Restriction of Rigid Meromorphic Cocycles from O(3,1) to O(2,1)

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Abstract

Darmon and Vonk introduced rigid meromorphic cocycles for $O(2,1)$ as real quadratic analogues of classical Borcherds products for the same group. To a weakly holomorphic modular form f of weight $1/2$, they associate a rigid meromorphic cocycle Ψ_f whose divisor is controlled by the negative coefficients of f. Darmon, Gehrmann, and Lipnowski proposed the construction of cocycles for an orthogonal group of arbitrary signature. Following their work, we obtain cocycles for $O(3,1)$. While we are not able to establish a lift from the space of weakly holomorphic modular forms, nonetheless we construct it from a space similar to the space of principal parts of weakly holomorphic modular forms. More precisely, let $\delta \in \mathbb{Z}[q^{-1}]$ be a Laurent polynomial such that its coefficients give rise to a linear relation among the Fourier coefficients of a modular form of weight two and a suitable level. With some additional technical assumptions on δ, we obtain a rigid meromorphic cocycle Ψ_δ whose divisor is governed by the coefficients of δ. Since the singularities of Borcherds products for $O(2,2)$ are encoded in the principal parts of weakly holomorphic modular forms of weight 0, the resulting cocycle Ψ_δ can be viewed as a p-adic analogue of a Borcherds product for the orthogonal group $O(3,1)$ instead of $O(2,2)$.

We flesh out the analogy further by restricting the cocycle Ψ_δ to an embedded $O(2,1)$. We prove that the singularities of the restriction are encoded in the principal part of the formal Laurent series $\delta \times \theta$, where θ denotes a unary theta series. The analogous operation in the case of classical Borcherds products is called the quasi-pullback, and similar formulae for the input modular form of the quasi-pullback have been obtained in several cases ([2, 3, 11, 14, 18]). Our result further contributes to the proposed analogy between classical Borcherds products and rigid meromorphic cocycles for orthogonal groups.

Abrégé

Darmon et Vonk ont introduit les cocycles rigides méromorphes pour $O(2,1)$ dans le but d'obtenir un analogue réel quadratique des produits de Borcherds classiques pour ce groupe. À chaque forme modulaire faiblement holomorphe f de poids 1/2, ils associent un cocycle rigide méromorphe Ψ_f dont le diviseur est contrôlé par les coefficients négatifs de f. Darmon, Gehrmann et Lipnowski ont ensuite proposé une construction de cocycles rigides méromorphes pour des groupes orthogonaux de signature arbitraire. En suivant leur approche, nous construisons des cocycles pour $O(3,1)$. Cependant, nous ne parvenons pas à relever toutes les formes modulaires faiblement holomorphes à un cocycle. Pour remédier à ce problème, nous recourons à une certaine collection de polynômes de Laurent dans $\mathbb{Z}[q^{-1}]$. Hormis certaines conditions techniques, les coefficients de ces polynômes doivent engendrer des relations linéaires entre les coefficients de Fourier de formes modulaires de poids 2 d'un niveau approprié. En ce sens, les cocycles rigides méromorphes pour $O(3,1)$ peuvent être considérés comme des analogues p-adiques des produits de Borcherds pour $O(2,2)$.

Étant donné un cocycle rigide méromorphe Ψ_δ pour $O(3,1)$, nous pouvons le restreindre à $O(2,1)$. Nous montrons que cette restriction peut être vue comme un relèvement de la partie principale d'une série formelle $\delta \times \theta$, où θ est une série thêta unaire. Le processus analogue dans le cas des produits de Borcherds classiques se nomme quasi-tiré-en-arrière et des formules similaires pour les arguments de ces quasi-tirés-en-arrière ont été obtenues dans d'autres cas. Le résultat de cette thèse s'inscrit dans cette direction et contribue à cimenter l'analogie entre les produits de Borcherds classiques et les cocycles rigides méromorphes pour les groupes orthogonaux.

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1 Introduction

The primary goal of this thesis is to flesh out the analogy between classical Borcherds lifts [2] and rigid meromorphic cocycles for orthogonal groups in the sense of [5]. The latter, originally introduced in [7] and generalized in [5], are 1-cocycles for a p-arithmetic group with values in a space of rigid meromorphic functions on a suitable p-adic analytic space with singularities at special divisors.

Darmon and Vonk [6] proved that for a weakly holomorphic modular form of weight $1/2$, there exists a rigid meromorphic cocycle for $\mathrm{SL}_2(\mathbb{Z}[1/p])$ whose divisor is controlled by the principal part of the input modular form. This result can be seen as a p-adic analogue of the Borcherds lift for the case of signature $(2,1)$ via the accidental isogeny between SL_2 and the orthogonal group of a quadratic form of signature $(2,1)$. The introduction in *loc. cit.* provides a compelling account of the analogy between the two worlds. Later, Darmon, Gehrmann, and Lipnowski [5] generalized the notion of rigid cocycles to other signatures. In this thesis, we focus on the construction in the setting of the p-arithmetic Bianchi group $\mathrm{SL}_2(\mathcal{O}_K[1/p])$, where K is an imaginary quadratic field and p splits in K. This group is closely related to the orthogonal group of signature $(3,1)$, i.e., automorphisms of the Minkowski space. Our first main result (Theorem 6.11) provides strong evidence for the existence of a similar lift from a weakly holomorphic modular form of weight 0 to rigid meromorphic Bianchi cocycles.

This case can be seen as an analogue of the Borcherds lift for signature $(2, 2)$. We elaborate on the analogy in Section 1.2.

Let f be a weakly holomorphic form of weight 0 and of a suitable level, and Ψ_f be the corresponding rigid cocycle for the associated congruence subgroup $\Gamma_K \subset \mathrm{SL}_2(\mathcal{O}_K[1/p])$. Since a rigid meromorphic cocycle for Γ_K is a 1-cocycle, we can restrict Ψ_f to obtain a 1-cocycle $\mathrm{Res}(\Psi_f)$ for $\Gamma_K \cap \mathrm{SL}_2(\mathbb{Z}[1/p])$. Our calculations suggest that the restricted cocycle $\mathrm{Res}(\Psi_f)$ is essentially a lift of the modular form $f \times \theta$, which is a weakly holomorphic modular form of weight 1/2. In other words, the restriction map in the target of the p-adic lift should correspond to multiplication of the input weakly holomorphic forms by a unary theta function. This is strongly reminiscent of a classical phenomenon, in the setting of classical Borcherds lifts, governing the diagonal restrictions of Hilbert modular forms.

The operation analogous to the restriction of cocycles is known as the quasi-pullback of Borcherds products, first examined systematically by Borcherds in [2]. In some special cases, it is essentially a pullback: restricting a Borcherds lift Ψ_L (associated to a rank four lattice L) to a symmetric domain $\mathfrak{D}_M$ (associated to a ternary sublattice $M \subset L$). A result of Ma [18, Theorem 1] (see also Examples 3.13 and 3.14) establishes that if Ψ_L is a lift of f, then the restriction $\Psi_L \mid \mathfrak{D}_M$ is a lift of $f \times \theta$.

1.1 Classical Borcherds Lifts

We first provide a high-level overview of Borcherds lifts since this provides a good motivation for the setup needed for p-arithmetic Bianchi groups. The statements presented here are intentionally kept informal, since precise formulations require additional technical hypotheses that might obscure the themes we wish to convey. In general, Borcherds gave a systematic method of generating modular forms on orthogonal groups from elliptic modular forms. Let $V_{\mathbb{R}}$ be a quadratic space of signature $(2, n)$ and $L \subset V_{\mathbb{R}}$ be an even unimodular lattice. Borcherds discovered a lifting from weakly holomorphic modular forms of weight $1 - (n/2)$ for $\mathrm{Mp}_2(\mathbb{Z})$ to meromorphic modular forms for the orthogonal group $\Gamma(L)$ of the lattice L (see [2]).

In the specific case of $n = 1$, the associated complex analytic space of $O(2, 1)$ is the Poincaré upper half-plane $\mathcal{H}$. The group $\mathrm{SL}_2(\mathbb{R})$ acts on this space. Thus, the Borcherds lift in this case takes a weakly holomorphic form of weight 1/2 to a meromorphic modular form on $\mathcal{H}$ for discrete arithmetic subgroups of $\mathrm{SL}_2(\mathbb{R})$ such as $\mathrm{SL}_2(\mathbb{Z})$ and its congruence subgroups. The zeros and poles of the lifted modular form are concentrated on CM points and are determined by the negative coefficients of the input modular form. For a precise statement, see [1, Theorem 14.1].

Turning to the case where $n = 2$, the complex analytic space associated with $V_{\mathbb{R}}$ is $\mathcal{H} \times \mathcal{H}$, on which the group $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ acts. Given a real quadratic field F, the group $\mathrm{SL}_2(\mathcal{O}_F) \hookrightarrow \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ is isogenous to the orthogonal group of a certain $\mathbb{Z}$-lattice in $V_{\mathbb{R}}$. Thus, for a weakly holomorphic form of weight 0, there exists a meromorphic Hilbert modular form for the group $\mathrm{SL}_2(\mathcal{O}_F)$. The zeros and poles of this modular form are supported on a special collection of divisors known as Hirzebruch-Zagier divisors. The multiplicities of the divisors appearing in the lift are determined by the negative coefficients of the input modular form of weight 0. For a precise statement, see Theorem 3.44 in [4]. This theorem allows one to recover the earlier results of Hirzebruch and Zagier [13]. While established before the development of the Borcherds framework, their result is presented in [4, Theorem 3.51] as a direct consequence of the modern theory.

1.2 Rigid Meromorphic Cocycles: A p-adic Analogue

Rigid meromorphic cocycles were first introduced by Darmon and Vonk in [7] for the orthogonal group of signature $(2, 1)$. These are classes in the first cohomology of the Ihara group $\mathrm{SL}_2(\mathbb{Z}[1/p])$ with values in the space of meromorphic functions on the p-adic upper half-plane $\mathcal{H}_p = \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$. Darmon and Vonk conjectured that evaluating these cocycles at real quadratic irrationalities, or equivalently restricting them to orthogonal groups of signature $(1, 1)$, yields elements in abelian extensions of real quadratic fields. This framework is analogous to the well-understood setting of imaginary quadratic extensions, where $\mathrm{SL}_2(\mathbb{Z})$-invariant functions are evaluated at imaginary quadratic points to produce elements in abelian extensions of imaginary quadratic fields. The authors also provided convincing numerical evidence supporting their conjecture.

In a follow-up article [6], Darmon and Vonk associate a rigid meromorphic cocycle to a weakly holomorphic modular form of weight 1/2. The zeros and poles of the cocycle are real quadratic points whose multiplicities are determined by the negative Fourier coefficients of the input modular form. Their result is stated informally here, omitting certain technical hypotheses (see Section 1.1 in [6] for full details). This statement closely resembles the case of the Borcherds lift for signature $(2, 1)$, although the divisors are now real quadratic instead of imaginary quadratic points. Darmon and Vonk put forward the viewpoint that rigid cocycles can be envisaged as real quadratic analogues of Borcherds lifts. We record the analogy in Table 1.1.

Darmon, Gehrmann, and Lipnowski [5] introduced a systematic procedure to construct

Table 1.1: The Classical and Real Quadratic Borcherds Lift

	Classical	**Real Quadratic**
Input	weakly hol form (wt = 1/2)	weakly hol form (wt = 1/2)
Output Object	meromorphic modular form	rigid meromorphic cocycle
Domain	complex upper half-plane $\mathcal{H}$	p-adic upper half-plane $\mathcal{H}_p$
Divisor Support	CM points (imaginary quadratic)	RM points (real quadratic)
Negative Coefficients	dictate poles/zeros	dictate poles/zeros

cocycles for a wider class of orthogonal groups, namely those arising from $\mathbb{Z}[1/p]$-lattices with arbitrary signatures. The case considered by Darmon and Vonk can then be seen as a special case of the generalized framework, as the group $\mathrm{SL}_2(\mathbb{Z}[1/p])$ is a subgroup of the orthogonal group of a rank three $\mathbb{Z}[1/p]$-lattice L where $L \otimes \mathbb{R}$ has signature $(2, 1)$. Given a quadratic space of signature (r, s), the authors in [5] postulated the existence of classes in s-th cohomology that are valued in spaces of functions on suitable p-adic spaces depending on the rank of the quadratic space. Furthermore, the authors proved lifting theorems analogous to results within Borcherds theory, strengthening the viewpoint that rigid meromorphic cocycles can be viewed as p-adic analogues of Borcherds products. In this thesis, we take a specific case of their general construction to obtain cocycles for certain subgroups of $\mathrm{SL}_2(\mathcal{O}_K[1/p])$, where K is imaginary quadratic and the prime p splits in K. Since $\mathrm{SL}_2(\mathcal{O}_K)$ is known as the Bianchi group, we refer to $\mathrm{SL}_2(\mathcal{O}_K[1/p])$ as a p-arithmetic Bianchi group. A large part of this thesis is devoted to explaining the construction of cocycles for certain subgroups of this group.

We now explain how the cocycles for Bianchi groups can be viewed as p-adic analogues of Borcherds lifts on $\mathcal{H} \times \mathcal{H}$. Consider a rational quadratic space $V_{\mathbb{Q}}$ of rank four such that the space $V_{\mathbb{Q}} \otimes \mathbb{Q}_p$ decomposes as a sum of two hyperbolic spaces. This makes the space "similar to signature $(2, 2)$" but now over a non-Archimedean place instead of an Archimedean place. For instance, this happens when the discriminant of the quadratic form is a square in $\mathbb{Q}_p$ and its Hasse invariant is 1. Taking this p-adic perspective further, we see that the group $\mathrm{SL}_2(\mathbb{Q}_p) \times \mathrm{SL}_2(\mathbb{Q}_p)$ acts on $\mathcal{H}_p \times \mathcal{H}_p$. Furthermore, the group $\mathrm{SL}_2(\mathcal{O}_K[1/p])$ injects into $\mathrm{SL}_2(\mathbb{Q}_p) \times \mathrm{SL}_2(\mathbb{Q}_p)$ via the two embeddings, provided the prime p splits in K. This assumption of p splitting is crucial and mirrors the assumption of F being real quadratic in the signature $(2, 2)$ case of the Borcherds lift. However, this group does not act discretely. Thus, the space of $\mathrm{SL}_2(\mathcal{O}_K[1/p])$-invariant functions on $\mathcal{H}_p \times \mathcal{H}_p$, or equivalently, the 0-th cohomology, is not interesting. In fact, the "right" degree of cohomology is dictated by the behaviour of $V_{\mathbb{Q}}$ in the Archimedean world. If the signature of $V_{\mathbb{Q}} \otimes \mathbb{R}$ is (r, s) with $r \geq s$, then the rigid cocycles should be classes in the s-th cohomology. In this thesis, we consider the case of signature $(3,1)$. The assumption that K is an imaginary quadratic field ensures that the group $\mathrm{SL}_2(\mathcal{O}_K[1/p])$ can be identified with the orthogonal group of a $\mathbb{Z}[1/p]$-lattice in $V_{\mathbb{Q}} \otimes \mathbb{R}$ via the exceptional isomorphism between $\mathrm{PSL}_2(\mathbb{C})$ and $\mathrm{PSO}(V_{\mathbb{Q}} \otimes \mathbb{R})$.

We now present the first of our two main theorems. This theorem is inspired by Theorem 3.24 of [5], albeit formulated more precisely.

Let ℓ be a positive integer such that the prime p splits in $K = \mathbb{Q}(\sqrt{-\ell})$. For an integer $N > 1$, we construct cocycles for the subgroup $(\Gamma_1^K(N))^p \subset \mathrm{SL}_2(\mathcal{O}_K[1/p])$ defined by

$$(\Gamma_1^K(N))^p := \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}[\sqrt{-\ell}][1/p]) : \begin{pmatrix} x & y \\ z & w \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

Let f be a weakly holomorphic modular form of weight 0 for $\Gamma_0(4\ell Np)$, holomorphic at all cusps except possibly at ∞. Suppose its q-expansion at ∞ is given by

$$f(\tau) = \sum_{m=1}^{m_0} c_m q^{-m} + c_0 + O(q).$$

Let g be a cusp form of weight 2 for $\Gamma_0(4\ell Np)$. Then $f(\tau)g(\tau)\,d\tau$ is a meromorphic differential on $X_0(4\ell Np)$ with a possible pole at ∞. The residue of $f(\tau)g(\tau)\,d\tau$ at ∞ is the constant term of the q-expansion of fg, which is exactly $\sum_{m=1}^{m_0} c_m \times (m\text{-th coeff of } g)$. Since the sum of residues on a compact Riemann surface must be zero, and ∞ is the only pole, this residue must vanish. By Serre duality, we also have the converse. The polynomial $\sum_{m=1}^{m_0} c_m q^{-m}$ occurs as the principal part of a weakly holomorphic modular form of weight 0 for $\Gamma_0(4\ell Np)$ that is regular away from ∞ if and only if $\sum_{m=1}^{m_0} c_m \times (m\text{-th coeff of } g) = 0$ for all cusp forms g of weight 2 for $\Gamma_0(4\ell Np)$.

We now give two definitions that are inspired by the discussion of the previous paragraph and are useful in stating our theorem. Let $M_2(\Gamma_0(4\ell Np), \left(\frac{\cdot}{\ell}\right))$ denote the space of modular forms of weight 2 for $\Gamma_0(4\ell Np)$ and character $\left(\frac{\cdot}{\ell}\right)$.

Definition 1.1. *We use* $\mathrm{PP}_0(M, \chi)$ *to denote the set of all finite collections of integers* $\{c_m\}_{m=1}^{m_0}$ *satisfying*

$$\sum_{m=1}^{m_0} c_m \times m\text{-th coefficient of } f = 0,$$

for all $f \in M_2(\Gamma_0(M), \chi)$.

Definition 1.2. *We define* $\mathrm{PP}_0(M,\chi)^\dagger$ *to be the subset of* $\mathrm{PP}_0(M,\chi)$ *consisting of collections* $\{c_m\}_{m=1}^{m_0}$ *satisfying*

1. $c_m = 0$ *if* p *divides* m*, and*

2. $c_m = 0$ *if* m *is equal to* $a^2+\ell b^2$ *for some rational numbers* a *and* b*.*

Our theorem establishes the map

$$\mathrm{BL}_p : \mathrm{PP}_0(4\ell Np, \left(\tfrac{\cdot}{\ell}\right))^\dagger \to \mathrm{Z}^1((\Gamma_1^K(N))^p, \mathcal{M}(\mathcal{H}_p\times\mathcal{H}_p)^\times/\mathbb{Z}_p^\times)$$

that takes $\delta = \{c_m\}_{m=1}^{m_0} \in \mathrm{PP}_0(4\ell Np, \left(\tfrac{\cdot}{\ell}\right))^\dagger$ as input and produces a rigid meromorphic cocycle whose divisor is completely determined by the integers $\{c_m\}_{m=1}^{m_0}$.

Theorem 1.3. *Let* L *be a* $\mathbb{Z}[1/p]$*-lattice equipped with a quadratic form* Q *such that* $L\otimes\mathbb{R}$ *has signature* $(3,1)$*. Suppose that the group of isometries of* L *is* $(\Gamma_1^K(N))^p$*. For each* $v\in L$*, define a* p*-adic analogue of the Hirzebruch-Zagier divisor*

$$\Delta_{v,p} := \{(z_1,z_2)\in\mathcal{H}_p\times\mathcal{H}_p : a_1z_1z_2 + a_2z_1 + a_3z_2 + a_4 = 0\},$$

where $a_1,a_2,a_3,a_4\in\mathbb{Z}[1/p]$ *are the coordinates of* v *with respect to a fixed basis of* L*. Let* $\delta = \{c_m\}_{m=1}^{m_0} \in \mathrm{PP}_0(4\ell Np, \left(\tfrac{\cdot}{\ell}\right))^\dagger$*. Then there exists a rigid meromorphic cocycle* $\Psi_\delta \in \mathrm{Z}^1((\Gamma_1^K(N))^p, \mathcal{M}(\mathcal{H}_p\times\mathcal{H}_p)^\times/\mathbb{Z}_p^\times)$ *such that for each* $\gamma\in(\Gamma_1^K(N))^p$*, the divisor of* $\Psi_\delta(\gamma)$ *is given by*

$$\sum_{m=1}^{m_0} c_m \times \sum_{\substack{v\in L\\ Q(v)=m}} \epsilon_v(\gamma)\Delta_{v,p},$$

where $\epsilon_v : \mathbb{P}_1(K)\times\mathbb{P}_1(K)\to\mathbb{Z}$ *is a modular symbol associated to each* $v\in L$*.*

Our theorem provides a partial analogue of classical Borcherds lifts, as we construct a map from $\mathrm{PP}_0(4\ell Np, \left(\tfrac{\cdot}{\ell}\right))^\dagger$ instead of $\mathrm{PP}_0(4\ell Np, \left(\tfrac{\cdot}{\ell}\right))$. The former is a proper subset of the latter, defined by imposing two extra conditions. The first condition is a technical assumption made to simplify our calculations. However, the second condition is more substantial. For the lengths that are norms of elements in K, a suitable candidate for the divisor of a rigid meromorphic cocycle remains elusive.

