_k(\phi, \mathbb{1}) \in M_k(\Gamma_1(N), \phi)$. By [Oza17, Prop.1.1]

$$A_\delta = 0 \text{ if } (N, c) > 1, \qquad A_\delta = -\frac{\tau(\phi)}{2N^k} \phi^{-1}(|c|) L(\phi^{-1}, 1-k) \text{ if } (N, c) = 1$$

where $\tau(\phi)$ denotes the Gauss sum of ϕ. Thus, if $(c, N) > 1$ then $A_\delta^{(p)}$ vanishes, while if $(c, N) = 1$ we obtain

$$\begin{aligned} A_\delta^{(p)} = A_\delta - p^{k-1} A_{p\delta} &= -\frac{\tau(\phi)}{2N^k} \phi(|c|)(1 - p^{k-1}) L(\phi^{-1}, 1-k) \\ &= -\frac{\tau(\phi)}{2N^k} \phi(|c|) L_p(\phi^{-1}\omega_p, 1-k). \end{aligned}$$

The form $g^{(p)}$ is the weight k ordinary specialization of $\mathcal{E}_{\phi,\mathbb{1}}$. Since $(p, N) = 1$ and $\omega_p^{k-1} = \mathbb{1}$, N^k is the weight k specialization of an element in $\mathbb{Z}_p[\![X]\!]^\times$, while $L_p(\phi^{-1}\omega_p, 1-k)$ is the weight k specialization of $\zeta_{\phi^{-1}}$. Thus, the constant term of $\mathcal{E}_{\phi,\mathbb{1}}$ vanishes at ∞ and is a multiple of $\zeta_{\phi^{-1}}$ by a unit of Λ at $\Gamma \left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right] \in D_\Gamma^{\mathrm{can}}$ if $(c, N) = 1$. In particular, let u, v be integers satisfying $pv + uN = 1$. Then $\Gamma \left[\begin{smallmatrix} 1 \\ pv \end{smallmatrix}\right] \in D_\Gamma^{\mathrm{can}}$ and $\Gamma_1(N) \left[\begin{smallmatrix} 1 \\ pv \end{smallmatrix}\right] = \Gamma_1(N) \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, so the constant term of the q-expansion at 0 is non-zero.

$\square$

Corollary 3.1.8. *We have* $\mathrm{Res}_\Lambda(\mathcal{E}_{\mathbb{1},\phi}) \in \zeta_\phi \mathscr{C}_\Lambda$ *and* $\mathrm{Res}_\Lambda(\mathcal{E}_{\phi,\mathbb{1}}) \in \zeta_{\phi^{-1}} \mathscr{C}_\Lambda$.

Proof. By Proposition 3.1.7, $\mathrm{Res}_\Lambda(\mathcal{E}_{\mathbb{1},\phi}) \in \left(\zeta_\phi \bigoplus_{\delta\in D} \Lambda\right) \cap \mathscr{C}_\Lambda$. Since by Corollary 3.1.2, $\mathscr{C}_\Lambda$ is a direct sum of $\bigoplus_{\delta\in D}\Lambda$, it follows that $\mathrm{Res}_\Lambda(\mathcal{E}_{\mathbb{1},\phi}) \in \zeta_\phi\mathscr{C}_\Lambda$. Similarly for $\mathrm{Res}_\Lambda(\mathcal{E}_{\phi,\mathbb{1}})$. $\square$

Theorem 3.1.9. *The Kubota-Leopoldt p-adic L-functions $L_p(\phi\omega_p, s)$ and $L_p(\phi^{-1}\omega_p, s)$ have a simple zero at $s = 0$.*

Proof. Up to replacing ϕ by ϕ^{-1}, it suffices to show the statement for $L_p(\phi\omega_p, s)$. By the interpolation property $L_p(\phi\omega_p, s)$ has a trivial zero at $s = 0$. From (2.5), the zero is simple if and only if $\zeta_\phi \notin (X^2)$. Since $\mathscr{C}_{\Lambda,\mathfrak{m}_f}$ is a free Λ-module, tensoring (3.3) with $\Lambda/(\zeta_\phi)$ yields an exact sequence of $\Lambda/(\zeta_\phi)$-modules

$$0 \to S^{\dagger,0}_{\Lambda,\mathfrak{m}_f} \otimes \Lambda/(\zeta_\phi) \to M^{\dagger,0}_{\Lambda,\mathfrak{m}_f} \otimes \Lambda/(\zeta_\phi) \xrightarrow{\mathrm{Res}_{\Lambda/\zeta_\phi}} \mathscr{C}_{\Lambda,\mathfrak{m}_f} \otimes \Lambda/(\zeta_\phi) \to 0.$$

The image of $\mathrm{Res}_{\Lambda/\zeta_\phi}(\mathcal{E}_{\mathbb{1},\phi})$ is zero by Corollary 3.1.8. Thus, there exists $\mathcal{G} \in S^{\dagger,0}_{\Lambda,\mathfrak{m}_f}$ such that

$$\mathcal{G} - \mathcal{E}_{\mathbb{1},\phi} \in \zeta_\phi M^{\dagger,0}_{\Lambda,\mathfrak{m}_f}.$$

In particular, the q-expansion of $\mathcal{G} - \mathcal{E}_{\mathbb{1},\phi}$ satisfies $(\mathcal{G} - \mathcal{E}_{\mathbb{1},\phi})(q) \in \zeta_\phi\Lambda[\![q]\!]$. By Proposition 3.4, $\mathcal{G}$ is of the form $\alpha\mathcal{F}$ where $\mathcal{F}$ is the unique normalized eigenform in $S^{\dagger,0}_{\Lambda,\mathfrak{m}_f}$, and $\alpha \in \Lambda$. Moreover, since $\mathcal{F}$ and $\mathcal{E}_{\mathbb{1},\phi}$ are both normalized, it follows that $1-\alpha \in (\zeta_\phi)$; hence $(\mathcal{F} - \mathcal{E}_{\mathbb{1},\phi}) \in \zeta_\phi\Lambda[\![q]\!]$. The first order derivatives of $\mathcal{F}$ and $\mathcal{E}_{\mathbb{1},\phi}$ are distinct by Proposition 3.1.5 and Corollary 2.4.1. Hence, we must have $(\zeta_\phi) = (X)$ and $L_p(\phi\omega_p, s)$ has a simple zero at $s = 0$. $\square$

3.2 Duality for the Hecke algebra

In this section, we establish a duality result between the space of ordinary Λ-adic modular forms $M^{\dagger,0}_{\Lambda,\mathfrak{m}_f}$ and the Hecke algebra $\mathcal{T}$. The duality between the cuspidal Hecke algebra $\mathcal{T}^{\mathrm{cusp}}$ and $S^{\dagger,0}_{\Lambda,\mathfrak{m}_f}$ follows from the general theory of ordinary Hida families (after showing an isomorphism between $\mathcal{T}^{\mathrm{cusp}}$ and the full Hecke algebra $\mathcal{T}^{\mathrm{cusp}}_{\mathrm{full}}$). However, Hida theory only guarantees a perfect duality between the Hecke algebra $\mathcal{T}$ and an a priori larger space containing forms whose constant coefficient is not necessarily Λ-integral (see [Hid93, Thm 5, Sec. 7.3]).

Recall that $\mathcal{F}$ is the generator of $S^{\dagger,0}_{\Lambda,\mathfrak{m}_f}$, with q-expansion $\mathcal{F}(q) = \sum_{n\geq 1} a_n(\mathcal{F})q^n$, normalized so that $a_1(\mathcal{F}) = 1$. Denote

$$\mathcal{F}_{\mathbb{1},\phi} = \frac{\mathcal{E}_{\mathbb{1},\phi} - \mathcal{F}}{X}, \qquad \mathcal{F}_{\phi,\mathbb{1}} = \frac{\mathcal{E}_{\phi,\mathbb{1}} - \mathcal{F}}{X}$$

Note that $\mathcal{F}_{\mathbb{1},\phi}, \mathcal{F}_{\phi,\mathbb{1}} \in M^{\dagger,0}_{\Lambda,\mathfrak{m}_f}$ because the reduction modulo (X) of $\mathcal{F}, \mathcal{E}_{\mathbb{1},\phi}, \mathcal{E}_{\phi,\mathbb{1}}$ are equal to f. We calculate a basis of $M^{\dagger,0}_{\Lambda,\mathfrak{m}_f}$ [BDP, Prop. 5.4].

Proposition 3.2.1. *$\{\mathcal{F}, \mathcal{F}_{\mathbb{1},\phi}, \mathcal{F}_{\phi,\mathbb{1}}\}$ is a basis of $M^{\dagger,0}_{\Lambda,\mathfrak{m}_f}$ as a Λ-module.*

Proof. The exact sequence (3.3) splits, because $\mathscr{C}_{\Lambda,\mathfrak{m}_f}$ is a free Λ-module. Since $S^{\dagger,0}_{\Lambda,\mathfrak{m}_f} = \Lambda\mathcal{F}$, it suffices to check that

$$\mathrm{Res}_\Lambda(\langle \mathcal{F}_{\mathbb{1},\phi}, \mathcal{F}_{\phi,\mathbb{1}}\rangle) = \mathscr{C}_{\Lambda,\mathfrak{m}_f}.$$

Let $K(\Lambda)$ be the fraction field of Λ. Then $\{\mathrm{Res}_\Lambda(\mathcal{E}_{\mathbb{1},\phi}), \mathrm{Res}_\Lambda(\mathcal{E}_{\phi,\mathbb{1}})\}$ is a basis of $\mathscr{C}_{\Lambda,\mathfrak{m}_f} \otimes_\Lambda K(\Lambda)$, so $\{\mathrm{Res}_\Lambda(\mathcal{F}_{\mathbb{1},\phi}), \mathrm{Res}_\Lambda(\mathcal{F}_{\phi,\mathbb{1}})\}$ is also a basis over $K(\Lambda)$. Thus, it suffices to show that $\mathrm{Res}_\Lambda(\langle \mathcal{F}_{\mathbb{1},\phi}, \mathcal{F}_{\phi,\mathbb{1}}\rangle)$ is a direct summand of $\bigoplus_{\delta\in D} \Lambda$. By Proposition 3.1.7,

$$A_\infty(\mathcal{F}_{\mathbb{1},\phi}) \in \Lambda^\times, \qquad A_\infty(\mathcal{F}_{\phi,\mathbb{1}}) = 0 \qquad \text{and} \qquad A_0(\mathcal{F}_{\phi,\mathbb{1}}) \in \Lambda^\times.$$

This guarantees that the span of $\mathrm{Res}_\Lambda(\mathcal{F}_{\mathbb{1},\phi})$ and $\mathrm{Res}_\Lambda(\mathcal{F}_{\phi,\mathbb{1}})$ is a direct summand. $\square$

We now establish a duality result [BDP, Prop.5.5].

Theorem 3.2.2. *The pairing*

$$(-,-)\colon M^{\dagger,0}_{\Lambda,\mathfrak{m}_f}\times\mathcal{T}\to\Lambda,\qquad(\mathcal{G},T)=a_1(T\mathcal{G})\tag{3.7}$$

is perfect.