The following table summarizes our discussion.

Table 1.2: Rank Four: Borcherds Lift and p-adic Analogue

	Borcherds Lift	**p-adic Analogue**
Quadratic Space	$V_{\mathbb{R}}$ signature $(2,2)$	$V_{\mathbb{Q}_p}$ split; $V_{\mathbb{R}}$ signature $(3,1)$
Arithmetic Group	$\mathrm{SL}_2(\mathcal{O}_F)$	$\mathrm{SL}_2(\mathcal{O}_K[1/p])$
Condition on Field	$F \otimes \mathbb{R} = \mathbb{R} \oplus \mathbb{R}$	$K \otimes \mathbb{Q}_p = \mathbb{Q}_p \oplus \mathbb{Q}_p$
Domain	$\mathcal{H} \times \mathcal{H}$	$\mathcal{H}_p \times \mathcal{H}_p$
Input Type	Weakly holom. form (wt. 0)	$\mathrm{PP}_0(4\ell N p, (\frac{\cdot}{\ell}))^\dagger$
Output Type	Hilbert modular forms	Classes in H^1

1.3 Our Contribution

The quasi-pullback of Borcherds products was first introduced in [2] and [3]. It produces a meromorphic modular form on a lower-dimensional symmetric domain, which is again a Borcherds product. This operation has been considered in many special cases by several authors (see the Introduction in [18] for a comprehensive overview).

Ma [18, Theorem 1] proved an explicit formula for the weakly holomorphic modular form whose Borcherds lift is the quasi-pullback of a given Borcherds product. We describe the result in a specific setting. Let L be a lattice of signature $(2,2)$ such that it splits as $L = M \oplus M'$ where M is of signature $(2,1)$ and M' is negative definite. Let f be a weakly holomorphic modular form of weight 0, and let $\Psi_f(L)$ denote the corresponding Borcherds lift. Let $\mathfrak{D}_M$ denote the symmetric domain embedded in $\mathfrak{D}_L = \mathcal{H} \times \mathcal{H}$. The embedding of $\mathfrak{D}_M$ inside $\mathfrak{D}_L$ corresponds to an embedding of $\mathcal{H}$ into $\mathcal{H} \times \mathcal{H}$. Assume that $\Psi_f(L)$ does not vanish identically on $\mathfrak{D}_M$. Then the pullback, which is the same as the quasi-pullback under the non-vanishing assumption, is $\Psi_f(L) \mid \mathfrak{D}_M$. The divisor of the restriction is the same as that of the Borcherds lift of $f \times \theta$. In other words, the restriction is the same as the lift of $f \times \theta$ up to a holomorphic, non-vanishing $\Gamma(M)$-invariant function on $\mathcal{H}$. This can be described informally in the following diagram.

$$\begin{tikzcd}
\mathrm{WHF}_0 \arrow[r, "\mathrm{BL}_\infty"] \arrow[d, "\substack{\text{Multiplication by}\\ \text{a unary } \theta \text{ series}}"'] & \substack{\text{Meromorphic Hilbert}\\ \text{Modular Forms}} \arrow[d, "\substack{\text{Pullback along embedding}\\ \text{of } \mathcal{H} \text{ into } \mathcal{H}\times\mathcal{H}}"] \\
\mathrm{WHF}_{1/2} \arrow[r, "\mathrm{BL}_\infty"] & \substack{\text{Meromorphic Modular Forms for}\\ \text{congruence subgroup of } \mathrm{SL}_2(\mathbb{Z})}
\end{tikzcd}$$

Remark 1. *Theorem 1 of [18] proves exact equality of the restriction and the lift of $f \times \theta$, but it requires the sublattice M to be anisotropic and thus we cannot use their statement directly. However, our argument in the previous paragraph follows from their proof of Proposition 3.4.*

We now move to the second main theorem of this thesis, which can be seen as a p-adic analogue of the result discussed in the previous two paragraphs.

Let L be the $\mathbb{Z}[1/p]$-lattice that appears in Theorem 1.3. It splits as $M \oplus M'$, where M' is a one-dimensional lattice such that the discriminant of $M' \otimes \mathbb{Q}_p$ is equal to $-\alpha^2$ for some $\alpha \in \mathbb{Q}_p^*$. The p-adic space corresponding to M is the diagonal $\{(\tau, \tau) : \tau \in \mathcal{H}_p\}$ embedded in $\mathcal{H}_p \times \mathcal{H}_p$. We only describe the main features of the theorem here (for the full statement, see Theorem 7.13). Let $\Psi_L(\delta)$ denote the rigid meromorphic cocycle associated to $\delta \in \mathrm{PP}_0(4\ell Np, \left(\frac{\cdot}{\ell}\right))^\dagger$. Then for any $\gamma \in (\Gamma_1^{\mathbb{Q}}(N))^p := (\Gamma_1^K(N))^p \cap \mathrm{SL}_2(\mathbb{Z}[1/p])$, the diagonal restriction $\Psi_L(\delta)(\gamma) \mid (\tau, \tau)$ has divisors encoded in the collection $\{d_n\}_{n=1}^{m_0}$, where

$$d_n = \sum_{\ell j^2 \leq m_0 - n} c_{n+\ell j^2}. \tag{1.1}$$

This collection of integers possesses an interesting property. For any $h \in S_{3/2}\big(\Gamma_0(Mp), \left(\frac{\cdot}{\ell}\right)\big)$ (see Definition 7.10 for a precise definition), we have $\sum_{n=1}^{m_0} d_n \times n$-th coefficient of $h = 0$. In fact, the polynomial $\sum_{n=1}^{m_0} d_n q^{-n}$ is obtained by taking the principal part of the formal Laurent series

$$\theta(\ell\tau) \times \left(\sum_{m=1}^{m_0} c_m q^{-m} \right).$$

In this sense, the collection $\{d_n\}_{n=1}^{m_0}$ can be seen as an element that annihilates the space of cusp forms of weight 3/2 for $\Gamma_0(4\ell Np)$ and character $\left(\frac{\cdot}{\ell}\right)$. We give a precise description of the set of such collections in Section 7.4 (see Definition 7.11) and denote the set by $\mathrm{PP}_{1/2}(4\ell Np, \left(\frac{\cdot}{\ell}\right))^\dagger$. The following informal diagram captures the theme of our two main theorems.

$$\begin{tikzcd}
\mathrm{PP}_0(4\ell Np, \left(\frac{\cdot}{\ell}\right))^{\dagger} \arrow[r, "\mathrm{BL}_p"] \arrow[d, "\begin{array}{c}\text{Multiplication by}\\ \theta(\ell\tau)\end{array}"'] & Z^1((\Gamma_1^K(N))^p, \mathcal{M}(\mathcal{H}_p \times \mathcal{H}_p)/\mathbb{Z}_p^{\times}) \arrow[d, "\begin{array}{c}\text{Pullback along embedding}\\ \text{of } \mathcal{H}_p \text{ into } \mathcal{H}_p \times \mathcal{H}_p\end{array}"] \\
\mathrm{PP}_{1/2}(4\ell Np, \left(\frac{\cdot}{\ell}\right))^{\dagger} \arrow[r, "\mathrm{BL}_p"] & Z^1((\Gamma_1^{\mathbb{Q}}(N))^p, \mathcal{M}(\mathcal{H}_p)/\mathbb{Z}_p^{\times})
\end{tikzcd}$$

Remark 2. *The map in the bottom row is a generalization of the results of Darmon and Vonk [6] to congruence subgroups of* $\mathrm{SL}_2(\mathbb{Z}[1/p])$*. We are not able to find a statement as clean as their statement. However, Theorem 3.24 of [5] clearly applies to signature* $(2,1)$*, yielding a lifting of this kind under additional assumptions similar to those in Theorem 1.3.*

What if? If there were no assumptions in Theorem 1.3, we would have a lift from the space of weakly holomorphic modular forms of weight 0 to the space of rigid meromorphic Bianchi cocycles. Combining this with the calculations for the diagonal restriction would yield the following diagram.

$$\begin{tikzcd}
\mathrm{WHF}_0 \arrow[r, "\mathrm{BL}_p"] \arrow[d, "\begin{array}{c}\text{Multiplication by}\\ \text{a unary } \theta\text{-series}\end{array}"'] & \begin{array}{c}\text{Rigid cocycles for}\\ p\text{-arithmetic Bianchi groups}\end{array} \arrow[d, "\begin{array}{c}\text{Pullback along embedding}\\ \text{of } \mathcal{H}_p \text{ into } \mathcal{H}_p \times \mathcal{H}_p\end{array}"] \\
\mathrm{WHF}_{1/2} \arrow[r, "\mathrm{BL}_p"] & \begin{array}{c}\text{Rigid cocycles for}\\ \text{congruence subgroups of } \mathrm{SL}_2(\mathbb{Z}[1/p])\end{array}
\end{tikzcd}$$

It would be worthwhile to prove more precise and general statements of this sort, thereby bringing the analogy between rigid meromorphic Bianchi cocycles and Hilbert modular forms into sharper focus.

2

Vector-Valued Modular Forms

In the first section, we define vector-valued modular forms. These are functions taking values in the group ring $\mathbb{C}[D]$, where D is a finite quadratic module, or equivalently, a finite abelian group equipped with a quadratic form. These functions satisfy nice transformation properties under the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$, which is a double cover of $\mathrm{SL}_2(\mathbb{Z})$. This section is inspired by [5, Sections 3.2.1 and 3.2.6].

We discuss the decomposition of a finite quadratic module in the second section. A finite quadratic module D is, in particular, a finite abelian group and thus decomposes into its p-primary components. We show that each component can be equipped with a quadratic structure such that the decomposition as an abelian group respects the quadratic structure. The decomposition of D allows us to write $\mathbb{C}[D]$ as a tensor product of vector spaces, each having an action of $\mathrm{Mp}_2(\mathbb{Z})$. The main result is that the tensor product of representations of components is isomorphic to the representation of $\mathbb{C}[D]$. The underlying arguments are already present in [23]. We give complete proofs because the statements we require do not follow immediately from *loc. cit.* The main results are fairly straightforward to grasp but the proofs are quite lengthy. We advise the reader to read the main results first and look at the proofs only if they are interested in verification.

The third section details the explicit action of certain subgroups of $\mathrm{Mp}_2(\mathbb{Z})$ on finite quadratic modules, following [21]. This provides the requisite machinery for constructing scalar-valued modular forms from vector-valued modular forms that we need in Chapter 4.

2.1 Vector-Valued Modular Forms

Let $\mathcal{H} \subset \mathbb{C}$ be the usual complex upper-half plane. The group $\mathrm{GL}_2(\mathbb{R})^+ \subset \mathrm{GL}_2(\mathbb{R})$ of matrices with positive determinant acts on $\mathcal{H}$ by Möbius transformations. That is,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$, let

$$j(g, \tau) : \mathcal{H} \to \mathbb{C}, \qquad \tau \mapsto c\tau + d$$

be the usual automorphy factor. The metaplectic two-fold cover $\widetilde{\mathrm{GL}}_2(\mathbb{R})^+$ of $\mathrm{GL}_2(\mathbb{R})^+$ consists of pairs $(g, \phi_g(\tau))$, where $\phi_g : \mathcal{H} \to \mathbb{C}$ a holomorphic function such that $\phi_g(\tau)^2 = c\tau + d$. The multiplication is given by the law $(g, \phi(\tau)) \cdot (g', \psi(\tau)) = (gg', \phi(g'\tau)\psi(\tau))$. We have a group homomorphism $\pi : \widetilde{\mathrm{GL}}_2(\mathbb{R})^+ \to \mathrm{GL}_2(\mathbb{R})^+$ such that $\pi((g, \phi_g)) = g$. The inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under this homomorphism, denoted by $\mathrm{Mp}_2(\mathbb{Z})$, is generated by

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right), \qquad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right),$$

where $\sqrt{\tau}$ denotes the principal branch of the complex square root, defined such that $-\pi/2 < \arg(\sqrt{\tau}) \leq \pi/2$. The fact that these two elements generate $\mathrm{Mp}_2(\mathbb{Z})$ can be established using the standard generators of $\mathrm{SL}_2(\mathbb{Z})$, and noting that

$$S^4 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, -1 \right).$$

Definition 2.1. *A finite quadratic module (D, Q) is a finite abelian group D equipped with a function $Q : D \to \mathbb{Q}/\mathbb{Z}$ satisfying*

1. *$Q(nx) = n^2 Q(x)$ for all $x \in D$.*

2. *The pairing $B(x, y) := Q(x + y) - Q(x) - Q(y)$ is a symmetric bilinear pairing.*

The level N_D of D is the smallest positive integer such that $N_D(Q(x)) = 0$ for all $x \in D$. Its signature $sign(D) \in \mathbb{Z}/8\mathbb{Z}$ is defined by the formula

$$e^{2\pi i \ \mathrm{sign}(D)/8} = \frac{1}{\sqrt{|D|}} \sum_{x \in D} e^{-2\pi i Q(x)}. \tag{2.1}$$

We denote the right hand side by $\mathfrak{g}(D)$. Recall the Weil representation $\widetilde{\varrho_D} : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}[D])$. We denote by $\vec{\boldsymbol{e}}_x$, for $x \in D$, the standard basis of $\mathbb{C}[D]$. The Weil representation is determined by the action of the generators T and S, given by

$$\begin{aligned}
\widetilde{\varrho_D}(T)\vec{\boldsymbol{e}}_x &= e^{2\pi i Q(x)}\vec{\boldsymbol{e}}_x, \\
\widetilde{\varrho_D}(S)\vec{\boldsymbol{e}}_x &= \frac{1}{\sqrt{|D|}}\mathfrak{g}(D)^{-1}\sum_{y \in D} e(-B(x,y))\vec{\boldsymbol{e}}_y.
\end{aligned}$$

Action of the Metaplectic Group on Functions. Let D be a finite quadratic module and $k \in \frac{1}{2}\mathbb{Z}$ a half-integer. The metaplectic group $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ acts on functions $f : \mathcal{H} \to \mathbb{C}$ from the right via

$$(f \mid_k (g, \phi))(\tau) := \det(g)^{k/2}\, \phi(\tau)^{-2k}\, f(g \cdot \tau), \qquad \forall\ (g, \phi) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R}).$$

Any function $f : \mathcal{H} \to \mathbb{C}[D]$ can be written as

$$f = \sum_{x \in D} f_x \cdot \vec{\boldsymbol{e}}_x,$$

for some functions $f_x : \mathcal{H} \to \mathbb{C}$. For $\gamma \in \mathrm{Mp}_2(\mathbb{Z})$, we define

$$f \mid_k \gamma := \sum_{x \in D} (f_x \mid_k \gamma) \cdot \rho_D(\gamma)^{-1}(\vec{\boldsymbol{e}}_x).$$

Let $w \geq 1$ be an integer, and suppose that f satisfies $f \mid_k T^w = f$, or in other words,

$$f_x(\tau + w)\, e^{-2\pi i\, wQ(x)} = f_x(\tau) \quad \text{for all } x \in D.$$

Then the function

$$\tau \longmapsto f_x(\tau)\, e^{-2\pi i Q(x)\tau}$$

is periodic with period w, and thus f has a Fourier expansion of the form

$$f(\tau) = \sum_{x \in D} \sum_{m \in \mathbb{Q}} a_f(m, x)\, e^{2\pi i m \tau}\, \vec{\boldsymbol{e}}_x.$$

Moreover, in the case $w = 1$, the Fourier coefficients $a_f(m, x)$ vanish unless $m \in Q(x) + \mathbb{Z}$.

Definition 2.2. *Let $\mathcal{G} \subseteq \mathrm{Mp}_2(\mathbb{Z})$ be a subgroup. A vector-valued modular form of weight k, type ρ_D, and level $\mathcal{G}$ is a function $f : \mathcal{H} \longrightarrow \mathbb{C}[D]$ such that*

1. *$f\,|_k\,\gamma = f$ for all $\gamma \in \mathcal{G}$ and $\tau \in \mathcal{H}$,*
2. *f is holomorphic,*
3. *f is holomorphic at the cusps, that is, for every $\gamma \in \mathrm{Mp}_2(\mathbb{Z})$ the Fourier coefficients of $f\,|_k\,\gamma$ satisfy*
$$a_{f|_k\gamma}(m, x) = 0 \quad \textit{for all } m < 0.$$

2.2 Finite Quadratic Modules associated with Lattices

The results of this section are summarized below to allow the reader to bypass the proofs. The proofs involve technical details that are not utilized elsewhere in this thesis, since only the resulting statements are useful for the following chapters.

One source of finite quadratic modules is quadratic lattices. Let M be a quadratic $\mathbb{Z}$-lattice contained in its dual M^*. We use D_M to denote the quotient M^*/M. It is equipped with a natural $\mathbb{Q}/\mathbb{Z}$-valued quadratic form.

The lattices $M \otimes \mathbb{Z}_p$ and $M^* \otimes \mathbb{Z}_p$ are equipped with quadratic forms as well. Consequently, the quotient module $D_M \otimes \mathbb{Z}_p$ is equipped with a $\mathbb{Q}_p/\mathbb{Z}_p$-valued quadratic form. Similarly, $D_M \otimes \mathbb{Z}[1/p]$ is equipped with a $\mathbb{Q}/\mathbb{Z}[1/p]$-valued quadratic form.

The abelian group $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the direct sum $\mathbb{Q}_p/\mathbb{Z}_p \oplus \mathbb{Q}/\mathbb{Z}[1/p]$ (see Proposition 2.7). Therefore, both $D_M \otimes \mathbb{Z}_p$ and $D_M \otimes \mathbb{Z}[1/p]$ are equipped with a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form via the isomorphism. As a consequence, $D_M \otimes \mathbb{Z}[1/p] \oplus D_M \otimes \mathbb{Z}_p$ is naturally equipped with a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form such that the direct sum is orthogonal.

As abelian groups, D_M and $D_M \otimes \mathbb{Z}_p \oplus D_M \otimes \mathbb{Z}[1/p]$ are isomorphic. We prove that the isomorphism extends to that of $\mathbb{Q}/\mathbb{Z}$-valued quadratic modules (Proposition 2.5).

In Theorem 2.9, we establish that the Weil representation associated with D_M is isomorphic to the tensor product of the Weil representations associated with $D_M \otimes \mathbb{Z}[1/p]$ and $D_M \otimes \mathbb{Z}_p$.

2.2.1 Finite Quadratic Module over Rationals

Let M, M^*, and D_M be as before. We use $Q(\cdot)$ to denote the quadratic form on $M \otimes \mathbb{Q}$ and $B(\cdot,\cdot)$ to denote the corresponding bilinear form. Assume that $Q(M) \subseteq \mathbb{Z}$. Let $x \in M^*$ and $\overline{x}$ be its image in D_M. Define a $\mathbb{Q}/\mathbb{Z}$-valued quadratic function on D_M,

$$\begin{aligned} \overline{Q} : D_M &\to \mathbb{Q}/\mathbb{Z} \\ \overline{x} &\mapsto Q(x) + \mathbb{Z}. \end{aligned}$$

This map is well defined because if we replace x by $x+y$ for some $y \in M$, then $Q(x+y)-Q(x) = Q(y) + B(x,y) \in \mathbb{Z}$ since $Q(M) \subseteq \mathbb{Z}$ and $x \in M^*$.

2.2.2 Finite Quadratic Module over the p-adics

Consider $M \otimes \mathbb{Z}_p$ and $M^* \otimes \mathbb{Z}_p$. They are both free $\mathbb{Z}_p$-modules of finite rank, since $M \cong \mathbb{Z}^n$ implies $M \otimes \mathbb{Z}_p \cong \mathbb{Z}_p^n$ and $M^* \otimes \mathbb{Z}_p \cong \mathbb{Z}_p^n$. We also have the inclusion $M \otimes \mathbb{Z}_p \subset M^* \otimes \mathbb{Z}_p$. Furthermore, each element in $M^* \otimes \mathbb{Z}_p$ can be uniquely written as $\displaystyle\sum_{i=1}^{n} \alpha_i(f_i \otimes 1)$, where $\alpha_i \in \mathbb{Z}_p$ and $\{f_i\}_{i=1}^n$ is a basis of M^*.