Proof. Recall that, by Theorem 2.5.3, the map $(\pi^{\mathrm{eis}}_{\mathbb{1},\phi},\pi^{\mathrm{eis}}_{\phi,\mathbb{1}},\pi^{\mathrm{cusp}})\colon\mathcal{T}\to\Lambda\times\Lambda\times\Lambda$ defines an isomorphism between $\mathcal{T}$ and $S=\Lambda\times_{\bar{\mathbb{Q}}_p}\Lambda\times_{\bar{\mathbb{Q}}_p}\Lambda$. As a Λ-module, $S\simeq 1\Lambda\oplus(X,0,0)\Lambda\oplus(0,X,0)\Lambda$. Under the identification $\mathcal{T}\simeq S$ as above, the matrix of the pairing is given by

$$\begin{bmatrix}a_1(\mathcal{F}) & a_1((X,0,0)\mathcal{F}) & a_1((0,X,0)\mathcal{F})\\ a_1(\mathcal{F}_{\mathbb{1},\phi}) & a_1((X,0,0)\mathcal{F}_{\mathbb{1},\phi}) & a_1((0,X,0)\mathcal{F}_{\mathbb{1},\phi})\\ a_1(\mathcal{F}_{\phi,\mathbb{1}}) & a_1((0,X,0)\mathcal{F}_{\phi,\mathbb{1}}) & a_1((0,X,0)\mathcal{F}_{\phi,\mathbb{1}})\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.\tag{3.8}$$

Thus, the pairing is perfect and $(1,(X,0,0),(0,X,0))$ is the dual basis of $(\mathcal{F},\mathcal{F}_{\mathbb{1},\phi},\mathcal{F}_{\phi,\mathbb{1}})$.

□

Corollary 3.2.3. *(i) There exist elements T_∞ and T_0 in $\mathcal{T}$ such that $A_\infty(\mathcal{G})=a_1(T_\infty\mathcal{G})$ and $A_0(\mathcal{G})=a_1(T_0\mathcal{G})$ for every $\mathcal{G}\in M^{\dagger,0}_{\Lambda,\mathfrak{m}_f}$.*

(ii) $\mathcal{T}^{\mathrm{ord}}_\rho=\mathcal{T}/(T_0)$ and $\mathcal{T}^{\mathrm{ord}}_{\rho'}=\mathcal{T}/(T_\infty)$.

Proof. The constant terms of the q-expansion at the cusps 0 and ∞ define linear morphisms $M^{\dagger,0}_{\Lambda,\mathfrak{m}_f}\to\Lambda$, thus the Hecke operators T_0 and T_∞ exist by duality. More precisely, comparing the proof of Proposition 3.1.7, with the pairing (3.8) we obtain

$$T_\infty=\frac{\zeta_\phi}{2X}(X,0,0)\in\Lambda^\times\cdot(X,0,0)\qquad\text{and}\qquad T_0=-\frac{\tau(\phi)\zeta_{\phi^{-1}}}{2NX}(0,X,0)\in\Lambda^\times\cdot(0,X,0).$$

Since $\mathcal{T}^{\mathrm{ord}}_{\rho'}\simeq\mathcal{T}/((X,0,0))$ and $\mathcal{T}^{\mathrm{ord}}_\rho\simeq\mathcal{T}/((0,X,0))$, the conclusion follows.

□

As a corollary of the explicit description of the pairing, we obtain duality statements for the quotients $\mathcal{T}_\rho^{\mathrm{ord}}$ and $\mathcal{T}_{\rho'}^{\mathrm{ord}}$ of $\mathcal{T}$. Denote $M_{\mathbb{1},\phi}^{\dagger,0} = \Lambda\mathcal{F} \oplus \Lambda\mathcal{F}_{\mathbb{1},\phi}$ and $M_{\phi,\mathbb{1}}^{\dagger,0} = \Lambda\mathcal{F} \oplus \Lambda\mathcal{F}_{\phi,\mathbb{1}}$.

Corollary 3.2.4. *The pairing* (3.7) *induces a perfect pairing between $\mathcal{T}_\rho^{\mathrm{ord}}$ and $M_{\mathbb{1},\phi}^{\dagger,0}$ and between $\mathcal{T}_{\rho'}^{\mathrm{ord}}$ and $M_{\phi.\mathbb{1}}^{\dagger,0}$.*

Remark 3.2.5. In light of Proposition 3.2.3, we obtain an alternative characterization of the quotient $\mathcal{T}_\rho^{\mathrm{ord}}$ and $\mathcal{T}_{\rho'}^{\mathrm{ord}}$ of $\mathcal{T}$. Indeed, $\mathcal{T}_\rho^{\mathrm{ord}}$ is the completed local ring of the vanishing locus of the constant term at 0, while $\mathcal{T}_{\rho'}^{\mathrm{ord}}$ of the constant term at ∞.

Remark 3.2.6. The residue map and its connection with duality for Hida families was extensively studied by Ohta [Oht03](although the case of irregular characters is treated by Lafferty [Laf]). The results of these sections could have been derived directly combining their works with our characterization of the structure of $\mathcal{T}$.

3.2.1 Λ-adic modular forms as Hecke modules

Once we established the duality between $\mathcal{T}$ and $M_{\Lambda,\mathfrak{m}_f}^{\dagger,0}$, we examine the structure of $M_{\Lambda,\mathfrak{m}_f}^{\dagger,0}$ as a module over the Hecke algebra, which is closely related to the ring theoretic properties of $\mathcal{T}$ discussed in the previous chapter. In particular, the Gorenstein property can be phrased in terms of self-duality of certain algebras, thus explaining the interest for Hecke rings. The failure of Gorensteinness of $\mathcal{T}$ implies that the module $M_{\Lambda,\mathfrak{m}_f}^{\dagger,0}$ is not free of rank one over $\mathcal{T}$. However, this property holds for the submodules $M_{\mathbb{1},\phi}^{\dagger,0}$ and $M_{\phi,\mathbb{1}}^{\dagger,0}$ and the corresponding quotients of the Hecke algebras, reflecting the fact that the latter are of complete intersection.

Proposition 3.2.7. *(i) $\{\mathcal{F}_{\mathbb{1},\phi}, \mathcal{F}_{\phi,\mathbb{1}}\}$ is a minimal set of generators for $M_{\Lambda,\mathfrak{m}_f}^{\dagger,0}$ over $\mathcal{T}$.*

(ii) $M_{\mathbb{1},\phi}^{\dagger,0}$ and $M_{\phi,\mathbb{1}}^{\dagger,0}$ are free of rank one over $\mathcal{T}_\rho^{\mathrm{ord}}$ and $\mathcal{T}_{\rho'}^{\mathrm{ord}}$ respectively.

Proof. By Nakayama's Lemma, it suffices to show that the image of $\{\mathcal{F}_{\mathbb{1},\phi}, \mathcal{F}_{\phi,\mathbb{1}}\}$ is a minimal set of generators of $M^{\dagger,0}_{\Lambda,\mathfrak{m}_f} \otimes_{\mathcal{T}} \mathcal{T}/\mathfrak{m}_{\mathcal{T}}$. The ideal $\mathfrak{m}_{\mathcal{T}}$ is generated by

$$(X, 0, 0), (0, X, 0), (0, 0, X)$$

over Λ. A direct computation shows that

$$\begin{aligned}
(X, 0, 0)M^{\dagger,0}_{\Lambda,\mathfrak{m}_f} =&(X\mathcal{F}_{\mathbb{1},\phi} + \mathcal{F})\Lambda \\
(0, X, 0)M^{\dagger,0}_{\Lambda,\mathfrak{m}_f} =&(X\mathcal{F}_{\phi,\mathbb{1},} + \mathcal{F})\Lambda \\
(0, 0, X)M^{\dagger,0}_{\Lambda,\mathfrak{m}_f} =&\mathcal{F}\Lambda.
\end{aligned}$$

Therefore, $\mathfrak{m}_{\mathcal{T}} M^{\dagger,0}_{\Lambda,\mathfrak{m}_f} = \mathcal{F}\Lambda \oplus X\mathcal{F}_{\mathbb{1},\phi}\Lambda \oplus X\mathcal{F}_{\phi,\mathbb{1}}\Lambda$. In particular, $M^{\dagger,0}_{\Lambda,\mathfrak{m}_f} \otimes_{\mathcal{T}} \mathcal{T}/\mathfrak{m}_{\mathcal{T}}$ is a $\bar{\mathbb{Q}}_p$-vector space of dimension 2 generated by $\mathcal{F}_{\mathbb{1},\phi}$ and $\mathcal{F}_{\phi,\mathbb{1}}$. The same computation shows that $M^{\dagger,0}_{\mathbb{1},\phi}$ is generated by $\mathcal{F}_{\mathbb{1},\phi}$ over $\mathcal{T}^{\mathrm{ord}}_{\rho}$. The map $\mathcal{T}^{\mathrm{ord}}_{\rho} \to M^{\dagger,0}_{\mathbb{1},\phi}$ sending x to $x\mathcal{F}_{\mathbb{1},\phi}$ is necessarily injective, because it is a surjective map between two free Λ-modules of rank two. A similar argument applies to $M^{\dagger,0}_{\phi,\mathbb{1}}$ as a $\mathcal{T}^{\mathrm{ord}}_{\rho'}$-module. □

3.3 q-expansion of a basis of overconvergent weight one generalized eigenforms

In this section, we apply the strategy of [DLR15a] to compute the q-expansion of a basis of the generalized eigenspace for the system of eigenvalues of f. This generalized eigenspace is the $\bar{\mathbb{Q}}_p$-dual of the relative Hecke algebra $\mathcal{T}_{w_f} = \mathcal{T}/\mathfrak{m}_{\Lambda}\mathcal{T}$. The existence of non-trivial elements of this eigenspace is guaranteed by the fact that the eigencurve is not etale over the weight space at f. As in *loc.cit*, the key ingredient for this computation is the isomorphism between the local ring of the cuspidal Hecke algebra and a deformation ring, together with the description of the tangent space of the latter via group cohomology. However, our approach diverges slightly from *loc.cit*; rather than computing the relative tangent space directly, we use the calculations of the first derivatives of the coefficients of the q-expansion of the cuspidal family obtained in Proposition 3.1.5. These derivatives are interesting in their own right, since they provide the q-expansion of a cuspform of over an infinitesimal neighborhood of w_f on the weight space.

As a corollary of these results we obtain an alternative proof of Gross's formula for the derivative of the p-adic L-function at 0, in a similar spirit to [DDP11]. Our argument relies crucially on identifying the classical subspace in the overconvergent generalized eigenspace.

3.3.1 The generalized eigenspace of f and the Gross-Stark Conjecture

Let $M^{\dagger,0}_{w_f}$ (resp. $S^{\dagger,0}_{w_f}$) be the space of ordinary overconvergent p-adic modular forms (resp. cuspforms) of weight 1 and character ϕ with coefficients in $\bar{\mathbb{Q}}_p$. The relative Hecke algebra $\mathcal{T}_{w_f} = \mathcal{T}/\mathfrak{m}_\Lambda\mathcal{T}$ is an artinian $\bar{\mathbb{Q}}_p$-algebra. Denote by $\mathfrak{m}_f$ the maximal ideal of $\mathcal{T}$ corresponding to the system of eigenvalues of f. The space $M^{\dagger,0}_{w_f}$ has an action of $\mathcal{T}_{w_f}$. For every $i > 0$, denote $M^{\dagger,0}_{w_f}[\mathfrak{m}_f^i]$ (resp. $S^{\dagger,0}_{w_f}[\mathfrak{m}_f^i]$) the subspace of $M^{\dagger,0}_{w_f}$ (resp. $S^{\dagger,0}_{w_f}$) annihilated by $\mathfrak{m}_f^i$. Let

$$M^{\dagger,0}_{w_f}[\![f]\!] = \bigcup_{i>0} M^{\dagger,0}_{w_f}[\mathfrak{m}_f^i] \qquad \text{and} \qquad S^{\dagger,0}_{w_f}[\![f]\!] = \bigcup_{i>0} S^{\dagger,0}_{w_f}[\mathfrak{m}_f^i]$$

be the generalized eigenspace of f. They are isomorphic to the completions of $M^{\dagger,0}_{w_f}$ and $S^{\dagger,0}_{w_f}$ at the $\mathfrak{m}_f$.