The bilinear form on M^* extends naturally to $M^* \otimes \mathbb{Z}_p$. We denote it by $B_p(\cdot,\cdot)$. Also, for any $x \in M^* \otimes \mathbb{Z}_p$, we define $Q_p(x) := \dfrac{1}{2} B_p(x,x)$. We always assume that the prime p is odd.

Tensoring the exact sequence $0 \to M \to M^* \to D_M \to 0$ with $\mathbb{Z}_p$ gives

$$0 \to M \otimes \mathbb{Z}_p \to M^* \otimes \mathbb{Z}_p \to D_M \otimes \mathbb{Z}_p \to 0, \tag{2.2}$$

since $\mathbb{Z}_p$ is torsion-free and thus flat over $\mathbb{Z}$. We define a quadratic function on $D_M \otimes \mathbb{Z}_p$ that is valued in $\mathbb{Q}_p/\mathbb{Z}_p$. Let $\overline{x} \in D_M \otimes \mathbb{Z}_p$ be the image of some $x \in M^* \otimes \mathbb{Z}_p$. Define

$$\overline{Q}_p(\overline{x}) := Q_p(x) + \mathbb{Z}_p \in \mathbb{Q}_p/\mathbb{Z}_p.$$

This map is well-defined. If we replace x with $x+y$ for some $y \in M \otimes \mathbb{Z}_p$, then the difference $Q_p(x+y) - Q_p(x) = Q_p(y) + B_p(x,y)$ lies in $\mathbb{Z}_p$ because $Q_p(y) \in \mathbb{Z}_p$, and $B_p(x,y) \in \mathbb{Z}_p$, which is justified by the following calculation. Let $\{e_i\}_{i=1}^n$ denote a basis of M and $\{f_j\}_{j=1}^n$ denote the corresponding dual basis satisfying $B(e_i, f_j) = \delta_{ij}$. Using the bases, we write $y = \sum_{i=1}^n \alpha_i(e_i \otimes 1) \in M \otimes \mathbb{Z}_p$ and $x = \sum_{j=1}^n \beta_j(f_j \otimes 1) \in M^* \otimes \mathbb{Z}_p$. Since $B(e_i, f_j) \in \mathbb{Z}$ and

$\alpha_i, \beta_j \in \mathbb{Z}_p$ for all i, j, it follows that $B_p(x,y) = \sum_{i,j=1}^{n} \alpha_i \beta_j B(e_i, f_j) \in \mathbb{Z}_p$.

From the exact sequence (2.2), we have a natural map $\overline{i_p} : D_M \to D_M \otimes \mathbb{Z}_p$ such that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "i_p"] & M^* \arrow[r] \arrow[d, "i_p"] & D_M \arrow[r] \arrow[d, dotted, "\overline{i_p}"] & 0 \\
0 \arrow[r] & M \otimes \mathbb{Z}_p \arrow[r] & M^* \otimes \mathbb{Z}_p \arrow[r] & D_M \otimes \mathbb{Z}_p \arrow[r] & 0
\end{tikzcd}$$

For any $x \in M^*$ we have $\overline{x} \in D_M$ and $i_p(x) \in M^* \otimes \mathbb{Z}_p$. Using the commutativity of the diagram, we have $\overline{i_p}(\overline{x}) = \overline{i_p(x)}$, where $\overline{i_p(x)}$ is the image under the map $M^* \otimes \mathbb{Z}_p \to D_M \otimes \mathbb{Z}_p$.

The module $D_M \otimes \mathbb{Z}_p$ can be seen as a $\mathbb{Q}/\mathbb{Z}$-valued quadratic module by using the map $\mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z}$ defined below.

Definition 2.3. *Any $v \in \mathbb{Q}_p/\mathbb{Z}_p$ has a representative of the form a/p^n for some integer a coprime to p. We define the map j_p as*

$$\begin{aligned} j_p : \mathbb{Q}_p/\mathbb{Z}_p &\to \mathbb{Q}/\mathbb{Z} \\ \frac{a}{p^n} + \mathbb{Z}_p &\mapsto \frac{a}{p^n} + \mathbb{Z} \end{aligned}$$

The map j_p is well-defined. Suppose $\frac{a}{p^n} \equiv \frac{b}{p^m} \pmod{\mathbb{Z}_p}$. Then the difference $D = \frac{a}{p^n} - \frac{b}{p^m}$ is in $\mathbb{Z}_p$. Since the difference $D \in \mathbb{Q}$ has only powers of p in its denominator, it must be an integer. Hence the image is well-defined in $\mathbb{Q}/\mathbb{Z}$.

2.2.3 Finite Quadratic Module over $\mathbb{Z}[1/p]$

We consider similar ideas but with the ring $\mathbb{Z}[1/p]$ instead of $\mathbb{Z}_p$. Both $M \otimes \mathbb{Z}[1/p]$ and $M^* \otimes \mathbb{Z}[1/p]$ are free $\mathbb{Z}[1/p]$-modules of finite rank. Since $M^* \otimes \mathbb{Z}[1/p]$ is a subset of the vector space $M^* \otimes \mathbb{Q}$, the bilinear form is well-defined. We use $B^p(\cdot,\cdot)$ to denote the bilinear form on $M^* \otimes \mathbb{Z}[1/p]$.

The exact sequence $0 \to M \to M^* \to D_M \to 0$ gives

$$0 \to M \otimes \mathbb{Z}[1/p] \to M^* \otimes \mathbb{Z}[1/p] \to D_M \otimes \mathbb{Z}[1/p] \to 0,$$

since $\mathbb{Z}[1/p]$ is torsion-free and thus a flat module over $\mathbb{Z}$. We now define a quadratic function on $D_M \otimes \mathbb{Z}[1/p]$. Let $u \in D_M \otimes \mathbb{Z}[1/p]$ and let $x \in M^* \otimes \mathbb{Z}[1/p]$ be a preimage of u. Define

$$\overline{Q}^p(u) := Q(x) + \mathbb{Z}[1/p] \in \mathbb{Q}/\mathbb{Z}[1/p].$$

The definition makes sense as can be seen using the arguments previously stated and the following calculation. Let $\{e_i\}_{i=1}^n$ denote a basis of M, and let $\{f_j\}_{j=1}^n$ be the corresponding dual basis of M^*. Suppose that $y = \sum_{i=1}^n \alpha_i e_i \in M \otimes \mathbb{Z}[1/p]$ and $x = \sum_{j=1}^n \beta_j f_j \in M^* \otimes \mathbb{Z}[1/p]$, we have

$$B^p(x, y) = \sum_{i,j=1}^n \alpha_i \beta_j B(e_i, f_j) \in \mathbb{Z}[1/p]$$

as $B(e_i, f_j) \in \mathbb{Z}$ for all i, j.

From the exact sequences seen before, we have a natural map $\overline{i^p} : D_M \to D_M \otimes \mathbb{Z}[1/p]$ such that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "i^p"] & M^* \arrow[r] \arrow[d, "i^p"] & D_M \arrow[r] \arrow[d, dotted, "\overline{i^p}"] & 0 \\
0 \arrow[r] & M \otimes \mathbb{Z}[1/p] \arrow[r] & M^* \otimes \mathbb{Z}[1/p] \arrow[r] & D_M \otimes \mathbb{Z}[1/p] \arrow[r] & 0
\end{tikzcd}$$

For any $x \in M^*$, we have $\overline{x} \in D_M$ and $i^p(x) \in M^* \otimes \mathbb{Z}[1/p]$. Using the commutativity of the digaram, we have $\overline{i^p}(\overline{x}) = \overline{i^p(x)}$ where $\overline{i^p(x)}$ is the image of $i^p(x)$ under the map $M^* \otimes \mathbb{Z}[1/p] \to D_M \otimes \mathbb{Z}[1/p]$.

The module $D_M \otimes \mathbb{Z}[1/p]$ can be seen as a $\mathbb{Q}/\mathbb{Z}$-valued quadratic module by using a map $\mathbb{Q}/\mathbb{Z}[1/p] \to \mathbb{Q}/\mathbb{Z}$ defined below.

Lemma 2.4. *For any $u \in \mathbb{Q}/\mathbb{Z}[1/p]$, there exists a representative $r/s \in \mathbb{Q}$ such that $\gcd(s, p) = 1$. Furthermore, the map*

$$\begin{aligned} j^p : \mathbb{Q}/\mathbb{Z}[1/p] &\to \mathbb{Q}/\mathbb{Z} \\ u &\mapsto \frac{r}{s} + \mathbb{Z} \end{aligned}$$

is well-defined.

Proof. Let $x = r/s$ be an arbitrary representative of u with $\gcd(r, s) = 1$. We write the denominator s as $p^\alpha N$ such that $\gcd(p, N) = 1$. We seek a representative whose denominator is purely N. Since N is coprime to p^α, there exists an integer r' satisfying the congruence $Nr' \equiv r \pmod{p^\alpha}$. It follows that $r - Nr' = kp^\alpha$ for some integer k. We then have

$$\frac{r}{p^\alpha N} - \frac{r'}{p^\alpha} = \frac{r - Nr'}{p^\alpha N} = \frac{kp^\alpha}{p^\alpha N} = \frac{k}{N}.$$

Since $r'/p^\alpha \in \mathbb{Z}[1/p]$, the rational k/N represents the same class as r/s in the quotient $\mathbb{Q}/\mathbb{Z}[1/p]$. As $\gcd(N,p) = 1$ by construction, we have the required representative.

We now establish that the definition is independent of the representative. Suppose $\frac{r}{s} \equiv \frac{r'}{s'} \pmod{\mathbb{Z}[1/p]}$ with $\gcd(s,p) = 1$ and $\gcd(s',p) = 1$. This means the difference $D = \frac{r}{s} - \frac{r'}{s'}$ belongs to $\mathbb{Z}[1/p]$. Calculating the difference directly gives $D = (rs' - r's)/ss'$. Since p divides neither s nor s', the denominator ss' is coprime to p and $D \in \mathbb{Z}$. This means both r/s and r'/s' give the same coset in $\mathbb{Q}/\mathbb{Z}$. $\square$

2.2.4 Decomposition of Finite Quadratic Modules

We discuss the decomposition of D_M into two finite quadratic modules $D_M \otimes \mathbb{Z}_p$ and $D_M \otimes \mathbb{Z}[1/p]$. The group $D_M \otimes \mathbb{Z}_p$ is isomorphic to the p-primary part of D_M. Furthermore, $D_M \otimes \mathbb{Z}[1/p]$ is isomorphic to the subgroup of elements of D_M that have order relatively prime to p. Hence as an abelian group, D_M is isomorphic to the direct sum of these two groups. We establish the isomorphism as quadratic modules as well. The main result of this section is the following proposition.

Proposition 2.5. *The map $\overline{i_p} \oplus \overline{i^p} : D_M \to (D_M \otimes \mathbb{Z}_p) \oplus (D_M \otimes \mathbb{Z}[1/p])$ is an isometry where the quadratic form for the right-hand side is $j_p \circ \overline{Q}_p \oplus j^p \circ \overline{Q}^p$.*

We first need to discuss the decomposition of the abelian group $\mathbb{Q}/\mathbb{Z}$ into $\mathbb{Q}_p/\mathbb{Z}_p$ and $\mathbb{Q}/\mathbb{Z}[1/p]$. We already have maps $j_p : \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z}$ and $j^p : \mathbb{Q}/\mathbb{Z}[1/p] \to \mathbb{Q}/\mathbb{Z}$. We now give definitions for maps in the opposite direction. The proposition following the definition establishes the desired decomposition.

Definition 2.6. *Let $x = r/s$ be a rational number and use $\overline{x}$ to denote its image in $\mathbb{Q}/\mathbb{Z}$. We can write the denominator as $s = p^\alpha N$, where N is an integer coprime to p. Let t_1, t_2 be integers such that $t_1 p^\alpha + t_2 N = 1$. We then have*

$$\frac{r}{s} = \frac{rt_1}{N} + \frac{rt_2}{p^\alpha}$$

and thus define

$$\overline{x}_p := \frac{rt_2}{p^\alpha} + \mathbb{Z}_p, \qquad \overline{x}^p := \frac{rt_1}{N} + \mathbb{Z}[1/p].$$

It is clear that the choice of t_1, t_2 does not affect the definition. Therefore we have defined maps $(\ )_p : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}_p/\mathbb{Z}_p$ and $(\ )^p : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}[1/p]$.

Proposition 2.7. *The map $(\ )_p \oplus (\ )^p : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}_p/\mathbb{Z}_p \oplus \mathbb{Q}/\mathbb{Z}[1/p]$ is an isomorphism.*

Proof. We show that $j_p + j^p$ defines an inverse to the map in the statement. This is a direct calculation. We only show one composition.

Let $x = r/s$ be a rational number for some integers r, s. We write $s = p^\alpha N$, where N is coprime to p. Let t_1, t_2 be integers such that $t_1 p^\alpha + t_2 N = 1$. Then we have

$$\begin{aligned} j_p(\overline{x}_p) + j^p(\overline{x}^p) &= j_p\left(\frac{rt_2}{p^\alpha} + \mathbb{Z}_p\right) + j^p\left(\frac{rt_1}{N} + \mathbb{Z}[1/p]\right) \\ &= \left(\frac{rt_2}{p^\alpha} + \mathbb{Z}\right) + \left(\frac{rt_1}{N} + \mathbb{Z}\right) = \frac{r(t_1p^\alpha + t_2N)}{s} + \mathbb{Z} = \overline{x}. \end{aligned}$$

Thus, we have established that the map $\mathbb{Q}/\mathbb{Z} \to \mathbb{Q}_p/\mathbb{Z}_p \oplus \mathbb{Q}/\mathbb{Z}[1/p] \to \mathbb{Q}/\mathbb{Z}$ is identity. The other composition follows similarly by applying definitions of the maps involved. $\square$

We now consider the abelian group $(D_M \otimes \mathbb{Z}_p) \oplus (D_M \otimes \mathbb{Z}[1/p])$ and the $\mathbb{Q}/\mathbb{Z}$-valued quadratic function

$$\begin{aligned} j_p \circ \overline{Q}_p \oplus j^p \circ \overline{Q}^p : (D_M \otimes \mathbb{Z}_p) \oplus (D_M \otimes \mathbb{Z}[1/p]) &\to \mathbb{Q}/\mathbb{Z} \\ (u, v) &\mapsto j_p(\overline{Q}_p(u)) + j^p(\overline{Q}^p(v)), \end{aligned}$$

By definition, the direct sum is orthogonal. The norm of (u, v) equals the sum of norm $(u, 0)$ and that of $(0, v)$.

We are almost ready to prove Proposition 2.5. There is only one technical lemma we require.

Lemma 2.8. *Let $\phi_1 : \mathbb{Q} \to \mathbb{Q}_p/\mathbb{Z}_p$ be the natural projection map defined by $x \mapsto x + \mathbb{Z}_p$. Let $\phi_2 : \mathbb{Q} \to \mathbb{Q}_p/\mathbb{Z}_p$ be the composition $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \xrightarrow{(\ )_p} \mathbb{Q}_p/\mathbb{Z}_p$. Then $\phi_1 = \phi_2$.*

Proof. Let $x = r/s \in \mathbb{Q}$ and write $s = p^\alpha N$ with $\gcd(N, p) = 1$. By Bézout's identity, there exist integers t_1, t_2 such that $t_1 p^\alpha + t_2 N = 1$. We decompose x as:

$$x = \frac{r(t_1 p^\alpha + t_2 N)}{p^\alpha N} = \frac{rt_1}{N} + \frac{rt_2}{p^\alpha}$$

By the definition of the map $(\ )_p$ on $\mathbb{Q}/\mathbb{Z}$, we have $\phi_2(x) = \overline{x}_p = \frac{rt_2}{p^\alpha} + \mathbb{Z}_p$. Now consider the image under ϕ_1. In the field $\mathbb{Q}_p$, the integer N is a p-adic unit because $p \nmid N$, hence $1/N \in \mathbb{Z}_p$. It follows that $rt_1/N \in \mathbb{Z}_p$. Thus:

$$\phi_1(x) = \left(\frac{rt_1}{N} + \frac{rt_2}{p^\alpha}\right) + \mathbb{Z}_p = \left(\frac{rt_1}{N} + \mathbb{Z}_p\right) + \left(\frac{rt_2}{p^\alpha} + \mathbb{Z}_p\right)$$

Since $rt_1/N \in \mathbb{Z}_p$, it vanishes in the quotient, yielding $\phi_1(x) = \frac{rt_2}{p^\alpha} + \mathbb{Z}_p$. Therefore, $\phi_1(x) = \phi_2(x)$ for all $x \in \mathbb{Q}$. □

Proof of Proposition 2.5. Let $x \in M^*$ with image $\overline{x} \in D_M$. We want to prove that the map $\overline{i_p} \oplus \overline{i^p} : D_M \to (D_M \otimes \mathbb{Z}_p) \oplus (D_M \otimes \mathbb{Z}[1/p])$ is an isometry. In other words, we must prove that

$$\overline{Q}(\overline{x}) = j_p \circ \overline{Q}_p(\overline{i_p}(\overline{x})) + j^p \circ \overline{Q}^p(\overline{i^p}(\overline{x})). \tag{2.3}$$

We prove this by establishing two equalities separately. We show that $j_p \circ \overline{Q}_p(\overline{i_p}(\overline{x})) = j_p((\overline{Q}(\overline{x}))_p)$ and $j^p \circ \overline{Q}^p(\overline{i^p}(\overline{x})) = j^p((\overline{Q}(\overline{x}))^p)$. Thus using the identity in Proposition 2.7 we get (2.3). The proofs of the two equalities mentioned involve some tedious verifications.We do this step by step for the first equality. Consider $j_p \circ \overline{Q}_p(\overline{i_p}(\overline{x}))$.

1. We know that $\overline{i_p}(\overline{x}) = \overline{i_p(x)}$ by the definition of $\overline{i_p}$.

2. By the definition of $\overline{Q}_p$ it is clear that $\overline{Q}_p(\overline{i_p(x)}) = \overline{Q_p(i_p(x))}$ where the overline over the right-hand side means that we are considering the image of $Q_p(i_p(x))$ under the map $\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p$.

3. It is obvious that $Q_p(i_p(x)) = Q(x)$.

4. Using the lemma 2.8, which states that $\mathbb{Q} \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p$ and $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}_p/\mathbb{Z}_p$ coincide, we have

$$\overline{Q_p(i_p(x))} = \overline{Q(x)}_p$$

5. Finally the term $j_p \circ \overline{Q}_p(\overline{i_p}(\overline{x}))$ is equal to $j_p(\overline{Q(x)}_p)$.

Each step of the above reasoning can be applied to the case of $\mathbb{Z}[1/p]$, and thus $j^p \circ \overline{Q}^p(\overline{i^p}(\overline{x})) = j^p(\overline{Q(x)}^p)$. Hence by Proposition 1.6, we conclude that $\overline{i_p} \oplus \overline{i^p}$ is an isomorphism of quadratic modules. □

2.2.5 The Weil Representation

In this section, we establish the following theorem.

Theorem 2.9. *The Weil representation associated with D_M is isomorphic to the tensor product of representations associated with $D_M \otimes \mathbb{Z}_p$ and $D_M \otimes \mathbb{Z}[1/p]$.*

Before proceeding to the proof, we prove a proposition from which the theorem follows easily.

Proposition 2.10. *Let D be a finite quadratic module associated with a quadratic lattice such that it is isomorphic to $D_1 \oplus D_2$ where D_1, D_2 are also finite quadratic modules associated with lattices and the direct sum is orthogonal. We then have*

$$\varrho_{D_1 \oplus D_2} \cong \varrho_{D_1} \otimes \varrho_{D_2}.$$

Proof. The underlying vector space $\mathbb{C}[D]$ is isomorphic to $\mathbb{C}[D_1] \otimes \mathbb{C}[D_2]$. This can be established by using the isomorphism $D = D_1 \oplus D_2$. It remains to show that the action of the metaplectic group is compatible with this identification.

$$\begin{aligned}\varrho_D(T)(\vec{e}_{(x_1,x_2)}) &= e^{2\pi i Q(x_1,x_2)}\vec{e}_{(x_1,x_2)} \\ &= e^{2\pi i (Q_1(x_1)+Q_2(x_2))}\vec{e}_{(x_1,x_2)}\end{aligned}$$

and this maps to $e^{2\pi i Q_1(x_1)}\vec{e}_{x_1} \otimes e^{2\pi i Q_2(x_2)}\vec{e}_{x_2}$ under the morphism of representations.

The calculation for S is slightly more complicated.

$$\varrho_{D_1}(S)(\vec{e}_{x_1}) \otimes \varrho_{D_2}(S)(\vec{e}_{x_2}) = C \sum_{\substack{y_1 \in D_1 \\ y_2 \in D_2}} A(y_1, y_2)\vec{e}_{y_1} \otimes \vec{e}_{y_2}$$

where

$$C = \frac{1}{\sqrt{|D_1|}}\mathfrak{g}(D_1)^{-1} \times \frac{1}{\sqrt{|D_2|}}\mathfrak{g}(D_2)^{-1}$$

and

$$A(y_1, y_2) = e(-B_1(x_1, y_1) - B_2(x_2, y_2)) = e(-B(x_1 + x_2, y_1 + y_2))$$

It is clear that $|D_1 \oplus D_2| = |D_1| \times |D_2|$. Futher we compute

$$\begin{aligned}\sum_{x_1 \in D_1} e^{-2\pi i Q_1(x_1)} \times \sum_{x_2 \in D_2} e^{-2\pi i Q_2(x_2)} &= \sum_{\substack{x_1 \in D_1 \\ x_2 \in D_2}} e^{-2\pi i Q_1(x_1)} \times e^{-2\pi i Q_2(x_2)} \\ &= \sum_{(x_1, x_2) \in D} e^{-2\pi i Q((x_1, x_2))}.\end{aligned}$$

Therefore $\mathfrak{g}(D_1)\mathfrak{g}(D_2) = \mathfrak{g}(D)$. □

Proof of Theorem 2.9. By Proposition 2.5, the direct sum $D_M \otimes \mathbb{Z}[1/p] \oplus D_M \otimes \mathbb{Z}_p$ is isomorphic to the quadratic module D_M. Thus their associated Weil representations are isomorphic. By Proposition 2.10, the Weil representation associated with the orthogonal direct sum is isomorphic to the tensor product of the summands. Consequently, ϱ_{D_M} is isomorphic to the tensor product of the Weil representations associated with the modules $D_M \otimes \mathbb{Z}_p$ and $D_M \otimes \mathbb{Z}[1/p]$. □

2.3 Explicit Action of the Metaplectic Group

The Weil representation is completely determined by the action of the generators.

$$\begin{aligned}\widetilde{\varrho_D}(T)\vec{\boldsymbol{e}}_x &= e^{2\pi i Q(x)}\vec{\boldsymbol{e}}_x, \qquad \text{and} \\ \widetilde{\varrho_D}(S)\vec{\boldsymbol{e}}_x &= \frac{1}{\sqrt{|D|}}\mathfrak{g}(D)^{-1}\sum_{y \in D} e(-B(x, y))\vec{\boldsymbol{e}}_y.\end{aligned}$$

It is known that the element $\tilde{Z} := S^2$ generates the center of $\mathrm{Mp}_2(\mathbb{Z})$ and has order four (i.e., $\tilde{Z}^4 = I$). The explicit action of the center is given by the following lemma.

Lemma 2.11. *Let $\widetilde{\varrho_D}$ be the Weil representation. Then*

$$\widetilde{\varrho_D}(\tilde{Z})\vec{\boldsymbol{e}}_x = \mathfrak{g}(D)^2\vec{\boldsymbol{e}}_{-x}.$$

Proof. See Equation 5.1 and Lemma 5.2 in [21]. □

Remark 3. *Note that if* $\mathrm{sign}(D)$ *is even then* $\mathfrak{g}(D)^4 = 1$*. In that case* $\tilde{Z}^2$ *acts trivially and the action of* $\mathrm{Mp}_2(\mathbb{Z})$ *induces an action of* $\mathrm{SL}_2(\mathbb{Z})$*.*

We restrict ourselves to the case where $\mathrm{sign}(D)$ is even.

Lemma 2.12. *Let D be a finite quadratic module for which* $\mathrm{sign}(D)$ *is even and level is ℓ. Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\ell)$*. Then*

$$\widetilde{\varrho_D} \vec{\boldsymbol{e}}_\alpha = e(bdQ(\alpha))\ \chi_D(A)\ \vec{\boldsymbol{e}}_{d\alpha},$$

where

1. *If* $4 \nmid \ell$*, then*

$$\chi_D(A) = \left(\frac{d}{|D|}\right).$$

2. *If* $4 \mid \ell$*, then*

$$\chi_D(A) = \left(\frac{d}{|D| 2^{\mathrm{sign}(D)}}\right) \times \begin{cases} \left(\frac{-1}{d}\right) & \mathrm{sign}(D) + \left(\frac{-1}{|D|}\right) \equiv 1 (mod\ 4) \\ 1 & \mathrm{sign}(D) + \left(\frac{-1}{|D|}\right) \equiv 3 (mod\ 4). \end{cases}$$

Proof. See Lemma 5.6 of [21] and combine it with Definition 5.4 for even $\mathrm{sign}(D)$ to obtain this result. See also Lemma 5.13 and Definition 5.10 in [22]. □

3

Fourier Coefficients of Theta Series associated to Signature (3,1)

In this chapter, we present the work of Funke and Millson [9] and Kudla [15, 16] on theta series associated to lattices of signature $(3,1)$. The Fourier coefficients of these theta series are essential ingredients in the construction of rigid meromorphic cocycles. Our presentation is a specialization of their results so that they fit our objectives and yield transparent theorems.

The first section of this chapter introduces a theta series associated to a quadratic lattice L of signature $(3,1)$. This series is a smooth differential 1-form on the three-dimensional manifold $\mathcal{H}^{(3)} := \mathbb{C} \times \mathbb{R}_{>0}$. Integrating this 1-form along a geodesic joining $(s,0)$ to $(t,0)$ in $\mathcal{H}^{(3)}$ yields a modular form. We are interested in the cases where s and t are elements of an imaginary quadratic field K whose discriminant is equal to that of the vector space $L \otimes \mathbb{Q}$. Computations in the second section reveal that the Fourier coefficients of the modular form are composed of two types of quantities. The first type consists of sums of integrals of 1-forms associated to vectors of a fixed length, while the second type is a combination of special values of the Hurwitz zeta function. In the third section, we utilize the coefficients of the first type to construct modular symbols invariant under suitable congruence subgroups of $\mathrm{SL}_2(\mathcal{O}_K)$. However, it remains a mystery to us how one can use the second type of coefficients to construct cocycles.

3.1 Symmetric Space and Theta Series

Let $V_{\mathbb{R}}$ be a quadratic space over $\mathbb{R}$ with signature $(n,1)$ for $n \geq 1$. Fix an orthonormal basis $\{e_1, \ldots, e_n, f\}$ such that $(e_i, e_i) = 1$ and $(f,f) = -1$. Define $\widetilde{B} := \{Z \in V_{\mathbb{R}} : (Z,Z) = -1\}$. Using this basis, we express $\widetilde{B}$ as the locus

$$\widetilde{B} = \left\{ \sum_{i=1}^{n} x_i e_i + yf \mid y^2 = 1 + \sum_{i=1}^{n} x_i^2 \right\}.$$

The condition $y^2 = 1 + \sum_{i=1}^{n} x_i^2$ shows that $\widetilde{B}$ consists of two connected components. We choose one connected component and denote it by B. This is a real manifold of dimension n, as can be seen by sending each $Z = \sum_{i=1}^{n} x_i e_i + yf \in B$ to the tuple $(x_1, x_2, \ldots, x_n)$.

Let $T_Z(B)$ denote the tangent space at $Z \in B$. We identify $T_Z(B)$ with $Z^{\perp}$. Since $V_{\mathbb{R}}$ is a quadratic space of signature $(n,1)$ and Z is of negative length, the bilinear form on $Z^{\perp}$ is positive definite. Therefore, the tangent space $T_Z(B)$ is a positive definite quadratic space and B has a natural Riemannian structure induced from $V_{\mathbb{R}}$.

We now describe a family of 1-forms on B. For $X \in V_{\mathbb{R}}$ and $Z \in B$, we define $\eta(X)_Z$ as

$$\begin{aligned} \eta(X)_Z : T_Z(B) &\to \mathbb{R} \\ W &\mapsto (X,W) e^{-\pi (X,X)_Z}, \end{aligned} \tag{3.1}$$

where $(X,X)_Z$ is defined as follows. For $X \in V_{\mathbb{R}}$, we use X_Z to denote the projection of X onto $\operatorname{span}(Z)$ and $X_{Z^{\perp}}$ to denote the projection onto $\operatorname{span}(Z)^{\perp}$. We then have $X = X_{Z^{\perp}} + X_Z$ and define

$$(X,X)_Z := (X_{Z^{\perp}}, X_{Z^{\perp}}) - (X_Z, X_Z) > 0.$$

Note that $(X,X)_Z$ is non-negative because the bilinear form is positive definite on $Z^{\perp}$ and $(X_Z, X_Z) \leq 0$. Therefore, for a vector $W \in T_Z(B)$, the function $\eta(\cdot)_Z(W) : V_{\mathbb{R}} \to \mathbb{R}$ is an infinitely differentiable and rapidly decaying function on $V_{\mathbb{R}}$. We modify $\eta(X)$ to define a 1-form valued in functions on $\mathcal{H}$. Let $\tau = u + iv \in \mathcal{H}$ and define

$$\begin{aligned} \eta(\tau, X)_Z : T_Z(B) &\to \mathbb{C} \\ W &\mapsto \sqrt{v}\ (X,W)\ e^{\pi i u (X,X)}\ e^{-\pi v (X,X)_Z}. \end{aligned} \tag{3.2}$$

The 1-form $\eta(X)$ in (3.1) is $\eta(\tau, X)$ evaluated at $\tau = i$. Therefore we have a family of 1-forms in B that are valued in functions on $\mathcal{H}$ and indexed by $X \in V_{\mathbb{R}}$. The following theorem packages these 1-forms into a theta series and establishes its modularity.

Theorem 3.1. *Let L be a lattice in $V_{\mathbb{R}}$. Let D_L be the finite module L^*/L. Let $h \in L^*$ and define*

$$\Theta(\tau, h + L) := \sum_{X \in h+L} \eta(\tau, X).$$

This is a real analytic function valued in 1-forms on B and depends only on the coset of L in L^*. The series

$$\sum_{h \in D_L} \Theta(\tau, h + L)\,\vec{\mathbf{e}}_h \tag{3.3}$$

transforms like a vector-valued modular form of weight $n/2$ and type ϱ_L, where ϱ_L is the Weil representation associated to the lattice L.

There are well-known exceptional isomorphisms between low-dimensional orthogonal groups and special linear groups. We focus our attention on the case of the orthogonal group associated to real quadratic spaces of signature $(3,1)$. For the remainder of this chapter, we fix the signature of $V_{\mathbb{R}}$ to be $(3,1)$. The following quadratic space serves as a model for $V_{\mathbb{R}}$:

$$\left(\left\{\begin{pmatrix} x+iy & b \\ c & -x+iy \end{pmatrix} : x, y, b, c \in \mathbb{R}\right\}, -\det\right).$$

The group $\mathrm{SL}_2(\mathbb{C})$ acts on $V_{\mathbb{R}}$ in the following sense. Let M be a matrix in $V_{\mathbb{R}}$. Then the action of a $\gamma \in \mathrm{SL}_2(\mathbb{C})$ is given by $M \mapsto \gamma M \overline{\gamma}^{-1}$. This is an isometry as the determinant remains unchanged. Therefore $\mathrm{SL}_2(\mathbb{C})$ acts by isometries on a quadratic space of signature $(3,1)$. We refer the reader to Section 1.3 of [8, Proposition 3.8-3.12] for more details on this connection.

The associated 3-manifold $B(V_{\mathbb{R}})$ is in bijection with the upper half-space $\mathcal{H}^{(3)} := \mathbb{C} \times \mathbb{R}_{>0}$. To see this, note that each vector of length -1 is of the form

$$\begin{pmatrix} x+iy & b \\ c & -x+iy \end{pmatrix}, \qquad \textit{where} \qquad x^2 + y^2 + bc = -1.$$

Consider the matrices for which b is positive (which is equivalent to choosing one of the two connected components). Mapping each such matrix to $(x+iy, b)$ gives us the required bijection.

We extend this bijection to a larger collection of vectors. The vectors of length zero are the matrices for which $x^2+y^2+bc=0$, where x, y, b, c are not all zero. The map defined by

$$\begin{pmatrix} x+iy & b \\ c & -x+iy \end{pmatrix} \mapsto \begin{cases} [x+iy : b] & b \neq 0, \\ [-c : x-iy] & \text{otherwise}, \end{cases}$$

establishes a bijection between isotropic lines in $V_{\mathbb{R}}$ and $\mathbb{P}_1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$. We use ℓ_z to denote the isotropic line that is the preimage of $z \in \mathbb{C} \cup \{\infty\}$. For distinct $s, t \in \mathbb{C} \cup \{\infty\}$, the lines ℓ_s and ℓ_t are not orthogonal. Let $H\{s,t\}$ denote the hyperbolic plane generated by ℓ_s and ℓ_t, and consider the associated symmetric space $B(H\{s,t\})$. This is a real manifold of dimension one. The inclusion of the quadratic space $i : H\{s,t\} \hookrightarrow V_{\mathbb{R}}$ induces an embedding of manifolds $i : B(H\{s,t\}) \hookrightarrow B(V_{\mathbb{R}})$. Under the identification of $B(V_{\mathbb{R}})$ with $\mathcal{H}^{(3)}$, the submanifolds $B(H\{s,t\})$ are geodesics in $\mathcal{H}^{(3)}$.

We want to integrate the theta series $\Theta(\tau, h+L)$, which is composed of the 1-forms $\eta(\tau, X)$, on the submanifolds $B(H\{s,t\})$. For this reason, we devote the next section to the case of signature $(1,1)$.

3.1.1 Signature (1,1)

Let $H_{\mathbb{R}}$ be a quadratic space of signature $(1,1)$. The associated manifold $B(H_{\mathbb{R}})$ is a real manifold of dimension one. For each $X \in H_{\mathbb{R}}$, we have a 1-form $\eta(\tau, X)$ defined by (3.1). We state three propositions about integrals of these 1-forms. However, we first describe the setup required to state them. Let $\{f_1, f_2\}$ be an oriented basis of $H_{\mathbb{R}}$ such that $(f_1, f_2) = -1$ and f_1, f_2 are isotropic vectors. We orient the manifold $B(H_{\mathbb{R}})$ by declaring that a non-zero tangent vector W at a point $Z \in B(H_{\mathbb{R}})$ points in the positive direction if and only if the ordered basis $\{Z, W\}$ is positively oriented for $H_{\mathbb{R}}$.

Proposition 3.2 (Proposition 2.3, [15]). *Let $X = x_1 f_1 + x_2 f_2$ be a positive length vector. This means $x_1 x_2 < 0$ and we have*

$$e^{2\pi i \tau x_1 x_2} \int_{B(H_{\mathbb{R}})} \eta(\tau, X) = \begin{cases} +1 & x_2 > 0, \\ -1 & x_2 < 0. \end{cases}$$

Proposition 3.3 (Proposition 2.4, [15])**.** *Let X be a negative length vector. Then*

$$\int_{B(H_{\mathbb{R}})} \eta(\tau, X) = 0.$$

Proposition 3.4 (Proposition 2.5, [15])**.** *Let $X = x_1 f_1 + x_2 f_2$ be an isotropic vector. This means $x_1 x_2 = 0$ and*

$$\int_{B(H_{\mathbb{R}})} \eta(\tau, X) = \begin{cases} \frac{1}{2}\mathrm{sgn}(x_2) & x_1 = 0, \\ -\frac{1}{2}\mathrm{sgn}(x_1) & x_2 = 0. \end{cases}$$

The proofs of the preceding propositions are largely computational and are presented in the following section. The proofs follow the presentations in [15, Section 2] and [17, Section 2].

3.1.2 Proofs of Propositions

Proof. The vectors of length -1 are $\{af_1 + bf_2 : ab = 1/2\}$. Thus the set $\left\{ \left(\frac{1}{\sqrt{2}t}, \frac{t}{\sqrt{2}} \right) : t > 0 \right\}$ is in bijection with $B(H_{\mathbb{R}})$. Consider the mapping

$$\begin{aligned} \mathbb{R}_{>0} &\to B(H_{\mathbb{R}}) \\ t &\mapsto \frac{1}{\sqrt{2}} \left(\frac{1}{t} f_1 + t f_2 \right). \end{aligned}$$

Thus $B(H_{\mathbb{R}})$ is parametrized as a function of the variable $t \in \mathbb{R}_{>0}$. Let $Z_t := \frac{1}{\sqrt{2}} \left(\frac{1}{t} f_1 + t f_2 \right)$ and $W_t := \frac{1}{\sqrt{2}} \left(-\frac{1}{t} f_1 + t f_2 \right)$. Thus the pullback of the 1-form $\eta(\tau, X)$ is

$$\eta(\tau, X)(t) = \sqrt{v}\ (X, W_t)\ \exp(\pi i u(X, X) - \pi v(X, X)_{Z_t})\ \frac{dt}{t}.$$

For $X = x_1 f_1 + x_2 f_2$, we have

$$\begin{aligned} (X, W_t) &= \frac{1}{\sqrt{2}} \left(-x_1 t + \frac{x_2}{t} \right), \\ (X, X) &= -2x_1 x_2, \text{ and} \\ (X, X)_{Z_t} &= -2x_1 x_2 + (X, Z_t)^2 = -2x_1 x_2 + \frac{1}{2}(x_1 t + x_2/t)^2. \end{aligned}$$

Therefore

$$\begin{aligned}\pi i u(X,X) - \pi v(X,X)_{Z_t} &= \pi i u(-2x_1x_2) - \pi v(-2x_1x_2 + (x_1t + x_2/t)^2/2)\\ &= -\pi i\tau(2x_1x_2) - \frac{1}{2}\pi v(x_1t + x_2/t)^2.\end{aligned}$$

Thus we now need to compute

$$\int_0^\infty \sqrt{v}\frac{1}{\sqrt{2}}\left(-x_1t + \frac{x_2}{t}\right)\exp\left(-\pi v\frac{1}{2}\left(x_1t + x_2/t\right)^2\right)\frac{dt}{t}.$$

We now take several cases and compute the integral for each case.

Case 1 Assume that $x_1x_2 < 0$. Let $f = \frac{1}{\sqrt{2}}\left(x_1t + \frac{x_2}{t}\right)$. Then as $t \to 0$, $f \to \operatorname{sgn}(x_2)\infty$ and as $t \to \infty$, $f \to \operatorname{sgn}(x_1)\infty$. Further $df = \frac{1}{\sqrt{2}}\left(x_1 - \frac{x_2}{t^2}\right)dt = \frac{1}{\sqrt{2}}\left(tx_1 - \frac{x_2}{t}\right)\frac{dt}{t}$.

We first assume that $x_1 > 0$ and $x_2 < 0$. Then the integral becomes

$$-\int_{-\infty}^\infty \sqrt{v}\ \exp\left(-\pi v f^2\right)df = -1.$$

The other case of $x_1 < 0$ and $x_2 > 0$ works exactly the same way.

Case 2 We assume that $x_1x_2 > 0$. Then let $T = x_2/x_1t$ and we have $dT = -(x_2/x_1)\frac{dt}{t^2} = -T\frac{dt}{t}$. Then the integral becomes

$$\int_0^\infty \frac{\sqrt{v}}{\sqrt{2}}\left(-\frac{x_2}{T} + x_1T\right)\exp\left(-\pi\frac{v}{2}\left(\frac{x_2}{T} + x_1T\right)^2\right)\frac{dT}{T},$$

which is exactly the negative of the original integral and therefore the integral in this case is zero.

Case 3.1 We assume $x_1 = 0$. Then the integral is

$$\int_0^\infty \frac{\sqrt{v}}{\sqrt{2}}\left(\frac{x_2}{t}\right)\exp\left(-\pi\frac{v}{2}(x_2/t)^2\right)\frac{dt}{t}.$$

Let $f = \frac{x_2}{\sqrt{2}t}$, then we have

$$\int_0^{\operatorname{sgn}(x_2)\infty} \sqrt{v}\exp\left(-\pi v f^2\right)df = \operatorname{sgn}(x_2)\frac{1}{2}.$$

Case 3.2 We assume $x_2 = 0$. Then the integral is

$$\int_0^\infty \frac{\sqrt{v}}{\sqrt{2}}(-x_1 t)\exp\left(-\pi\frac{v}{2}(x_1 t)^2\right)\frac{dt}{t}.$$

Let $f = \frac{x_1 t}{\sqrt{2}}$, then we have

$$-\int_0^{\mathrm{sgn}(x_1)\infty} \sqrt{v}\exp\left(-\pi v f^2\right)df = -\mathrm{sgn}(x_1)\frac{1}{2}.$$

$\square$

3.2 Integration of Theta Series

The theta series in (3.3) is a general gadget with modularity properties for a wide class of lattices, as detailed in [9]. However, we fix a specific family of lattices for the rest of this thesis. The choice is primarily made to simplify the setting enough to have clean statements for modularity and construction of cocycles.