Let $M_1(\Gamma,\phi)$ (resp. $S_1(\Gamma,\phi)$) be the space of classical modular forms (resp. cuspforms) of weight 1, level $\Gamma = \Gamma_1(N)\cap\Gamma_0(p)$ and character ϕ with coefficients in $\bar{\mathbb{Q}}_p$. Let $\mathcal{T}_1(N,p)$ be the classical Hecke algebra generated by all Hecke operators T_ℓ for primes $\ell \nmid Np$ and

U_ℓ for $\ell \mid Np$ acting on $M_1(\Gamma, \phi)$. Denote by $\mathfrak{n}_f$ the maximal ideal corresponding to f. Denote by $M_1^0(\Gamma, \phi)$ (resp. $S_1^0(\Gamma, \phi)$) the image of Hida ordinary projector e^{ord}. These subspaces are stable under the action of $\mathcal{T}_1(N, p)$. Let $M_1^0(\Gamma, \phi)[\mathfrak{n}_f^i]$ (resp. $S_1^0(\Gamma, \phi)[\mathfrak{n}_f^i]$) be the subspace annihilated by $\mathfrak{n}_f^i$ and denote

$$M_1^0(\Gamma, \phi)[\![f]\!] = \bigcup_{i>0} M_1^0(\Gamma, \phi)[\mathfrak{n}_f^i],$$
$$S_1^0(\Gamma, \phi)[\![f]\!] = \bigcup_{i>0} S_1^0(\Gamma, \phi)[\mathfrak{n}_f^i]$$

There are natural inclusions

$$\begin{tikzcd} S_1^0(\Gamma, \phi)[\![f]\!] \arrow[r, hook] \arrow[d, hook] & M_1^0(\Gamma, \phi)[\![f]\!] \arrow[d, hook] \\ S_{w_f}^{\dagger,0}[\![f]\!] \arrow[r, hook] & M_{w_f}^{\dagger,0}[\![f]\!] \end{tikzcd}$$

Since f is cuspidal-overconvergent but not cuspidal as a classical form, $S_1^0(\Gamma, \phi)[\![f]\!] = 0$. The generalized eigespace $M_1^0(\Gamma, \phi)[\![f]\!]$ is annihilated by $\mathfrak{n}_f^2$ and spanned by f and $E_1(\mathbb{1}, \phi)$, because f is the unique p-stabilization of the form $E_1(\mathbb{1}, \phi)$ of level N.

Proposition 3.3.1. *(i) $S_{w_f}^{\dagger,0}[\![f]\!] = S_{w_f}^{\dagger,0}[\mathfrak{m}_f] = \bar{\mathbb{Q}}_p f$. Moreover, the cuspidal-overconvergent generalized eigenspace is classical, in the sense that $S_{w_f}^{\dagger,0}[\![f]\!] \subset M_1^0(\Gamma, \phi)[\![f]\!]$;*

(ii) $M_{w_f}^{\dagger,0}[\![f]\!] = M_{w_f}^{\dagger,0}[\mathfrak{m}_f^2]$ is a 3-dimensional $\bar{\mathbb{Q}}_p$-vector space.

Proof. The module $S_{w_f}^{\dagger,0}[\![f]\!] = S_{\Lambda,\mathfrak{m}_f}^{\dagger,0} \otimes_\Lambda \bar{\mathbb{Q}}_p$ is spanned by the reduction modulo (X) of $\mathcal{F}$, which is f, that, in particular, belongs to the classical eigenspace $M_1^0(\Gamma, \phi)[\![f]\!]$. Since $M_{\Lambda,\mathfrak{m}_f}^{\dagger,0}$ is a free Λ-module of rank 3, $M_{w_f}^{\dagger,0}[\![f]\!] = M_{\Lambda,\mathfrak{m}_f}^{\dagger,0} \otimes_\Lambda \bar{\mathbb{Q}}_p$ is 3-dimensional over $\bar{\mathbb{Q}}_p$. From Theorem 2.5.3, it follows that

$$\mathcal{T}_{w_f} \simeq \bar{\mathbb{Q}}_p[\![T_1, T_2]\!]/(T_1 T_2, T_1^2, T_2^2),$$

so $M^{\dagger,0}_{\Lambda,\mathfrak{m}_f}$ is annihilated by $\mathfrak{m}_f^2$. $\square$

We are now ready to determine the q-expansion of a basis of the generalized eigenspace [BDP, Thm.C].

Theorem 3.3.2. *The generalized eigenspace $M^{\dagger,0}_{w_f}[\![f]\!]$ has a basis $\{f, f_{\mathbb{1},\phi}, f_{\phi,\mathbb{1}}\}$ over $\bar{\mathbb{Q}}_p$, where the q-expansions of $f_{\mathbb{1},\phi}$ and $f_{\phi,\mathbb{1}}$ are*

$$f^{\dagger}_{\phi,\mathbb{1}}(q) = \sum_{n\geqslant 1} q^n \sum_{d|n,\ p\nmid d} \phi(d)\Big(\mathrm{ord}_p(n)\mathcal{L}(\phi) - \log_p\left(\tfrac{d^2}{n}\right)\Big) \quad \textit{and}$$

$$f^{\dagger}_{\mathbb{1},\phi}(q) = -\tfrac{(\mathcal{L}(\phi)+\mathcal{L}(\phi^{-1}))L_p'(\phi\omega_p,0)}{2\mathcal{L}(\phi)} + \sum_{n\geqslant 1} q^n \sum_{d|n,\, p\nmid d} \phi(d)\Big(\mathrm{ord}_p(n)\mathcal{L}(\phi^{-1}) + \log_p\left(\tfrac{d^2}{n}\right)\Big).$$

Proof. The module $M^{\dagger,0}_{w_f}$ is spanned by the reductions of $\mathcal{F}, \mathcal{F}_{\mathbb{1},\phi}, \mathcal{F}_{\phi,\mathbb{1}}$ modulo (X). Thus,

$$f^{\dagger}_{\mathbb{1},\phi} = \tfrac{(\mathcal{L}(\phi)+\mathcal{L}(\phi^{-1}))\log_p(1+\mathbf{q})}{\mathcal{L}(\phi)}\mathcal{F}_{\mathbb{1},\phi}(0) = \tfrac{(\mathcal{L}(\phi)+\mathcal{L}(\phi^{-1}))\log_p(1+\mathbf{q})}{\mathcal{L}(\phi)}\ \tfrac{\partial}{\partial X}\Big|_{X=0}(\mathcal{E}_{\mathbb{1},\phi} - \mathcal{F}),$$

$$f^{\dagger}_{\phi,\mathbb{1}} = \tfrac{(\mathcal{L}(\phi)+\mathcal{L}(\phi^{-1}))\log_p(1+\mathbf{q})}{\mathcal{L}(\phi^{-1})}\mathcal{F}_{\phi,\mathbb{1}}(0) = \tfrac{(\mathcal{L}(\phi)+\mathcal{L}(\phi^{-1}))\log_p(1+\mathbf{q})}{\mathcal{L}(\phi^{-1})}\ \tfrac{\partial}{\partial X}\Big|_{X=0}(\mathcal{E}_{\phi,\mathbb{1}} - \mathcal{F}).$$

are a basis of the supplement of $\bar{\mathbb{Q}}_p f$ in $M^{\dagger,0}_{w_f}$. We first calculate the coefficients a_ℓ for ℓ prime. For $\ell \neq p$, from (2.15), it follows that $a_\ell(\mathcal{E}_{\mathbb{1},\phi}) = 1 + \phi(\ell)[\langle\!\langle \ell \rangle\!\rangle]$ and $a_\ell(\mathcal{E}_{\phi,\mathbb{1}}) = \phi(\ell) + [\langle\!\langle \ell \rangle\!\rangle]$. Combining this with Lemma 1.2.2 yields that

$$\tfrac{\partial}{\partial X}\Big|_{X=0} a_\ell(\mathcal{E}_{\mathbb{1},\phi}) = \phi(\ell)\tfrac{\log_p(\ell)}{\log_p(1+\mathbf{q})}, \quad \text{and} \quad \tfrac{\partial}{\partial X}\Big|_{X=0} a_\ell(\mathcal{E}_{\phi,\mathbb{1}}) = \tfrac{\log_p(\ell)}{\log_p(1+\mathbf{q})}.$$

For the p-th coefficient, since $a_p(\mathcal{E}_{\mathbb{1},\phi}) = a_p(\mathcal{E}_{\phi,\mathbb{1}}) = 1$, we have $\frac{\partial}{\partial X}\big|_{X=0} a_p(\mathcal{E}_{\mathbb{1},\phi}) = \frac{\partial}{\partial X}\big|_{X=0} a_p(\mathcal{E}_{\phi,\mathbb{1}}) = 0$. Combining these formulas with those given in Proposition 3.1.5 we obtain the desired formulas for the non-constant coefficients of $f^{\dagger}_{\mathbb{1},\phi}$ and $f^{\dagger}_{\phi,,1}$

$$a_p(f^{\dagger}_{\mathbb{1},\phi}) = \mathcal{L}(\phi^{-1}), \qquad a_\ell(f^{\dagger}_{\mathbb{1},\phi}) = (\phi(\ell) - 1)\log_p(\ell),\ \ \ell \neq p \tag{3.9}$$

$$a_p(f^{\dagger}_{\phi,\mathbb{1}}) = \mathcal{L}(\phi), \qquad a_\ell(f^{\dagger}_{\phi,\mathbb{1}}) = (1 - \phi(\ell))\log_p(\ell),\ \ \ell \neq p. \tag{3.10}$$

In order to compute the remaining coefficients a_n for $n > 0$, we work out the recursive relations for the Hecke operators (notice that $f_{\mathbb{1},\phi}$ and $f_{\phi,\mathbb{1}}$ are *not* eigenforms). Denote by $f^\dagger$ an element of $\{f^\dagger_{\mathbb{1},\phi}, f^\dagger_{\phi,\mathbb{1}}\}$. We observe that, since $\mathcal{E}_{\mathbb{1},\phi}$, $\mathcal{E}_{\phi,\mathbb{1}}$ and $\mathcal{F}$ are normalized eigenform for all Hecke operators $(T_n)_{n\geqslant 1}$, the classical relations between abstract Hecke operators imply that for every n, m coprime integers

$$a_{mn}(f^\dagger) = a_m(f)a_n(f^\dagger) + a_n(f)a_m(f^\dagger), \tag{3.11}$$

Similarly, the prime power coefficients satisfy the relations

$$a_{\ell^r}(f^\dagger) = ra_\ell(f)^{r-1}a_\ell(f^\dagger) = ra_\ell(f^\dagger) \tag{3.12}$$

for all primes $\ell \mid Np$ and integers $r \geq 1$. For primes $\ell \nmid Np$ and $r \geq 2$, we have instead

$$a_{\ell^r}(f^\dagger) = a_\ell(f)a_{\ell^{r-1}}(f^\dagger) + a_{\ell^{r-1}}(f)a_\ell(f^\dagger) - \phi(\ell)a_{\ell^{r-2}}(f^\dagger),$$

which combined with the formula $a_{\ell^r}(f) = \sum_{i=0}^r \phi(\ell)^i$ gives

$$a_{\ell^r}(f^\dagger) = a_\ell(f^\dagger)\sum_{i=0}^r (i+1)(r-i)\phi(\ell)^i.$$

This yields the explicit formulas for the prime power coefficients of f for $\ell \nmid Np$,

$$a_{\ell^r}(f^\dagger_{\mathbb{1},\phi}) = \sum_{i=0}^r (2i-r)\phi(\ell^i)\log_p(\ell) = -a_{\ell^r}(f^\dagger_{\phi,\mathbb{1}}),$$

which coincide with (3.12) for primes $\ell \mid N$. The formulas for prime power coefficients, together with (3.11), imply the desired result.