Let ℓ be a positive integer. Consider a four dimensional rational quadratic space V_ℓ generated by $\{e_1, e_2, e_3, e_4\}$ such that

$$Q(e_1) = 1,\ Q(e_2) = \ell,\ Q(e_3) = 0 = Q(e_4), \quad \textit{and}$$

$$\langle e_i, e_j\rangle = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2\ell & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We consider a particular family of lattices $L(\ell, N) := \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus N\mathbb{Z}e_3 \oplus \mathbb{Z}e_4$, where N can be any positive integer. Direct computation shows that the dual $(L(\ell,N))^*$ of $L(\ell,N)$ is $\frac{1}{2}\mathbb{Z}e_1 \oplus \frac{1}{2\ell}\mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \frac{1}{N}\mathbb{Z}e_4$. With this setup, $V_\ell \otimes \mathbb{R}$ is a quadratic space of signature $(3,1)$. The next step is to integrate the theta series associated to $L(\ell,N)$ along various one dimensional submanifolds of $\mathcal{H}^{(3)}$.

We first integrate along $B(H(0,\infty))$. The integration in this case yields a neat q-expansion that helps in understanding the integration along a general manifold $B(H\{s,t\})$.

3.2.1 Integration along $B(H(0,\infty))$

We first parametrize the path $B(H(0,\infty))$. Let t be a positive real number and define $Z_t := \frac{1}{\sqrt{2}}\left(\frac{1}{t}e_3 + te_4\right)$. We choose the tangent vector $W_t := \frac{1}{\sqrt{2}}\left(-\frac{1}{t}e_3 + te_4\right) \in T_{Z_t}(B)$. Thus we have a vector field W_t along the curve $\{Z_t : t \in \mathbb{R}_{>0}\}$. Let $h = ce_3 + \frac{d}{N}e_4 \in (L(\ell,N))^*$. Our goal is to integrate theta function $\Theta(\tau, h+L, Z_t, W_t)$ along the submanifold $\{Z_t : t \in \mathbb{R}^+\}$. Before integrating, we first explain how the function $\Theta(\tau, h+L(\ell,N), Z_t, W_t)$ factorises as a product of two theta series which allows us to use the results of Section 3.1.1. For any $X = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 \in h + L(\ell,N)$, the 1-form is given by

$$\eta(\tau,X)_{Z_t}(W_t) = \sqrt{2v}(X,W_t)\exp\left(-\pi v(X,Z_t)^2\right)\exp\left(\pi i\tau(X,X)\right).$$

We note that $Z_t, W_t \in \mathbb{R}e_3 + \mathbb{R}e_4$. Moreover, $V_{\mathbb{R}} = (\mathbb{R}e_1 + \mathbb{R}e_2) \oplus (\mathbb{R}e_3 + \mathbb{R}e_4)$ is an orthogonal direct sum. For ease of reading, let $D_X = x_1e_1 + x_2e_2$ and $H_X = x_3e_3 + x_4e_4$. Thus

$$\begin{aligned}\eta(\tau,X)_{Z_t}(W_t) &= \sqrt{2v}(X,W_t)\exp(-\pi v(H_X,Z_t)^2)\exp(\pi i\tau((D_X,D_X)+(H_X,H_X))) \\ &= \sqrt{2v}\times\left(q^{\frac{1}{2}(D_X,D_X)}\right)\times\left((H_X,W_t)\times\exp(-\pi v(H_X,Z_t)^2)\times q^{\frac{1}{2}(H_X,H_X)}\right).\end{aligned} \tag{3.4}$$

Expanding D_X and H_X shows that the theta series $\sum_{X\in h+L}\eta(\tau,X)_{Z_t}(W_t)$ is equal to

$$\sqrt{2v}\left(\sum_{x_1,x_2\in\mathbb{Z}} q^{x_1^2+\ell x_2^2}\right)\left(\sum_{\substack{x_3\in c+N\mathbb{Z}\\ x_4\in d/N+\mathbb{Z}}}\frac{1}{\sqrt{2}}\left(-x_3t+\frac{x_4}{t}\right)\exp\left(-\pi\frac{v}{2}\left(x_3t+\frac{x_4}{t}\right)^2\right)q^{-x_3x_4}\right). \tag{3.5}$$

We now integrate the above function over $\mathbb{R}_+^\times$, where t goes from 0 to ∞. Note that the sum over x_1, x_2 is independent of t and thus we only need to compute the integral

$$\int_0^\infty\left(\sum_{\substack{x_3\in c+N\mathbb{Z}\\ x_4\in d/N+\mathbb{Z}}}\frac{1}{\sqrt{2}}\left(-x_3t+\frac{x_4}{t}\right)\exp\left(-\pi\frac{v}{2}\left(x_3t+\frac{x_4}{t}\right)^2\right)q^{-x_3x_4}\right)\frac{dt}{t}.$$

We first state the Fourier expansion.

Theorem 3.5. *For a positive integer k, define $r_\ell(k) := \#\{(x_1,x_2)\in\mathbb{Z}^2 : x_1^2+\ell x_2^2 = k\}$.*

1. If $c,d\neq 0$, then the integral

$$\int_0^\infty \Theta\left(\tau, ce_3+\frac{d}{N}e_4+L, Z_t, W_t\right)\frac{dt}{t} = \sum_{m\in\mathbb{Q}_{\geq 0}}\left(\sum_{k=0}^{\lfloor m\rfloor} r_\ell(k)\sum_{\substack{x_3\in c+N\mathbb{Z}\\ x_4\in d/N+\mathbb{Z}\\ x_3x_4=k-m}}\operatorname{sgn}(x_3)\right)q^m.$$

2. If $d=0$ and $(c,N)=1$, then the integral $\int_0^\infty \Theta(\tau, ce_3+L, Z_t, W_t)\frac{dt}{t}$ is equal to

$$\sum_{m\in\mathbb{Q}_{\geq 0}}\left(r_\ell(m)\Theta_0(ce_3+L)+\sum_{k=0}^{m-1} r_\ell(k)\sum_{\substack{x_3\in c+N\mathbb{Z}\\ x_4\in \mathbb{Z}\setminus\{0\}\\ x_3x_4=k-m}}\operatorname{sgn}(x_3)\right)q^m,$$

where

$$\Theta_0(ce_3+L)=\zeta\left(1-\frac{c}{N},0\right)-\zeta\left(\frac{c}{N},0\right),$$

with $\zeta(s,a)$ denoting the Hurwitz zeta function.

Proof. We consider the two cases separately.

Case 1: $cd\neq 0$. The integral in this case can be computed using term-by-term integration, as noted in [15, Proposition 2.1]. Thus,

$$\int_0^\infty \Theta\left(\tau, ce_3+\frac{d}{N}e_4+L, Z_t, W_t\right)\frac{dt}{t} = \sum_{m\in\mathbb{Q}_{\geq 0}}\left(\sum_{k=0}^{\lfloor m\rfloor} r_\ell(k)\sum_{\substack{x_3\in c+N\mathbb{Z}\\ x_4\in d/N+\mathbb{Z}\\ x_3x_4=k-m}}\operatorname{sgn}(x_3)\right)q^m$$

We note that $r_\ell(k)$ is finite for each $0\leq k\leq m$, and the set $\{(x_3,x_4): x_3\in c+N\mathbb{Z}, x_4\in d/N+\mathbb{Z}, x_3x_4=k-m\}$ is finite as well. Therefore the term inside the brackets is finite for each m.

Case 2: $d = 0$ The integral of the entire theta series in this case cannot be computed term-by-term. We rewrite the integral in two parts.

$$\int_0^\infty \left(\sum_{\substack{x_3 \in c+N\mathbb{Z} \\ x_4 \in d/N+\mathbb{Z}}} \frac{1}{\sqrt{2}} \left(-x_3 t + \frac{x_4}{t}\right) \exp\left(-\pi \frac{v}{2} \left(x_3 t + \frac{x_4}{t}\right)^2\right) q^{-x_3 x_4} \right) \frac{dt}{t}$$
$$= \int_0^\infty \left(\sum_{x_3 \in c+N\mathbb{Z}} \frac{1}{\sqrt{2}} (-x_3 t) \exp\left(-\pi \frac{v}{2} (x_3 t)^2\right) \right) \frac{dt}{t}$$
$$+ \int_0^\infty \left(\sum_{\substack{x_3 \in c+N\mathbb{Z} \\ x_4 \in \mathbb{Z}\setminus\{0\}}} \frac{1}{\sqrt{2}} \left(-x_3 t + \frac{x_4}{t}\right) \exp\left(-\pi \frac{v}{2} \left(x_3 t + \frac{x_4}{t}\right)^2\right) q^{-x_3 x_4} \right) \frac{dt}{t}.$$

Following Kudla [16, Introduction], we note that the second integral can be computed term-by-term. So the second integral is

$$\sum_{m \in \mathbb{Q}_{\geq 0}} \left(\sum_{k=0}^{m-1} r_\ell(k) \sum_{\substack{x_3 \in c+N\mathbb{Z} \\ x_4 \in \mathbb{Z}\setminus\{0\} \\ x_3 x_4 = k-m}} \operatorname{sgn}(x_3) \right) q^m.$$

However, the first integral cannot be computed term-by-term (see the discussion on the second and third pages of the Introduction in [16]). We introduce an auxiliary complex variable r and consider the following general integral

$$F(r) := \int_0^\infty \left(\frac{1}{\sqrt{2}} \sum_{x_3 \in c+N\mathbb{Z}} (-x_3 t) \exp\left(-\pi \frac{v}{2} (x_3 t)^2\right) \right) t^r \frac{dt}{t}.$$

The function $F(r)$ is equal to the following function for large enough $\Re(r)$ (see Section 3.2.1.1).

$$F(r) = -\frac{1}{2} (\pi v)^{-\frac{r+1}{2}} 2^{\frac{r}{2}} \Gamma\left(\frac{r+1}{2}\right) N^{-r} \left[\zeta\left(\frac{c}{N}, r\right) - \zeta\left(1 - \frac{c}{N}, r\right)\right],$$

where $\zeta(s, a)$ is the Hurwitz zeta function. The right-hand side is a holomorphic function of r and is well-defined at $r = 0$. Therefore F can be evaluated at $r = 0$, and we get

$$F(0) = \frac{1}{2\sqrt{v}} \left(\zeta\left(1 - \frac{c}{N}, 0\right) - \zeta\left(\frac{c}{N}, 0\right)\right),$$

since $\Gamma(1/2)=\sqrt{\pi}$. Define $\Theta_0(ce_3+L):=\sqrt{2v}F(0)$. Then

$$\int_0^\infty \Theta(\tau, ce_3+L, Z_t, W_t)\frac{dt}{t} = \sum_{m\in\mathbb{Q}_{\geq 0}} \left(r_\ell(m)\Theta_0(ce_3+L) + \sum_{k=0}^{m-1} r_\ell(k) \sum_{\substack{x_3\in c+N\mathbb{Z} \\ x_4\in\mathbb{Z}\setminus\{0\} \\ x_3x_4=k-m}} \operatorname{sgn}(x_3) \right) q^m.$$

□

3.2.1.1 Computation of $F(r)$

We first compute $F(r)$ for $\Re(r)$ sufficiently large by term-by-term integration. For individual terms, we have

$$I(x_3)(r) := \int_0^\infty \frac{1}{\sqrt{2}}(-x_3t)\exp\left(-\pi\frac{v}{2}x_3^2t^2\right)t^r\frac{dt}{t}.$$

We apply the substitution $f=\frac{\pi v x_3^2 t^2}{2}>0 \implies t=\frac{\sqrt{2f}}{\sqrt{\pi v}|x_3|}$. The differential is then given by $\frac{dt}{t}=\frac{1}{2}\frac{df}{f}$. As $t\to 0$, $f\to 0$, and as $t\to\infty$, $f\to\infty$. Thus,

$$\begin{aligned}
I(x_3)(r) &= \int_0^\infty \left(-\frac{x_3}{\sqrt{2}}\frac{\sqrt{2f}}{\sqrt{\pi v}|x_3|}\right)\exp(-f)\left(\frac{\sqrt{2f}}{\sqrt{\pi v}|x_3|}\right)^r\left(\frac{1}{2}\frac{df}{f}\right) \\
&= -\frac{1}{2}\int_0^\infty \frac{x_3\sqrt{f}}{\sqrt{\pi v}|x_3|}\exp(-f)(\pi v)^{-\frac{r}{2}}2^{\frac{r}{2}}|x_3|^{-r}f^{\frac{r}{2}}f^{-1}df \\
&= -\frac{1}{2}\frac{x_3}{|x_3|}(\pi v)^{-\frac{r+1}{2}}2^{\frac{r}{2}}|x_3|^{-r}\int_0^\infty \exp(-f)f^{\frac{r+1}{2}-1}df.
\end{aligned}$$

Since $\frac{x_3}{|x_3|}=\operatorname{sgn}(x_3)$ and recalling the Gamma function $\Gamma(s)=\displaystyle\int_0^\infty e^{-f}f^{s-1}df$, we have

$$I(x_3)(r) = -\frac{1}{2}\operatorname{sgn}(x_3)(\pi v)^{-\frac{r+1}{2}}2^{\frac{r}{2}}|x_3|^{-r}\Gamma\left(\frac{r+1}{2}\right).$$

We obtain the explicit evaluation by summing over $x_3\in c+N\mathbb{Z}$.

$$F(r) = \sum_{x_3\in c+N\mathbb{Z}} I(x_3)(r) = -\frac{1}{2}(\pi v)^{-\frac{r+1}{2}}2^{\frac{r}{2}}\Gamma\left(\frac{r+1}{2}\right)\sum_{x_3\in c+N\mathbb{Z}}\operatorname{sgn}(x_3)|x_3|^{-r}.$$

The sum on the right-hand side can be expressed as a difference of two zeta functions.

We note that

$$\begin{aligned}
\sum_{x_3 \in c+N\mathbb{Z}} \operatorname{sgn}(x_3)|x_3|^{-r} &= \sum_{k=0}^{\infty} |c+Nk|^{-r} - \sum_{k=1}^{\infty} |c-Nk|^{-r} \\
&= \sum_{k=0}^{\infty} (c+Nk)^{-r} - \sum_{k=1}^{\infty} (-c+Nk)^{-r} \\
&= \sum_{k=0}^{\infty} (c+Nk)^{-r} - \sum_{k=0}^{\infty} (N-c+Nk)^{-r} \\
&= N^{-r}\left(\zeta\left(\frac{c}{N}, r\right) - \zeta\left(1-\frac{c}{N}, r\right)\right).
\end{aligned}$$

3.2.2 Integration along a General Geodesic

We now understand the calculations involved in integrating over the manifold associated to $\mathbb{R}e_3 \oplus \mathbb{R}e_4$. In this section, we integrate along the space generated by any two rational isotropic lines. In general, the theta function does not factorise, but we can still integrate at the cost of having a more complicated final sum. Since most of the steps are identical, we explain the differences and use the calculations of the previous section.

Let $K = \mathbb{Q}(\sqrt{-\ell})$. We now integrate $\eta(\tau, X)$ over $B(H(s,s'))$ for $s, s' \in \mathbb{P}_1(K)$. Let $e_s, e_{s'}$ be isotropic vectors associated to s, s' respectively such that $B(e_s, e_{s'}) < 0$. Let $\{e_s, e_{s'}\}$ be an oriented basis for $H(s,s')$. We then have

$$\frac{1}{q^{(X,X)}} \int_{B(H(s,s'))} \eta(\tau, X) = \begin{cases} -\operatorname{sgn}((X, e_{s'})) & (X,e_s)(X,e_{s'}) < 0 \\ -\frac{1}{2}\operatorname{sgn}((X, e_{s'})) & (X, e_s) = 0 \\ \frac{1}{2}\operatorname{sgn}((X, e_s)) & (X, e_{s'}) = 0 \\ 0 & (X,e_s)(X,e_{s'}) > 0 \end{cases} \tag{3.6}$$

For a general pair of isotropic vectors $e_s, e_{s'}$, we define $L' := L(\ell, N) \cap (\mathbb{Q}e_s \oplus \mathbb{Q}e_{s'})^{\perp}$ and $L'' := L(\ell, N) \cap (\mathbb{Q}e_s \oplus \mathbb{Q}e_{s'})$. Since the bilinear form is non-degenerate on the rational subspace $W = \mathbb{Q}e_s \oplus \mathbb{Q}e_{s'}$, we have an orthogonal direct sum decomposition $V_\ell = W \oplus W^{\perp}$. Because the orthogonal projection maps onto W and $W^{\perp}$ are rational linear transformations, clearing their denominators yields an integer $d > 0$ such that $dL(\ell, N) \subset L' \oplus L''$. Consequently, the direct sum $L' \oplus L''$ is a sublattice of finite index r in $L(\ell, N)$. Therefore, we get

$$h + L = \bigcup_{i=1}^{r} (h_i' + L') + (h_i'' + L'').$$

As before, we integrate over Z_t as t varies over $\mathbb{R}_+$. We only consider the case where $h = ce_3$ for some integer c coprime to N. The choice of h forces all the vectors in $h + L(\ell, N)$ to have integer lengths. The m-th Fourier coefficient is

$$\sum_{i=1}^{r}\left[\sum_{k=0}^{m-1}\left(\sum_{\substack{X_1\in h_i'+L' \\ Q(X_1)=k}}\ \sum_{\substack{X_2\in h_i''+L'' \\ Q(X_2)=m-k}} -\mathrm{sgn}((X_2, e_s))\right) + \sum_{\substack{X_1\in h_i'+L' \\ Q(X_1)=m}} \Theta_0(h_i''+L'')\right], \tag{3.7}$$

where $\Theta_0(h_i'' + L'')$ is given by the integral

$$\int_{B(H(s,s'))} \Theta(\tau, h_i'' + L'', Z_t, W_t)\frac{dt}{t}.$$

Using the results from the previous section, one could evaluate the above integral explicitly as a difference of zeta values depending on h_i'' and L''. Since the explicit value of this integral is not required for our subsequent results, we avoid that detour.

3.2.3 Isolating Fourier Coefficients useful for Cocycles

The expression for the m-th Fourier coefficient in (3.7) involves two types of sums. The first sum counts the contribution $\mathrm{sgn}(X, \cdot)$ for each vector X, and the second sum is composed of values of the Hurwitz zeta function. In this section, we show that there exist infinitely many Fourier coefficients that only involve the sum over individual vectors and the sum composed of zeta values does not contribute.

The second sum in (3.7) does not appear precisely when there is no vector $X_1 \in h_i' + L'$ whose length equals m. The following lemma guarantees that we can choose an m satisfying this condition for every L' associated with any pair $\{s, t\}$.

Lemma 3.6. *Let m be a rational number such that it is never equal to $x^2 + \ell y^2$ for any $x, y \in \mathbb{Q}$. Then for any $s, t \in \mathbb{P}_1(K)$, and $X_1 \in (\mathbb{Q}e_s \oplus \mathbb{Q}e_t)^\perp$, $Q(X_1)$ is not equal to m.*

Proof. Let $\{e_1, e_2, e_3, e_4\}$ be the basis of V_ℓ described in the beginning of Section 3.2. We write $X_1 = Y + Z$, where $Z \in \mathbb{Q}e_3 \oplus \mathbb{Q}e_4$ and $Y \in \mathbb{Q}e_1 \oplus \mathbb{Q}e_2$. If $X_1 \in (\mathbb{Q}e_3 \oplus \mathbb{Q}e_4)^\perp$, then $Q(X_1) = Q(Y)$ which is of the form $x^2 + \ell y^2$.

Now we prove the claim for a general pair (e_s, e_t). Let $\gamma \in \mathrm{SL}_2(K)$ such that $e_t = \gamma(e_4)$. Since the bilinear pairing is invariant under $\mathrm{SL}_2(K)$, we have $(\gamma^{-1}X_1, e_4) = 0$. It implies that $\gamma^{-1}X_1 = Y + ze_3$ for some $Y \in (\mathbb{Q}e_1 \oplus \mathbb{Q}e_2)$ and $z \in \mathbb{Q}$. It is clear that $Q(\gamma^{-1}X_1)$ is of the form $x^2 + \ell y^2$. But we also have $Q(\gamma^{-1}X_1) = Q(X_1)$ and thus $Q(X_1) \neq m$. $\square$

Definition 3.7. *A rational number m is called a* proper length with respect to V_ℓ *if it cannot be written as $x^2 + \ell y^2$ for any $x, y \in \mathbb{Q}$.*

In the following proposition, we show that there are infinitely many integers that satisfy the criterion of being a *proper length with respect to V_ℓ*. As a consequence, there are infinitely many Fourier coefficients that are composed exclusively of the terms of the form $\operatorname{sgn}(X, \cdot)$.