It remains to calculate the constant terms of the q-expansions of $f^\dagger_{\mathbb{1},\phi}$ and $f^\dagger_{\phi,\mathbb{1}}$. The constant terms of $\mathcal{F}$ and $\mathcal{E}_{\phi,\mathbb{1}}$ vanish at the cusp ∞, thus $a_0(f^\dagger_{\phi,\mathbb{1}}) = 0$. For $f^\dagger_{\mathbb{1},\phi}$, we have instead

$$\begin{aligned} a_0(f^\dagger_{\mathbb{1},\phi}) &= \tfrac{(\mathcal{L}(\phi)+\mathcal{L}(\phi^{-1}))\log_p(1+\mathbf{q})}{\mathcal{L}(\phi)} \tfrac{\partial}{\partial X}\Big|_{X=0} a_0(\mathcal{E}_{\mathbb{1},\phi}) \\ &= \tfrac{(\mathcal{L}(\phi)+\mathcal{L}(\phi^{-1}))\log_p(1+\mathbf{q})}{\mathcal{L}(\phi)} \tfrac{\partial}{\partial X}\Big|_{X=0} \frac{\zeta_\phi}{2} = -\tfrac{\mathcal{L}(\phi)+\mathcal{L}(\phi^{-1})}{2\mathcal{L}(\phi)} L_p'(\phi\omega_p, 0), \end{aligned}$$

where in the last equality we used the interpolation property $\zeta_\phi((1+\mathbf{q})^{s-1}-1) = L_p(\phi\omega_p, 1-s)$ □

We conclude by giving a proof of an instance of the Gross-Stark Conjecture relating the derivative of the Kubota-Leopoldt p-adic L-function with the $\mathcal{L}$-invariant.

Theorem 3.3.3 (Gross-Stark Conjecture)**.** *The p-adic L-function $L_p(\phi\omega_p, s)$ satisfies*

$$L_p'(\phi\omega_p, 0) = -\mathcal{L}(\phi)L(\phi, 0).$$

Proof. The classical form $E_1(\mathbb{1}, \phi)$ is an element of $M^{\dagger,0}_{w_f}$, hence we can write it as a linear combination of $f, f^\dagger_{\mathbb{1},\phi}, f^\dagger_{\phi,\mathbb{1}}$. Comparing the coefficients a_ℓ for $\ell \nmid p$ and a_p, it follows that

$$E_1(\mathbb{1}, \phi) = f + \tfrac{1}{\mathcal{L}(\phi)+\mathcal{L}(\phi^{-1})}(f^\dagger_{\mathbb{1},\phi} + f^\dagger_{\phi,\mathbb{1}})$$

Comparing the constant coefficients, we obtain

$$\frac{L(\phi,0)}{2} = a_0(E_1(\mathbb{1},\phi)) = a_0(f + \tfrac{1}{\mathcal{L}(\phi)+\mathcal{L}(\phi^{-1})}(f^\dagger_{\mathbb{1},\phi} + f^\dagger_{\phi,\mathbb{1}})) = -\tfrac{L_p'(\phi\omega_p,0)}{2\mathcal{L}(\phi)},$$

yielding the result. □

Conclusion

The results of this work give a satisfactory description of $\mathcal{T}$, the completed local ring of the eigencurve at f and its structure as a Λ-algebra. We show that three irreducible components of the eigencurve (two Eisenstein components and one cuspidal) meet transversally at f, and each is etale over the weight space. We also prove that the tangent space of the eigencurve at f is three-dimensional. The latter observation is crucial in order to determine the ring-theoretic properties of $\mathcal{T}$: we prove that $\mathcal{T}$ is Cohen-Macaulay, but not Gorenstein. Among ring-theoretic properties, Gorensteinness is especially relevant in the theory of Hecke algebra. We show a duality statement between $\mathcal{T}$ and an appropriate localization of the module of ordinary Λ-adic forms; the failure of the Gorenstein property implies that $\mathcal{T}$ is not self-dual, and in particular the module of modular forms is not free over $\mathcal{T}$. There are plenty of examples of failure of the Gorenstein property for the Hecke algebras in characteristic p, hence at height two primes of the ordinary Hecke algebra (see, for example [Wak15]). In our case, the Gorensteinness fails at a height one prime instead. The two phenomena are rather different in nature: the failure of Gorensteinness at height two primes is related to the vanishing of certain Iwasawa modules, and the existence of non-trivial zeros of the p-adic L-function. In this work, the phenomenon is geometric in nature and arises at a trivial zero. Understanding of the geometric picture in this settings has bearings on the order of vanishing of the p-adic L-function.

It is worth observing that the case in which the Dirichlet character ϕ is quadratic is far simpler than the general one from an arithmetic point of view, in the sense that no argument of linear independence of $\mathcal{L}$-invariants is required. This is unsurprising, given that in this case the cuspidal family specializing to f has the well-known explicit description of the CM ordinary family interpolating classical theta series. In addition to specializing to the the same form at weight one, the Eisenstein and the theta families satisfy another congruence, which we recover in our computation of the tangent space of $\mathcal{T}^{\mathrm{cusp}}$, namely

$$a_\ell(\mathcal{F}) = \frac{1}{2}(a_\ell(\mathcal{E}_{\mathbb{1},\phi}) + a_\ell(\mathcal{E}_{\phi,\mathbb{1}})) \pmod{X^2}$$

for every prime $\ell \neq p$. An analoguous congruence between Kato classes has been exploited in an upcoming work of Bertolini and Darmon to prove a conjecture of Perrin-Riou; given an elliptic curve E, the conjecture relates the position of the Kato class $K_E(\phi, \mathbb{1})$ in the ϕ-isotypic component of the Mordell-Weil group of E to a global point. One might hope that understanding the geometry of the eigencurve at all irregular Eisenstein weight one points could prove instrumental to extending their results beyond the quadratic case.