Proposition 3.8. *There are infinitely many integers that are not of the form $x^2 + \ell y^2$ for rationals x, y.*

Proof. Theorem 8 of [20, p. 40] says that a number m is represented by $x^2 + \ell y^2$ over rationals if and only if it is represented by the same form in all completions. It is clear that if $m \geq 0$ then it is represented over reals. Now we need to care about the p-adic places.

To decide for the p-adic places we use the Corollary following Theorem 6 of [20, page 37]. It says that for the rank two quadratic form $x^2 + \ell y^2$ over $\mathbb{Q}_q$ (for a prime q), we have that m is represented by the form if and only if $(m, -\ell)_q = (1, \ell)_q$ where $(\cdot, \cdot)_q$ is the Hilbert symbol corresponding to the prime q. Choose a prime $q > 2$ that satisfies

$$\left(\frac{-\ell}{q} \right) = -1.$$

It is clear that such primes exist.

The equation $z_1^2 - z_2^2 = \ell z_3^2$ always has a non-trivial solution of the form $(\alpha, \alpha, 0)$ and thus the Hilbert symbol $(1, \ell)_q$ is equal to 1. Choose $m = q^\beta$ where β is odd. By Theorem 1 of [20, page 20], we have

$$(m, -\ell)_q = \left(\frac{-\ell}{q} \right)^\beta .$$

The right-hand side is -1 by our choices and thus we have constructed infinitely many integers that are not of the form $x^2 + \ell y^2$ for rationals x, y.

$\square$

3.2.4 Integrals of 1-forms as Intersection of Submanifolds

There is a topological interpretation of integrals of 1-forms as intersections of submanifolds. We discuss it informally.

Submanifolds associated to a vector. For each vector $X \in V_{\mathbb{R}}$ we look at $X^{\perp}$ which is a three-dimensional space. Based on the general recipe of considering negative length lines, we define $B(X^{\perp}) := \{Z \in X^{\perp} : (Z, Z) = -1\}$. There are three different cases depending on the length of X:

1. $(X, X) > 0 :$ $X^{\perp}$ is of signature $(2, 1)$ and thus $B(X^{\perp})$ is a real manifold of dimension two. In fact we can identify it with the Poincaré upper half plane $\mathcal{H}$.

2. $(X, X) = 0 :$ $X^{\perp}$ is a positive semidefinite space and decomposes as an orthogonal direct sum $U \oplus \mathbb{R}X$ where U is a space of signature $(2, 0)$. In this case $B(X^{\perp})$ is empty.

3. $(X, X) < 0 :$ $X^{\perp}$ is of signature $(3, 0)$ and thus all vectors in $X^{\perp}$ have positive lengths. As a consequence $B(X^{\perp})$ is empty.

From this point of view, the only interesting vectors are those of positive length. Note that we have the inclusion of quadratic spaces $X^{\perp} \subset V_{\mathbb{R}}$ which gives us an embedding of manifolds as $B(X^{\perp}) \hookrightarrow B(V_{\mathbb{R}})$. Upon identifying $B(V_{\mathbb{R}})$ with $\mathcal{H}^{(3)}$, the embedded manifold $B(X^{\perp})$ is either a geodesic hemisphere or a plane in $\mathcal{H}^{(3)}$.

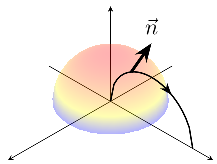
Let $B(H\{s, t\})$ be the geodesic in $\mathcal{H}^{(3)}$ associated to the pair of isotropic vectors $\{e_s, e_t\}$ and consider its intersection with $B(X^{\perp})$. If the submanifolds do not intersect, we count zero. If they intersect inside $\mathcal{H}^{(3)}$, we count $+1$ or -1 depending on the orientations. In these cases, the count we assign is equal to the integral of $\eta(\tau, X)$ on $B(H\{s, t\})$.

There is a third case where the intersection seems to occur at the boundary. This happens exactly when the projection of X along $H\{s, t\}$ is isotropic. However, the integral of $\eta(\tau, X)$ in such cases is either $1/2$ or $-1/2$. Thus we assign $\pm 1/2$ as the intersection count in this case.

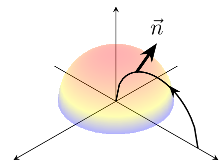
We note that restricting ourselves to vectors whose lengths are *proper lengths with respect to V_ℓ* ensures that we never run into the case of $\pm 1/2$. However with the interpretation here, we are able to prove that the values assigned to such "intersections" give rise to modular symbols (Proposition 3.11). We believe that this observation can be of use when constructing cocycles beyond the cases considered in this thesis.

Section 3.2.4 shows all five cases.

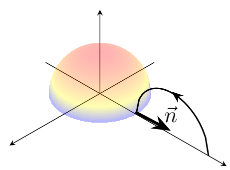
Figure 3.1: Intersection of Submanifolds in the upper half-space



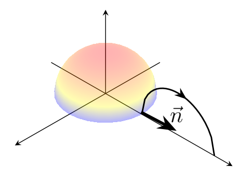
(a) Intersection number is +1



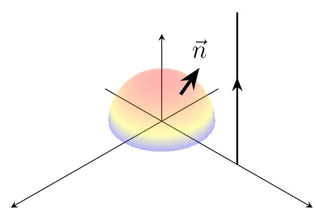
(b) Intersection number is -1



(c) Intersection number is -1/2



(d) Intersection number is +1/2



(e) Intersection number is 0

3.3 Integer-valued Modular Symbols

In this section, we use the objects discussed so far to produce modular symbols and thus integer-valued cocycles. This section is central to the construction of rigid cocycles. We present here an abstract definition of modular symbols that is best suited for our purposes.

Definition 3.9. *Let Ω be a $\mathbb{Z}$-module. An Ω-valued $\mathbb{Q}$-modular symbol is a function*

$$m : \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q}) \to \Omega$$

satisfying the following properties for all $r, s, t \in \mathbb{P}_1(\mathbb{Q})$:

1. $m\{s,t\} + m\{t,s\} = 0,$
2. $m\{r,s\} + m\{s,t\} + m\{t,r\} = 0.$

We need another definition that involves an imaginary quadratic field instead of the field of rational numbers. Let K be an imaginary quadratic field.

Definition 3.10. *An Ω-valued K-modular symbol is a function*

$$m : \mathbb{P}_1(K) \times \mathbb{P}_1(K) \to \Omega$$

satisfying the following properties for all $r, s, t \in \mathbb{P}_1(K)$:

1. $m\{s,t\} + m\{t,s\} = 0,$

2. $m\{r,s\} + m\{s,t\} + m\{t,r\} = 0.$

We use $\mathrm{MS}_K(\Omega)$ to denote the collection of Ω-valued K-modular symbols. Let Ω be equipped with a Γ-action where $\Gamma \subset \mathrm{SL}_2(K)$. The group $\Gamma \subset \mathrm{SL}_2(K)$ acts on $\mathbb{P}_1(K)$ by Möbius transformations. Given these actions, we define an action of Γ on the collection $\mathrm{MS}_K(\Omega)$ by $(m \mid \gamma)\{s,t\} := m\{\gamma s, \gamma t\} \mid \gamma$. We say that a modular symbol $m \in \mathrm{MS}_K(\Omega)$ is Γ-invariant if for all $\gamma \in \Gamma$ and all $s, t \in \mathbb{P}_1(K)$, we have $(m \mid \gamma)\{s,t\} = m\{s,t\}$ which is equivalent to $m\{\gamma s, \gamma t\} \mid \gamma = m\{s,t\}$. The same definition works for $\mathbb{Q}$-modular symbols when $\Gamma \subset \mathrm{SL}_2(\mathbb{Q})$. We denote the collection of all such Γ-invariant modular symbols by $\mathrm{MS}(\Gamma, \Omega)$. We omit the field from the notation, as it is implicit from Γ.

Modular symbols invariant under Γ give rise to cocycles for Γ. A 1-cocycle $\Phi \in Z^1(\Gamma, M)$ is a function Φ from the group Γ to a left Γ-module M such that for all $\gamma_1, \gamma_2 \in \Gamma$, we have $\Phi(\gamma_1\gamma_2) = \Phi(\gamma_1) + \gamma_1 \odot \Phi(\gamma_2)$. As Ω is equipped with a right Γ-action, we make it into a left Γ-module by defining $\gamma \odot \omega := \omega \mid \gamma^{-1}$ for $\gamma \in \Gamma$ and $\omega \in \Omega$. Given a modular symbol $m \in \mathrm{MS}(\Gamma, \Omega)$, define a function $\Phi_m : \Gamma \to \Omega$ by setting $\Phi_m(\gamma) := m\{\infty, \gamma\infty\}$. The following calculation shows that Φ_m is a cocycle.

$$\begin{aligned}
\Phi_m(\gamma_1\gamma_2) &= m\{\infty, \gamma_1\gamma_2\infty\} \\
&= m\{\infty, \gamma_1\infty\} + m\{\gamma_1\infty, \gamma_1\gamma_2\infty\} \\
&= m\{\infty, \gamma_1\infty\} + m\{\infty, \gamma_2\infty\} \mid \gamma_1^{-1} \\
&= \Phi_m(\gamma_1) + \Phi_m(\gamma_2) \mid \gamma_1^{-1} \\
&= \Phi_m(\gamma_1) + \gamma_1 \odot \Phi_m(\gamma_2).
\end{aligned}$$

Thus we have a map from $\mathrm{MS}(\Gamma, \Omega)$ to $Z^1(\Gamma, \Omega)$ given by $m \mapsto \Phi_m$. We obtain rigid meromorphic cocycles by constructing appropriate modular symbols and applying this map.

3.3.1 Intersection of Manifolds as Modular Symbols

We construct modular symbols from the integrals of 1-forms discussed earlier. We first define an integer-valued function on $\mathbb{P}_1(K) \times \mathbb{P}_1(K)$ associated to a vector $X \in V_\ell \otimes \mathbb{R}$. Let

s, t be two distinct points in $\mathbb{P}_1(K)$ and let $B(H\{s,t\})$ be the associated geodesic in $\mathcal{H}^{(3)}$. We define

$$\Delta_{X,\infty}\{s,t\} := \frac{1}{q^{(X,X)}} \int_{B(H\{s,t\})} \eta(\tau, X),$$

where $\{e_s, e_t\}$ is an oriented basis for $H\{s,t\}$ and $q = e^{2\pi i \tau}$. Therefore we have a function $\Delta_{X,\infty}\{\cdot,\cdot\}$ from $\mathbb{P}_1(K) \times \mathbb{P}_1(K)$ to $\mathbb{Z}$. We note that the order of s, t determines the orientation of the corresponding manifold $B(H\{s,t\})$ and thus influences the sign. We write a more explicit description of this function $\Delta_{X,\infty}\{s,t\}$ using the three propositions in Section 3.1.1.

$$\Delta_{X,\infty}\{s,t\} = \frac{1}{q^{(X,X)}} \int_{B(H\{s,t\})} \eta(\tau, X) = \begin{cases} -\mathrm{sgn}((X, e_t)) & (X, e_s)(X, e_t) < 0 \\ -\frac{1}{2}\mathrm{sgn}((X, e_t)) & (X, e_s) = 0 \\ \frac{1}{2}\mathrm{sgn}((X, e_s)) & (X, e_t) = 0 \\ 0 & (X, e_s)(X, e_t) > 0. \end{cases} \tag{3.8}$$

The following proposition lists the key properties of this function. The first two properties are exactly the same as those of a modular symbol, and thus $\Delta_{X,\infty}\{\cdot,\cdot\}$ defines an integer-valued modular symbol. The third property is used in defining modular symbols invariant under a subgroup of $\mathrm{SL}_2(K)$. This is discussed in Section 3.3.2.

Proposition 3.11. *The function $\Delta_{X,\infty}\{\cdot,\cdot\}$ satisfies*

1. $\Delta_{X,\infty}\{s,t\} + \Delta_{X,\infty}\{t,s\} = 0$ *for all* $s, t \in \mathbb{P}_1(K)$.
2. $\Delta_{X,\infty}\{r,s\} + \Delta_{X,\infty}\{s,t\} + \Delta_{X,\infty}\{t,r\} = 0$ *for all* $r, s, t \in \mathbb{P}_1(K)$.
3. $\Delta_{\gamma X,\infty}\{\gamma s, \gamma t\} = \Delta_{X,\infty}\{s,t\}$ *for all* $\gamma \in \mathrm{SL}_2(K)$.

The first property is clear, as changing the order reverses the orientation of the manifold, which consequently introduces a negative sign. The third relation is a consequence of a straightforward change-of-variable computation. Our proof of the second relation is dry, so we give the proof later in Section 3.3.3 without breaking the flow. In the next section, we present the construction of modular symbols invariant under a congruence subgroup of $\mathrm{SL}_2(K)$.

3.3.2 Modular Symbols Invariant under an Arithmetic Group

Let $s, t \in \mathbb{P}_1(K)$. Let m be a rational such that it is a *proper length with respect to* V_ℓ. Then the m-th Fourier coefficient of the modular form $\displaystyle\int_{B(H\{s,t\})} \Theta(\tau, h + L(\ell, N))$ is given by

$$Z(m, h + L(\ell, N))\{s, t\} := \sum_{Q(v)=m} \mathbb{1}(v \in h + L(\ell, N))\Delta_{v,\infty}\{s, t\}.$$

It is clear from Proposition 3.11 that the above sum satisfies the 2-term and 3-term relations. The following proposition shows that it is invariant under the group

$$\Gamma_1^K(N) := \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}[\sqrt{-\ell}]) : N \mid z; \quad x, w \equiv 1 \pmod{N} \right\}.$$

Proposition 3.12. *The group $\Gamma_1^K(N)$ preserves the lattice $L(\ell, N)$ and also preserves the cosets in its dual. Therefore $Z(m, h + L(\ell, N))\{\cdot, \cdot\}$ defines a $\Gamma_1^K(N)$-invariant modular symbol valued in integers, where the action of $\Gamma_1^K(N)$ on integers is trivial.*

Proof. We first show how the trivial action on cosets implies the invariance of the modular symbol. Let $\gamma \in \Gamma_1^K(N)$ and let $s, t \in \mathbb{P}_1(K)$. We have

$$\begin{aligned} Z(m, h + L(\ell, N))\{\gamma s, \gamma t\} \mid \gamma &= \sum_{Q(v)=m} \mathbb{1}(v \in h + L(\ell, N))\Delta_{\gamma^{-1}v,\infty}\{s, t\} \\ &= \sum_{Q(\gamma v)=m} \mathbb{1}(\gamma v \in h + L(\ell, N))\Delta_{v,\infty}\{s, t\} \\ &= \sum_{Q(v)=m} \mathbb{1}\left(v \in \gamma^{-1}(h + L(\ell, N))\right)\Delta_{v,\infty}\{s, t\} \\ &= Z(m, h + L(\ell, N))\{s, t\}. \end{aligned}$$

The first assertion in the proposition is a direct computation. We make use of the matrix model of the lattice. Define

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} \sqrt{-\ell} & 0 \\ 0 & \sqrt{-\ell} \end{pmatrix},$$
$$e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad e_4 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

For a $\gamma \in \Gamma_1^K(N)$, one can verify that $\gamma e_i \overline{\gamma}^{-1} \in L(\ell, N)$ for $i = 1, 2, 3, 4$. Therefore the lattice $L(\ell, N)$ is preserved under the action of $\Gamma_1^K(N)$.

We now prove that each coset $ce_3 + \frac{d}{N} e_4 + L(\ell, N)$ is also preserved. Let $\gamma = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \Gamma_1^K(N)$. We compute

$$\begin{aligned}\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 0 & d/N \\ c & 0 \end{pmatrix} \begin{pmatrix} \overline{w} & -\overline{y} \\ -\overline{z} & \overline{x} \end{pmatrix} &= \begin{pmatrix} yc & xd/N \\ wc & zd/N \end{pmatrix} \begin{pmatrix} \overline{w} & -\overline{y} \\ -\overline{z} & \overline{x} \end{pmatrix} \\ &= \begin{pmatrix} yc\overline{w} + xd\overline{z}/N & -y\overline{y}c + x\overline{x}d/N \\ w\overline{w}c + z\overline{z}d/N & -\overline{y}wc + z\overline{x}d/N. \end{pmatrix}\end{aligned}$$

We look at each term separately.

1. The term $yc\overline{w} + xd\overline{z}/N$ is an element of $\mathbb{Z}[\sqrt{-\ell}]$ because $y, w \in \mathbb{Z}[\sqrt{-\ell}]$ and $c \in \mathbb{Z}$. Since $z \in N\mathbb{Z}[\sqrt{-\ell}]$, we also have $xd\overline{z}/N \in \mathbb{Z}[\sqrt{-\ell}]$.

2. The term $w\overline{w}c + z\overline{z}d/N$ is of the form $c + \mathbb{Z}$. This is because $w - 1 \in N\mathbb{Z}[\sqrt{-\ell}]$ and thus $w\overline{w}$ is of the form $1 + Nm$ for some $m \in \mathbb{Z}$ and similarly $z\overline{z}$ is of the form $N^2 m'$ for some $m' \in \mathbb{Z}$. Hence

$$w\overline{w}c + z\overline{z}d/N = c + cNm + Ndm' \in c + N\mathbb{Z}.$$

3. The term $x\overline{x}d/N - y\overline{y}c$ is in $d/N + \mathbb{Z}$. This is true because $y\overline{y}c \in \mathbb{Z}$ and since $x - 1 \in N\mathbb{Z}[\sqrt{-\ell}]$ we have

$$x\overline{x}d/N - y\overline{y}c = (1 + Nm)\frac{d}{N} - y\overline{y}c = \frac{d}{N} + dm - y\overline{y}c \in \frac{d}{N} + \mathbb{Z}.$$

□

3.3.3 Proof of the Three-term Relation

Lemma 3.13. *There exists a collection of isotropic vectors $\{e_\alpha : \alpha \in \mathbb{P}_1(K)\}$ such that for any s, t, the pairing (e_s, e_t) is negative.*

Proof. We choose e_0, e_∞ such that $(e_0, e_\infty) = -1$. We then choose

$$e_s = (v_1 + e_0 + \frac{1}{2}(v_1, v_1)e_\infty), \qquad e_t = (v_2 + e_0 + \frac{1}{2}(v_2, v_2)e_\infty),$$

where v_1, v_2 are orthogonal to the space generated by e_0, e_∞. Then the pairing

$$\begin{aligned}(e_s, e_t) &= \left[(v_1, v_2) - \frac{1}{2}\left((v_1, v_1) + (v_2, v_2)\right)\right] \\ &= -\frac{1}{2}(v_1 - v_2, v_1 - v_2).\end{aligned}$$

It is clear that the pairing (e_s, e_t) is negative for distinct e_s, e_t. $\square$

We are now ready to prove three-term relations.

Proof. We rewrite the formula for $\Delta_{X,\infty}\{s, t\}$ but slightly differently. For $s, t \in \mathbb{P}_1(K)$, the vector space $V_\mathbb{R}$ decomposes as $(\mathbb{R}e_s + \mathbb{R}e_t)^\perp \oplus (\mathbb{R}e_s + \mathbb{R}e_t)$, where e_s, e_t are isotropic vectors associated to s, t respectively such that $(e_s, e_t) < 0$. We write $X = X_{s,t} + ae_s + be_t$ where $X_{s,t} \in (\mathbb{R}e_r + \mathbb{R}e_s)^\perp$. We then have

$$\Delta_{X,\infty}\{s, t\} = \begin{cases} \mathrm{sgn}(b) & ab < 0 \\ \frac{1}{2}\mathrm{sgn}(b) & a = 0 \\ -\frac{1}{2}\mathrm{sgn}(a) & b = 0 \\ 0 & ab > 0 \\ 0 & a = 0 = b \end{cases} \tag{3.9}$$

For each $s \in \mathbb{P}_1(K)$, we choose e_s by the recipe of the previous lemma. We fix such system of isotropic vectors for the whole proof. We write each vector X in three possible ways as

$$\begin{aligned} X &= X_{\{r,s\}} + a_r e_r + a_s e_s \\ &= X_{\{s,t\}} + b_s e_s + b_t e_t \\ &= X_{\{t,r\}} + c_t e_t + c_r e_r \end{aligned}$$

where $X_{\{r,s\}}$ is orthogonal to $H(e_r, e_s)$ and similarly for $X_{\{s,t\}}$ and $X_{\{t,r\}}$. We have

$$\begin{aligned} -(X, e_r) &= a_s(e_s, e_r) = c_t(e_t, e_r), \\ -(X, e_s) &= a_r(e_r, e_s) = b_t(e_t, e_s), \quad \textit{and} \\ -(X, e_t) &= b_s(e_s, e_t) = c_r(e_r, e_t). \end{aligned}$$

Thus we rewrite X as

$$\begin{aligned} X &= X_{\{r,s\}} + a_r e_r + a_s e_s \\ &= X_{\{s,t\}} + b_s e_s + a_r \frac{(e_r, e_s)}{(e_t, e_s)} e_t \\ &= X_{\{t,r\}} + a_s \frac{(e_s, e_r)}{(e_t, e_r)} e_t + b_s \frac{(e_s, e_t)}{(e_r, e_t)} e_r. \end{aligned}$$

Case I We assume that all three pairings (X, e_r), (X, e_s), and (X, e_t) are non-zero. By our previous derivations, the products of the relevant coefficients for the three pairs are given as follows.

1. For $\{r, s\}$, the product of coefficients is $a_r a_s$.
2. For $\{s, t\}$, the product is $b_s a_r \frac{(e_r, e_s)}{(e_t, e_s)}$.
3. For $\{t, r\}$, the product is $a_s b_s \frac{(e_s, e_r)(e_s, e_t)}{(e_t, e_r)(e_r, e_t)}$.

Because we established in the lemma that all pairings are strictly negative, any ratio of two pairings is strictly positive. Therefore, the signs of the three coefficient products are determined entirely by the signs of $a_r a_s$, $b_s a_r$, and $a_s b_s$, respectively.

Notice that the product of these three sign-determining terms is $(a_r a_s)(b_s a_r)(a_s b_s) = (a_r a_s b_s)^2 > 0$. Hence, it is impossible to have exactly one or exactly three negative terms among them. The number of negative products must be even. Thus, we only have two valid configurations to check.

1. Suppose that there are no negative products. All three products $a_r a_s$, $b_s a_r$, and $a_s b_s$ are positive. By the formula for intersection numbers, all three terms evaluate to 0. Thus

$$\Delta_{X,\infty}\{r, s\} + \Delta_{X,\infty}\{s, t\} + \Delta_{X,\infty}\{t, r\} = 0 + 0 + 0 = 0,$$

and the three-term relation holds.

2. Suppose there are exactly two negative products. Without loss of generality, assume that $a_r a_s > 0$. This forces the other two to be negative: $b_s a_r < 0$ and $a_s b_s < 0$. This implies that a_r and a_s share the same sign, which is opposite to the sign of b_s. Therefore, $\operatorname{sgn}(a_r) = -\operatorname{sgn}(b_s)$. Applying the formula, the sum becomes

$$\begin{aligned}&\Delta_{X,\infty}\{r,s\} + \Delta_{X,\infty}\{s,t\} + \Delta_{X,\infty}\{t,r\} \\ &= 0 + \operatorname{sgn}\left(a_r \frac{(e_r,e_s)}{(e_t,e_s)}\right) + \operatorname{sgn}\left(b_s \frac{(e_s,e_t)}{(e_r,e_t)}\right).\end{aligned}$$

Because the ratios of the negative pairings are positive, they can be pulled out of the sign function without altering it. The sum simplifies to

$$0 + \operatorname{sgn}(a_r) + \operatorname{sgn}(b_s) = 0.$$

This verifies the three-term relation.

Case II In this case we assume that exactly one out of the three pairings (X, e_r), (X, e_s), and (X, e_t) is zero. Without loss of generality, assume that $(X, e_r) = 0$. This implies that $a_s = 0$, and thus

$$\begin{aligned}X = X_{\{r,s\}} + a_r e_r &= X_{\{s,t\}} + b_s e_s + a_r \frac{(e_r,e_s)}{(e_t,e_s)} e_t \\ &= X_{\{t,r\}} + b_s \frac{(e_s,e_t)}{(e_r,e_t)} e_r,\end{aligned}$$

where a_r and b_s are non-zero. We now evaluate the sum for different subcases.

1. Assume $a_r > 0$ and $b_s > 0$. The product of the coefficients for the $\{s,t\}$ pair is strictly positive, which forces $\Delta_{X,\infty}\{s,t\}$ to be zero. Therefore we have

$$\begin{aligned}&\Delta_{X,\infty}\{r,s\} + \Delta_{X,\infty}\{s,t\} + \Delta_{X,\infty}\{t,r\} \\ &= -\frac{1}{2}\operatorname{sgn}(a_r) + 0 + \frac{1}{2}\operatorname{sgn}\left(b_s \frac{(e_s,e_t)}{(e_r,e_t)}\right) \\ &= -\frac{1}{2}\operatorname{sgn}(a_r) + 0 + \frac{1}{2}\operatorname{sgn}(b_s) \\ &= -\frac{1}{2} + 0 + \frac{1}{2} = 0.\end{aligned}$$

2. Assume that $a_r > 0$ but $b_s < 0$. The product of the coefficients for the $\{s,t\}$ pair is now negative. This forces $\Delta_{X,\infty}\{s,t\}$ to be non-zero and evaluate to the sign of the e_t

coefficient. Therefore the sum becomes

$$\begin{aligned}
&\Delta_{X,\infty}\{r,s\}+\Delta_{X,\infty}\{s,t\}+\Delta_{X,\infty}\{t,r\} \\
&= -\frac{1}{2}\mathrm{sgn}(a_r)+\mathrm{sgn}\left(a_r\frac{(e_r,e_s)}{(e_t,e_s)}\right)+\frac{1}{2}\mathrm{sgn}\left(b_s\frac{(e_s,e_t)}{(e_r,e_t)}\right) \\
&= -\frac{1}{2}\mathrm{sgn}(a_r)+\mathrm{sgn}(a_r)+\frac{1}{2}\mathrm{sgn}(b_s) \\
&= -\frac{1}{2}+1-\frac{1}{2}=0.
\end{aligned}$$

3. The two other cases $a_r<0, b_s>0$ and $a_r<0, b_s<0$ can be handled similarly and we thus have verified the three-term relation in this case as well.

Case III We now assume that exactly two out of the three pairings (X,e_r), (X,e_s), and (X,e_t) are zero. Without loss of generality, assume that $(X,e_r)=(X,e_s)=0$. Therefore,

$$X = X_{\{r,s\}} = X_{\{s,t\}}+b_s e_s = X_{\{t,r\}}+b_s\frac{(e_s,e_t)}{(e_r,e_t)}\;e_r$$

for $b_s\neq 0$. By (3.9), it is clear that $\Delta_{X,\infty}\{r,s\}=0$. Hence,

$$\Delta_{X,\infty}\{s,t\}+\Delta_{X,\infty}\{t,r\} = -\frac{1}{2}\mathrm{sgn}\,(b_s)+\frac{1}{2}\mathrm{sgn}\left(b_s\frac{(e_s,e_t)}{(e_r,e_t)}\right)=0.$$

Case IV We now assume that all the three pairings are zero. It is then clear that $\Delta_{X,\infty}\{r,s\}+\Delta_{X,\infty}\{s,t\}+\Delta_{X,\infty}\{t,r\}=0$.

□

4

Family of Lattices

The construction of cocycles for p-arithmetic groups involves summing over cosets of a $\mathbb{Z}[1/p]$-lattice instead of just a $\mathbb{Z}$-lattice. Since $\mathbb{Z}[1/p] = \bigcup_{r\in\mathbb{Z}_{\geq 0}} p^{-r}\mathbb{Z}$, every $\mathbb{Z}[1/p]$-lattice is an infinite union of $\mathbb{Z}$-lattices. The first section expresses cosets of a $\mathbb{Z}[1/p]$-lattice as a union of cosets of $\mathbb{Z}$-lattices.

In the second section, we specialize to the $\mathbb{Z}[1/p]$-lattice $L(\ell, N) \otimes \mathbb{Z}[1/p]$ and apply the calculations of the first section. We recall that for a *proper length with respect to* $L(\ell, N) \otimes \mathbb{Q}$, say $n \in \mathbb{Q}_{>0}$, and for $\alpha \in (L(\ell, N))^*$, the sum

$$\sum_{\substack{v\in\alpha+L(\ell,N)\\ Q(v)=n}} \Delta_{v,\infty}\{s,t\} \tag{4.1}$$

is the $(\overline{\alpha}, n)$-th coefficient of a vector-valued modular form of weight 2 and type $\varrho_{D(L(\ell,N))}$ for any pair $\{s,t\} \in \mathbb{P}_1(K) \times \mathbb{P}_1(K)$. We can rewrite $L(\ell, N) \otimes \mathbb{Z}[1/p]$ as a union of the lattices $p^{-r}L(\ell, N)$ for $r \geq 0$. This decomposition allows us to establish a modularity statement for a generating series of sums similar to (4.1) associated to the cosets of the lattices $p^r L(\ell, N)$ in their duals $p^{-r}(L(\ell, N))^*$. The statement says that the generating series is a vector-valued modular form of weight 2 and type $\varrho_{D(p^r L(\ell,N))}$. At the end of this section, we utilize the explicit action of the metaplectic group to derive modularity statements that involve scalar-valued modular forms instead of vector-valued ones.

4.1 Writing $\mathbb{Z}[1/p]$-Lattices as a Union of $\mathbb{Z}$-Lattices

Let L be a $\mathbb{Z}$-lattice. Define $L^p := L \otimes \mathbb{Z}[1/p]$ and $\Lambda := L \otimes \mathbb{Z}_p$. We then have $L = L^p \cap \Lambda$. We assume that L is contained in its dual, but we make no further assumptions on the lattice L until Proposition 4.3.

Denoting the dual of L^p by $(L^p)^*$, take an element $\beta \in (L^p)^*$ and consider the set $\{v \in \beta + L^p : Q(v) = m\}$ for a rational number m. Any vector $v \in \beta + L^p$ can be viewed as an element of $L \otimes \mathbb{Q}_p$. Thus, there exists a unique integer r such that $p^r v \in \Lambda - p\Lambda$. We denote this unique integer by $\mathrm{ord}_\Lambda(v)$. Consequently, the set admits the decomposition

$$\{v \in \beta + L^p : Q(v) = m\} = \bigsqcup_{r \in \mathbb{Z}} \{v \in \beta + L^p : \mathrm{ord}_\Lambda(v) = r,\, Q(v) = m\}. \tag{4.2}$$

Fix an integer r. Then the set $\{v \in \beta + L^p : \mathrm{ord}_\Lambda(v) = r,\ Q(v) = m\}$ is in bijection with $\{w \in p^r\beta + L^p : \mathrm{ord}_\Lambda(w) = 0,\ Q(w) = mp^{2r}\}$, where the bijection is given by $v \mapsto w := p^r v$. By definition of $\mathrm{ord}_\Lambda(\cdot)$, the set $\{w \in p^r\beta + L^p : \mathrm{ord}_\Lambda(w) = 0,\ Q(w) = mp^{2r}\}$ is equal to

$$\{w \in p^r\beta + L^p : w \in \Lambda - p\Lambda,\ Q(w) = mp^{2r}\}. \tag{4.3}$$

Lemma 4.1. *The set defined in* (4.3) *is contained in* L^*.

Proof. Let $w \in p^r\beta + L^p \subset (L^p)^*$ with $w \in \Lambda - p\Lambda$. Since $L \subset L^p$, we have $\langle w, L\rangle \subset \mathbb{Z}[1/p]$. Moreover, $w \in \Lambda - p\Lambda$ implies that we also have $\langle w, L\rangle \subset \mathbb{Z}_p$. Combining these facts, we obtain the set inclusion $\langle w, L\rangle \subset \mathbb{Z}_p \cap \mathbb{Z}[1/p] = \mathbb{Z}$. $\square$

We assume that $r \geq 0$. This condition implies that the lattice $p^{-r}L^*$—the dual of p^rL—contains L^*. Therefore by Lemma 4.1, the set in (4.3) is contained in $p^{-r}L^*$. Moreover, it can be written as a union of cosets of the lattice p^rL in its dual. The following lemma precisely specifies those cosets.

We use π_r to denote the isomorphism $D_{p^rL} \to D_{L^p} \oplus D_{p^r\Lambda}$ from Section 2.2.

Lemma 4.2. *Let* $S_{\mathrm{prim}}(p^r)$ *denote the set of representatives of classes in* $\Lambda/p^r\Lambda$ *that lie in*

$\Lambda - p\Lambda$. *Then the set* $\{w \in p^r\beta + L^p : w \in \Lambda - p\Lambda, Q(w) = mp^{2r}\}$ *is equal to*

$$\bigsqcup_{\lambda \in S_{\mathrm{prim}}(p^r)} \{w \in (p^rL)^* : \pi_r(\overline{w}) = (p^r\overline{\beta}, \overline{\lambda}), Q(w) = mp^{2r}\}$$

where $\overline{\cdot}$ *denotes the projection of an element into its respective discriminant module.*

Proof. We recall the following diagrams from Chapter 2.

$$\begin{array}{ccccc} (L^p)^* & \hookleftarrow & (p^rL)^* & \hookrightarrow & (p^r\Lambda)^* \\ \downarrow & & \downarrow & & \downarrow \\ D_{L^p} & \hookleftarrow & D_{p^rL} & \hookrightarrow & D_{p^r\Lambda} \end{array}$$

Let $w \in \{w \in p^r\beta + L^p : w \in \Lambda - p\Lambda, Q(w) = mp^{2r}\} \subset (p^rL)^*$. The projection $\overline{w} \in D_{p^rL}$ is mapped to an element of $D_{L^p} \oplus D_{p^r\Lambda}$ under the map π_r.

1. The first component of $\pi_r(\overline{w})$ is obtained by viewing w inside $(L^p)^*$ and then taking its image in D_{L^p}. This image is $p^r\overline{\beta}$ as $w \in p^r\beta + L^p$. Therefore the first coordinate of $\pi_r(\overline{w})$ is equal to $p^r\overline{\beta}$.

2. The second component of $\pi_r(\overline{w})$ is the image of $w \in (p^r\Lambda)^*$ under the map $(p^r\Lambda)^* \to D_{p^r\Lambda}$. Since $w \in \Lambda - p\Lambda$, the second component of $\pi_r(\overline{w})$ has to be $\overline{\lambda}$ for some $\lambda \in \Lambda - p\Lambda$.

This shows that the set on the left is contained in the set on the right. The reverse containment follows similarly from the diagram. $\square$

Remark 4. *The assumption of* $r \geq 0$ *is satisfied if we assume* $\mathrm{ord}_p(m) < 0$. *To see this, let* $v \in \beta + L^p$ *with* $\mathrm{ord}_\Lambda(v) = r$ *and* $Q(v) = m$. *Then* $p^rv \in \Lambda - p\Lambda$ *and thus* $\mathrm{ord}_pQ(v) + 2r \geq 0$, *which means that* $r \geq -\mathrm{ord}_p(m)/2 \geq 0$.

For the following proposition, we take L to be $L(\ell, N)$.

Proposition 4.3. *Let* $n \in \mathbb{Q}$ *be a* proper length with respect to $L(\ell, N) \otimes \mathbb{Q}$ *such that the numerator of* n *is coprime to* p. *Let* $S_{\mathrm{prim}}(p^r)$ *be the set defined in Lemma 4.2. The following* *equality holds for any integer $r \geq 0$.*

$$\begin{aligned}
&\sum_{\substack{Q(v)=n \\ \mathrm{ord}_\Lambda(v)=r}} \mathbb{1}(v \in \beta + L^p) \cdot \Delta_{v,\infty}\{s,t\} \\
&= \sum_{\lambda \in S_{\mathrm{prim}}(p^r)} \sum_{Q(w)=np^{2r}} \mathbb{1}(w \in (p^r L)^* : \pi_r(\overline{w}) = (p^r\overline{\beta}, \overline{\lambda})) \cdot \Delta_{w,\infty}\{s,t\}. \qquad (4.4)
\end{aligned}$$

Remark 5. *From our discussion in the first chapter, it is natural to expect that the sum on the right-hand side in (4.4) has some relation with Fourier coefficients of a vector-valued modular form. We make this precise in the next section. The above proposition then allows us to relate the sums over a $\mathbb{Z}[1/p]$-lattice to Fourier coefficients of modular forms.*

Proof. The bijection $v \mapsto w := p^r v$, discussed earlier, allows us to rewrite the left-hand side in (4.4) as

$$\sum_{\substack{Q(w)=np^{2r} \\ w \in \Lambda - p\Lambda}} \mathbb{1}(w \in p^r\beta + L^p) \cdot \Delta_{w,\infty}\{s,t\}.$$

It is crucial to note that $\Delta_{v,\infty}\{s,t\} = \Delta_{p^r v,\infty}\{s,t\} = \Delta_{w,\infty}\{s,t\}$ since scaling by a positive real does not change the sign of intersections. Finally using Lemma 4.2, we conclude that the above is equal to

$$\sum_{\lambda \in S_{\mathrm{prim}}(p^r)} \sum_{Q(w)=np^{2r}} \mathbb{1}(w \in p^{-r}L^* : \pi_r(\overline{w}) = (p^r\overline{\beta}, \overline{\lambda})) \cdot \Delta_{w,\infty}\{s,t\}$$

□

4.2 Modularity Statements

In this section, we only work with the lattice $L(\ell, N)$. We recall that $L(\ell, N) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus N\mathbb{Z}e_3 \oplus \mathbb{Z}e_4$ where

$$(e_i, e_j) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2\ell & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Lemma 4.4. *For the lattice $L(\ell, N)$, the discriminant module $D(\ell, N)$ has* sign 2*, size $4\ell N^2$, and level* $\operatorname{lcm}(4\ell, N)$.

Proof. The sign in this case is equal to $r - s \pmod 8$, where (r, s) is the signature of the lattice. This is proved in [19, Appendix 4 (page 127)]. The authors use the term "signature" for the difference $r - s$. The size is calculated via the index of $L(\ell, N)$ in $(L(\ell, N))^*$. For the level, consider an arbitrary $v \in (L(\ell, N))^*$. This vector is of the form

$$v = \frac{a}{2}e_1 + \frac{b}{2\ell}e_2 + ce_3 + \frac{d}{N}e_4$$

for $a, b, c, d \in \mathbb{Z}$. We then have $Q(v) = \dfrac{a^2}{4} + \dfrac{b^2}{4\ell} - \dfrac{cd}{N}$. Hence the level is equal to $\operatorname{lcm}(4\ell, N)$. $\square$

Assumption. From now on, we assume that the integers ℓ and N are coprime to p.

Define

$$L^p := L(\ell, N) \otimes \mathbb{Z}[1/p] = \mathbb{Z}[1/p]e_1 \oplus \mathbb{Z}[1/p]e_2 \oplus N\mathbb{Z}[1/p]e_3 \oplus \mathbb{Z}[1/p]e_4, \tag{4.5}$$

$$\Lambda := L(\ell, N) \otimes \mathbb{Z}_p = \mathbb{Z}_pe_1 \oplus \mathbb{Z}_pe_2 \oplus \mathbb{Z}_pe_3 \oplus \mathbb{Z}_pe_4 \tag{4.6}$$

where the equality in the second line follows as N is a unit in $\mathbb{Z}_p$ by our assumption.

Lemma 4.5. *The lattice Λ is a self-dual $\mathbb{Z}_p$-lattice.*

Proof. The result is clear since 2 and ℓ are invertible in $\mathbb{Z}_p$. $\square$

We denote the discriminant module corresponding to L^p by $D(\ell, N)^p$. From the discussion in Section 2.2 of Chapter 2, it is clear that $D(\ell, N)$ is isomorphic to $D(\ell, N)^p \oplus D_\Lambda$ as finite quadratic modules. Since Λ is self-dual, D_Λ is trivial. Thus $D(\ell, N)^p$ and $D(\ell, N)$ are isomorphic as finite quadratic modules. Therefore, the size and level are the same for both modules. Since the sign is defined by a formula depending only on the finite quadratic module (see (2.1)), the sign is also the same.

Let n be a *proper length with respect to $V_\ell = L(\ell, N) \otimes \mathbb{Q}$*, r be a non-negative integer, and let $\beta \in (L^p)^*$ and $\lambda \in \Lambda - p\Lambda$. Then we have the following proposition which states the modularity of intersection numbers similar to Chapter 3. The difference here is that we are now summing over cosets of $p^r L(\ell, N)$ instead of $L(\ell, N)$. The proof is also similar to the one in the first chapter, and thus we only provide a sketch.

Proposition 4.6. *Let $(p^r\beta, \lambda)$ denote an element in $(p^rL)^*$ satisfying*

$$\pi_r(\overline{(p^r\beta, \lambda)}) = (p^r\overline{\beta}, \overline{\lambda}) \in D_{p^rL(\ell,N)}.$$

Then the sum

$$S(r, \beta, \lambda, n) := \sum_{Q(w)=np^{2r}} \mathbb{1}(w \in (p^r\beta, \lambda) + p^rL)\ \Delta_{w,\infty}\{s, t\}$$

is the $((p^r\overline{\beta}, \overline{\lambda}), np^{2r})$-th coefficient of a vector-valued modular form of weight 2 and type $\varrho_{D_{p^rL(\ell,N)}}$.