Our modularity results could potentially be refined. We prove a modularity statement for the cuspidal Hecke algebra; we further extend it by showing the existence of an isomorphism $\mathcal{R}_\rho^{\mathrm{ord}} \simeq \mathcal{T}_\rho^{\mathrm{ord}}$ where $\mathcal{T}_\rho^{\mathrm{ord}}$ is the completed local ring of the Zariski closed subspace of the eigencurve given by the cuspidal eigencurve and one Eisenstein component and $\mathcal{R}_\rho^{\mathrm{ord}}$ is the ring of ordinary deformations of ρ. The construction is essentially symmetric in the two Eisenstein components, thus giving a Galois-theoretic interpretation of the quotients of $\mathcal{T}$ denoted by $\mathcal{T}_\rho^{\mathrm{ord}}$ and $\mathcal{T}_{\rho'}^{\mathrm{ord}}$. It would however be more satisfactory to obtain a modularity result for the ring $\mathcal{T}$ itself encompassing the previous statements. This raises a challenge from different points of view. Firstly, when analyzing the structure of $\mathcal{T}$, we realize that $\mathcal{T}$ is not generated by traces of representations, unlike its quotients

$\mathcal{T}_{\rho}^{\mathrm{ord}}$ and $\mathcal{T}_{\rho'}^{\mathrm{ord}}$. This is due to the fact that the U_p operator does not belong to the algebra generated by the Hecke operators of T_n for $(n,p)=1$. Secondly, even given a suitable candidate of deformation ring for a modularity statement, one would not be able to apply the techniques of this work to show the existence of an isomorphism. Indeed, Wiles' numerical criterion yields an isomorphism of rings of complete intersection, thus it would *a fortiori* fail because $\mathcal{T}$ is not Gorenstein. One could venture a guess for a suitable deformation ring for the modularity statement; through a Galois cohomology calculation as in Proposition 1.5.13 one can construct a surjective homomorphism $\mathcal{R}_{\rho}^{\mathrm{n.ord}} \to \mathcal{T}$. The corresponding Galois representation will not be ordinary, but only nearly ordinary. One can define a quotient of $\mathcal{R}_{\rho}^{\mathrm{n.ord}}$ by imposing the condition that the trace of the corresponding representation is compatible with being ordinary, in the sense that the restriction to the inertia group at p is $1+\kappa_{\Lambda}$.

The study of the relative local ring over the weight space $\mathcal{T}_{w_f}$ allows us to determine the generalized eigenspace of f in the space of weight one overconvergent modular forms. We determine a basis of a supplement of f given by two overconvergent non-classical forms and compute the q-expansion coefficients of these elements. As a byproduct of this calculation, we obtain a new proof of the Gross-Stark conjecture over $\mathbb{Q}$. In addition, these results have a second important arithmetic application, as they imply a classicality statement. By the work of Coleman [Col96], the space of ordinary cuspforms of weight $k \geq 2$ is always classical. While this result fails for weight one, the classicality of the generalized eigenspace attached to a given system of Hecke eigenvalues is worth investigating. In this work, we prove that the cuspidal generalized eigenspace $S_{w_f}^{\dagger,0}[\![f]\!]$ is spanned by f, a classical and cuspidal-overconvergent form, even though not cuspidal as a classical form. This result was originally conjectured in [DLR15b, Hypothesis C'] in view of the following application. Let E be an elliptic curve and g be another classical weight 1 form, denote by ρ_g the Artin representation attached to g. Under the assumption that the analytic rank of the pair (E, $\rho_g \otimes (\phi \oplus \mathbb{1})$) is greater than one, the elliptic Stark conjecture of *loc.cit.* predicts a relation between the values of some p-adic integrals and the formal group logarithms of global points of E. The assumption of classicality of the generalized eigenspace of f is required to define these p-adic iterated integrals. Thus, the results and numerical evidence towards the Elliptic Stark Conjecture stated in *loc.cit.* are now unconditional in light of our classicality result.

Further developments

Our study of the local geometry of the eigencurve at Eisenstein weight one points lead to a rather exhaustive picture. We would like to conclude by presenting some directions of generalization.

A related project is tackling the remaining weight one case *i.e.* studying the local ring of the eigencurve curve at *cuspidal irregular* weight one points. This scenario presents significantly more technical difficulties than both the present work and [BD16]. Indeed, the work of Bellaiche and Dimitrov builds on the calculation of the tangent space of certain deformation rings via Galois cohomology, and ours does as well, even though in a less direct manner. However, a similar strategy will not apply in the cuspidal irregular case. Let ρ be the Artin representation associated to a weight one irregular cuspform. One can define a deformation ring associated to ρ, and determine a quotient classifying ordinary lifts of ρ. Every first order deformation of ρ is automatically ordinary; in particular, a computation on the tangent space will not suffice to determine the ordinary deformation ring. Thus, it will be necessary to calculate higher order coefficients of the ideal defining the ordinary quotient. In [DLRar], the authors predict that the generalized eigenspace attached to an irregular weight one cuspform is four dimensional, which suggests a rather intricate geometric picture. One could again compute the q-expansion coefficients of a basis of the generalized space in terms of p-adic logarithm of units of a number field. We expect their expressions to involve products of logarithms, capturing the fact that the square of the maximal ideal of the relative local ring will be non-zero.

A second natural question is how to extend this work to the Hilbert case, *i.e.* for irregular parallel weight one Eisenstein series for a totally real field F. As in the $F = \mathbb{Q}$ case, one could define a cuspidal deformation ring as a quotient of the tensor products of certain nearly ordinary deformation rings attached to reducible indecomposable representations of G_F sharing the same semisimplification. This construction will depend on the ramification of p in F in a crucial way. For example, let F be a real quadratic field in which p is inert and let ϕ be a finite order character of G_F; under these assumptions, the cohomology group $\mathrm{H}^1(F, \phi)$ is two-dimensional. Our approach in the present work relied on the fact that the pair of reducible indecomposable representations associated to f were essentially unique. Thus, generalizing this strategy to higher dimension poses the question of understanding what residual representation can occur when considering different lattices for a given irreducible but residually reducible representation of G_F. Advances in this direction would provide a different (Galois theoretic) point of view on the Gross-Stark Conjecture proved in [DKV18].

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