Proof. The proof follows the same calculations as presented in Chapter 3. We consider the theta series

$$\Theta(\tau, p^rL(\ell, N)) = \sum_{\overline{\alpha}\in D_{p^rL(\ell,N)}} \sum_{w\in\alpha+p^rL(\ell,N)} \eta(\tau, w),$$

where $\eta(\tau, w)$ is the 1-form defined by (3.2). Using the general theory of Weil representations it can be established that this is a real analytic function that transforms like a vector-valued modular form of weight 2 and type $\varrho_{D_{p^rL}}$.

The next step is to integrate this theta series over various submanifolds of dimension one. We can integrate the sum

$$\sum_{Q(w)=np^{2r}} \eta(\tau, w)$$

term-by-term along any geodesic from s to t as long as all w in the sum have a non-isotropic component along the corresponding hyperbolic plane $H(e_s, e_t)$. If n is a proper length then np^{2r} is as well and term-by-term integration is possible. This yields the required statement.

$\square$

Note that the discriminant module $D_{p^rL(\ell,N)}$ decomposes as $D(\ell, N)^p \oplus D_{p^r\Lambda}$. We first use the explicit action to get a modular form for $D(\ell, N)^p$ (Proposition 4.7) and then a corresponding scalar-valued modular form (Theorem 4.8).

Proposition 4.7. *Let r be a positive integer with $\beta \in (L(\ell, N))^*$ such that $Q(\beta) \in \mathbb{Z}$. Let m be a* proper length with respect to $L(\ell, N) \otimes \mathbb{Q}$. *Let $\mathfrak{l}$ be a line in $\Lambda/p^r\Lambda$ such that it is* *generated by the class of some $\lambda \in S_{\mathrm{prim}}(p^r)$ and* $\mathrm{Gen}(\mathfrak{l})$ *denote the set of generators. Then the sum*

$$\sum_{\lambda \in \mathrm{Gen}(\mathfrak{l})} S(r, \beta, \lambda, m) \tag{4.7}$$

is the $(p^r\beta, mp^{2r})$-th coefficient of a modular form $\mathcal{F}(r, \mathfrak{l})$ of weight 2, type $D(\ell, N)^p$ for the group $\mathcal{G}_0(p^{2r}) := \pi^{-1}(\Gamma_0(p^{2r}))$.

Proof. It is clear from Proposition 4.6 that the sum $S(r, \beta, \lambda, m)$ is the $((p^r\overline{\beta}, \overline{\lambda}), mp^{2r})$-th coefficient of a modular form of weight 2 and type $\varrho_{D_{p^rL}}$ for the group $\mathrm{Mp}_2(\mathbb{Z})$. Let $F = \sum_{x \in D_{p^rL}} f_x \vec{\mathbf{e}}_x$ be that modular form. Therefore, the sum in (4.7) is $(p^r\beta, mp^{2r})$-th coefficient of the function

$$\sum_{x^{(p)} \in D(\ell,N)^p} \left[\sum_{\lambda \in \mathrm{Gen}(\mathfrak{l})} f_{(x^{(p)},\lambda)} \right] \vec{\mathbf{e}}_{x^{(p)}}.$$

We now establish that this function is a vector-valued modular form of weight 2, type $\varrho_{D(\ell,N)^p}$, and the group $\mathcal{G}_0(p^{2r})$.

Since the module $D(\ell, N)$ decomposes as $D(\ell, N)^p \oplus D_{p^r\Lambda}$, we need to understand the action of $\mathcal{G}_0(p^{2r})$ on $D_{p^r\Lambda}$. The sign is equal to 2 which means that the action reduces to that of $\Gamma_0(p^{2r})$. Since the level of this module is p^{2r}, we have for $x_p \in D_{p^r\Lambda}$ with $Q(x_p) = 0 \in \mathbb{Q}/\mathbb{Z}$,

$$\varrho_{D_{p^r\Lambda}}(\gamma)(\vec{\mathbf{e}}_{x_p}) = \left(\frac{d}{|D_{p^r\Lambda}|}\right) \vec{\mathbf{e}}_{dx_p} = \left(\frac{d}{p^{8r}}\right) \vec{\mathbf{e}}_{dx_p} = \vec{\mathbf{e}}_{dx_p} \qquad \textit{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p^{2r}).$$

The modularity of F implies that for $\gamma \in \mathcal{G}_0(p^{2r})$ we have

$$\begin{aligned} &\sum_{x_p \in D_{p^r\Lambda}} \sum_{x^{(p)} \in D(\ell,N)^p} f_{(x_p, x^{(p)})} \vec{\mathbf{e}}_{(x_p, x^{(p)})} \\ &= \sum_{x_p \in D_{p^r\Lambda}} \left[\sum_{x^{(p)} \in D(\ell,N)^p} f_{(x_p, x^{(p)})} \mid \gamma^{-1} \right] \left(\varrho_{D_{p^r\Lambda}}(\gamma) \vec{\mathbf{e}}_{x_p} \otimes \varrho_{D_{L^p}}(\gamma) \vec{\mathbf{e}}_{x^{(p)}} \right). \end{aligned} \tag{4.8}$$

Using the explicit action, the right-hand side is equal to

$$\sum_{x_p \in D_{p^r\Lambda}} \left[\sum_{x^{(p)} \in D(\ell,N)^p} f_{(x_p, x^{(p)})} \mid \gamma^{-1} \right] \vec{\mathbf{e}}_{dx_p} \otimes \varrho_{D(\ell,N)^p}(\gamma) \vec{\mathbf{e}}_{x^{(p)}}$$
$$= \sum_{x_p \in D_{p^r\Lambda}} \left[\sum_{x^{(p)} \in D(\ell,N)^p} f_{(x_p, x^{(p)})} \mid \gamma^{-1} \varrho_{D(\ell,N)^p}(\gamma) \vec{\mathbf{e}}_{x^{(p)}} \right] \otimes \vec{\mathbf{e}}_{dx_p}.$$

Since the set $\{d\lambda : \lambda \in \mathrm{Gen}(\mathfrak{l})\}$ is the same as the set $\mathrm{Gen}(\mathfrak{l})$ for any d coprime to p, we use (4.8) to conclude that for any $\gamma \in \mathcal{G}_0(p^{2r})$,

$$\sum_{x^{(p)} \in D(\ell,N)^p} \left[\sum_{x_p \in \mathrm{Gen}(\mathfrak{l})} f_{(x_p, x^{(p)})} \mid \gamma^{-1} \right] \varrho_{D^p}(\gamma) \vec{\mathbf{e}}_{x^{(p)}} = \sum_{x^{(p)} \in D(\ell,N)^p} \left[\sum_{x_p \in \mathrm{Gen}(\mathfrak{l})} f_{(x_p, x^{(p)})} \right] \vec{\mathbf{e}}_{x^{(p)}}$$

and this proves the modularity of the function $\displaystyle\sum_{x^{(p)} \in D(\ell,N)^p} \left[\sum_{\lambda \in \mathrm{Gen}(\mathfrak{l})} f_{(\lambda, x^{(p)})} \right] \vec{\mathbf{e}}_{x^{(p)}}.$ □

We now obtain a scalar-valued modular form using a similar averaging argument.

Theorem 4.8. *Let $\beta \in (L(\ell,N))^*$ such that $Q(\beta) \in \mathbb{Z}$. Let m be a* proper length *with respect to $L(\ell,N) \otimes \mathbb{Q}$. r be a positive integer, and $\mathfrak{l}$ be an isotropic line in $\Lambda/p^r\Lambda$. Define $M := \mathrm{lcm}(4\ell, N)$. Then the sum*

$$\sum_{\substack{0<u<M \\ (u,M)=1}} \left[\sum_{\lambda \in \mathrm{Gen}(\mathfrak{l})} \sum_{Q(w)=mp^{2r}} \Delta_{w,\infty}\{s,t\} \cdot \mathbb{1}(w \in (up^r\beta, \lambda) + p^r L(\ell,N)) \right] \tag{4.9}$$

is the mp^{2r}-th coefficient of a modular form of weight 2 for the group $\Gamma_0(Mp^{2r})$ and nebentypus $\left(\frac{\cdot}{4\ell N^2 p^2}\right) = \left(\frac{\cdot}{\ell}\right)$.

We first present a slightly abstract proposition which can then be used to prove the above theorem.

Proposition 4.9. *Let $\mathcal{G}$ be a subgroup of $\mathrm{Mp}_2(\mathbb{Z})$ and let $F = \sum_{x \in D} f_x \vec{\mathbf{e}}_x$ be an element of $M_2(D, \mathcal{G})$ for a discriminant module D such that the* sign(D) *is equal to 2, and the level is N_D. For $\beta \in D$ with $Q(\beta) = \overline{0} \in \mathbb{Q}/\mathbb{Z}$, define $\mathfrak{S}(\beta) := \{u\beta \in D : 0 < u < N_D, (u, N_D) = 1\}$. Then*

$\sum_{x\in\mathfrak{S}(\beta)} f_x$ *is a scalar-valued modular form of weight* 2 *for the group* $\pi(\mathcal{G})\cap\Gamma_0(N_D)\subset \mathrm{SL}_2(\mathbb{Z})$ *and nebentypus* $\left(\frac{\cdot}{|D|}\right)$ *where* $\pi:\mathrm{Mp}_2(\mathbb{Z})\to\mathrm{SL}_2(\mathbb{Z})$ *is the map discussed in Chapter 2.*

Proof. The proof of this proposition relies on understanding the action of $\Gamma_0(N_D)$ on the discriminant module. The modularity of $\sum_{x\in D} f_x\vec{\mathbf{e}}_x$ implies that for all $\delta\in\mathcal{G}$ we have

$$\sum_{x\in D}(f_x \mid \delta^{-1})\varrho_D(\delta)\vec{\mathbf{e}}_x = \sum_{x\in D} f_x\vec{\mathbf{e}}_x. \tag{4.10}$$

It is given that the sign of D is 2, and the level is N_D. Recalling our discussion around Lemma 2.12, we note that for any $x\in D$ such that $Q(x)=\overline{0}\in\mathbb{Q}/\mathbb{Z}$, the action of $\gamma\in\Gamma_0(N_D)$ on the basis vector $\vec{\mathbf{e}}_x$ is

$$\varrho_D(\gamma)(\vec{\mathbf{e}}_x)=\left(\frac{d}{|D|}\right)\vec{\mathbf{e}}_{dx},\qquad \textit{for } \gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}. \tag{4.11}$$

For x with $Q(x)\neq\overline{0}$, the vector $\vec{\mathbf{e}}_x$ is still sent to $\vec{\mathbf{e}}_{dx}$ under the action of γ but it also gets scaled by a root of unity along with the character. We plug $\delta=\gamma$ in (4.10) and use the explicit action of γ. Adding the coefficients corresponding to the vectors $\{\vec{\mathbf{e}}_x\}_{x\in\mathfrak{S}(\beta)}$ on both sides and using (4.11) gives

$$\left(\frac{d}{|D|}\right)\sum_{x\in\mathfrak{S}(\beta)}(f_x\mid\gamma^{-1})=\sum_{x\in\mathfrak{S}(\beta)} f_x,$$

which is the same as

$$\left(\frac{d}{|D|}\right)\sum_{x\in\mathfrak{S}(\beta)} f_x=\sum_{x\in\mathfrak{S}(\beta)}(f_x\mid\gamma).$$

□

Proof of Theorem 4.8. We specialize the proposition to prove the theorem. We do it in a bulleted fashion so that the argument is easy to check.

1. From Proposition 4.7, we have a vector-valued modular form $\mathcal{F}(r,\mathfrak{l})=\sum_{x\in D(\ell,N)^p} f_x\vec{\mathbf{e}}_x$ of weight 2, type $\varrho_{D(\ell,N)^p}$ and for the group $\mathcal{G}_0(p^{2r})\subseteq\mathrm{Mp}_2(\mathbb{Z})$. The discriminant module $D(\ell,N)^p$ is isomorphic to $D(\ell,N)$. Thus the sign is equal to 2, and the level is $M=\mathrm{lcm}(4\ell,N)$.

2. Consider the vector $p^r\beta\in(L^p)^*$ with $Q(p^r\beta)\in\mathbb{Z}$. Let $p^r\overline{\beta}\in D(\ell,N)^p$ denote its

image in the discriminant module. Because $Q(p^r\beta) \in \mathbb{Z}$, the quadratic form on the discriminant module satisfies $Q(p^r\overline{\beta}) = \overline{0} \in \mathbb{Q}/\mathbb{Z}$. We can therefore apply Proposition 2.10 to the element $p^r\overline{\beta}$ and the set $\{up^r\overline{\beta} \in D(\ell, N)^p : 0 < u < M, (u, M) = 1\}$.

We apply Proposition 4.9 in the above setup and obtain a scalar-valued modular form of weight 2. The group $\pi(\mathcal{G}_0(p^{2r})) \cap \Gamma_0(M)$ is $\Gamma_0(Mp^{2r})$ because $\pi(\mathcal{G}_0(p^{2r})) = \Gamma_0(p^{2r})$. The size of $D(\ell, N)^p$ is the same as that of $D(\ell, N)$, which is $4\ell N^2$. Therefore the character is

$$\left(\frac{\cdot}{4\ell N^2}\right) = \left(\frac{\cdot}{\ell}\right).$$

Hence, the function

$$\sum_{\substack{0<u<M \\ (u,M)=1}} f_{up^r\overline{\beta}}$$

is a modular form of weight 2 and for the group $\Gamma_0(Mp^{2r})$ with nebentypus $\left(\frac{\cdot}{\ell}\right)$ whose mp^{2r}-th coefficient is the sum (4.9).

□

Corollary 4.10. *The sum* (4.9) *is the m-th coefficient of a modular form of weight* 2 *for the group* $\Gamma_0(Mp)$ *with character* $\left(\frac{\cdot}{\ell}\right)$*, where* $M = \operatorname{lcm}(4\ell, N)$*.*

Proof. Let g be a modular form of weight 2 for the group $\Gamma_0(Mp^r)$ with character $\left(\frac{\cdot}{\ell}\right)$ for some positive integer $r \geq 2$. Then the Hecke operator T_p for $\Gamma_0(Mp^r)$ has the property that

1. $T_p(g) \in M_2(\Gamma_0(Mp^{r-1}), \left(\frac{\cdot}{\ell}\right))$, and
2. m-th coefficient of $T_p(g)$ is the mp-th coefficient of g.

These results can be deduced from standard references on modular forms. Moreover, if g is a modular form for the group $\Gamma_0(Mp)$, then $T_p(g)$ is again a modular form for $\Gamma_0(Mp)$ and the mp-th coefficient of g is the m-th coefficient of $T_p(g)$. Applying this to Theorem 4.8 completes the proof. □

5

Rigid Space and Meromorphic Functions

In this chapter, we describe a p-adic analytic space associated with the vector space $V_\ell \otimes \mathbb{Q}_p$. We also discuss some examples of meromorphic functions of the type that feature in our construction of rigid cocycles. The material in this chapter is taken from Sections 1 and 2 of [5].

5.1 p-adic Symmetric Spaces

We first discuss the archimedean symmetric space for $O(2,2)$ to motivate the definition of the p-adic space in Definition 5.1. Let (V, Q) be a non-degenerate quadratic space over $\mathbb{Q}$ such that $V_\mathbb{R} := V \otimes \mathbb{R}$ is of signature $(2,2)$. If $\mathcal{K} \subset O_V(\mathbb{R})$ is a maximal compact subgroup, then $O_V(\mathbb{R})/\mathcal{K}$ is a Riemannian manifold whose automorphism group acts transitively on it. We already saw this type of manifold in the first chapter for signature $(n,1)$, but in this case of signature $(2,2)$, we can also equip it with a complex structure using an alternative model. Consider the complex vector space $V_\mathbb{C} := V_\mathbb{R} \otimes \mathbb{C}$. Let $\mathcal{N}_\infty$ denote the set of isotropic lines in

$V_{\mathbb{C}}$. Consider the set

$$X_\infty := \{[v] \in \mathcal{N}_\infty : \langle v, \overline{v} \rangle \neq 0\}/\mathbb{C}^\times.$$

This set is a manifold of complex dimension 2 consisting of four connected components such that there is a real-analytic isomorphism between $O_V(\mathbb{R})/\mathcal{K}$ and each connected component.

Moreover, it is shown in [4, Section 2.7] that each component is isomorphic to $\mathcal{H} \times \mathcal{H}$. This is proved by calculating $\mathcal{N}_\infty$ and X_∞ for a suitable model of $V \otimes \mathbb{R}$. Our goal is to mimic this setup and corresponding calculations in the p-adic world.

Assumption. The prime p splits in the quadratic field $\mathbb{Q}(\sqrt{-\ell})$. It implies that the vector space $V_\ell \otimes \mathbb{Q}_p$ is isomorphic to the quadratic space $M_2(\mathbb{Q}_p)$ with the quadratic form given by $-\det$.

The following table provides a comparison of Archimedean and non-Archimedean setups and also mentions the results we are going to obtain via calculations in the p-adic world.

Archimedean viewpoint	Non-Archimedean viewpoint
F : *real quadratic field (∞ splits in F)*	K : *imaginary quadratic field (p splits in K)*
$F \xhookrightarrow{\sigma_1, \sigma_2} \mathbb{R}$	$K \xhookrightarrow{\sigma_1, \sigma_2} \mathbb{Q}_p$
Symmetric Space is $\mathcal{H} \times \mathcal{H}$	*Symmetric Space is* $\mathcal{H}_p \times \mathcal{H}_p$
$\mathrm{SL}_2(F) \hookrightarrow \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \curvearrowright \mathcal{H} \times \mathcal{H}$	$\mathrm{SL}_2(K) \hookrightarrow \mathrm{SL}_2(\mathbb{Q}_p) \times \mathrm{SL}_2(\mathbb{Q}_p) \curvearrowright \mathcal{H}_p \times \mathcal{H}_p$

Consider the set $\mathcal{N}_p := \{v \in V_\ell \otimes \mathbb{C}_p - \{0\} : Q(v) = 0\}/\mathbb{C}_p^\times$.

Definition 5.1. *Let $\mathfrak{Q}(\mathbb{Q}_p) = \{v \in V_\ell \otimes \mathbb{Q}_p : Q(v) = 0\}$ denote the set of isotropic vectors in $V_\ell \otimes \mathbb{Q}_p$. The p-adic symmetric space is defined as*

$$X_p := \{[v] \in \mathcal{N}_p : \langle v, w \rangle \neq 0 \textit{ for all } w \in \mathfrak{Q}(\mathbb{Q}_p)\}.$$

To compute X_p explicitly, we use the matrix model. Let $M_2(\mathbb{Q}_p)$ be the $\mathbb{Q}_p$-vector space of 2×2 matrices, equipped with the quadratic form given by the negative of the determinant. The collection of isotropic lines in $M_2(\mathbb{C}_p) = M_2(\mathbb{Q}_p) \otimes \mathbb{C}_p$ is

$$\mathcal{N}_p = \left\{ \begin{pmatrix} \tau_1\eta_2 & \tau_1\tau_2 \\ \eta_1\eta_2 & \eta_1\tau_2 \end{pmatrix} : (\tau_1, \tau_2, \eta_1, \eta_2) \in \mathbb{C}_p^4 - \{(0,0,0,0)\} \right\} / \mathbb{C}_p^\times.$$

The map sending $[v] \in \mathcal{N}_p$ to $([\tau_1 : \eta_1], [\tau_2 : \eta_2])$ is well-defined and establishes a bijection between $\mathcal{N}_p$ and $\mathbb{P}_1(\mathbb{C}_p) \times \mathbb{P}_1(\mathbb{C}_p)$. Similarly, we have

$$\mathfrak{Q}(\mathbb{Q}_p) = \left\{ \begin{pmatrix} x_1y_2 & x_1x_2 \\ y_1y_2 & y_1x_2 \end{pmatrix} : (x_1, x_2, y_1, y_2) \in \mathbb{Q}_p^4 - \{(0,0,0,0)\} \right\}.$$

It is easy to see that $\mathfrak{Q}(\mathbb{Q}_p)/\mathbb{Q}_p^\times$ is in bijection with $\mathbb{P}_1(\mathbb{Q}_p) \times \mathbb{P}_1(\mathbb{Q}_p)$ via the same map $\mathcal{N}_p \to \mathbb{P}_1(\mathbb{C}_p) \times \mathbb{P}_1(\mathbb{C}_p)$.

Choosing a representative v for the class $[v] \in \mathcal{N}_p$, we now compute the pairing $\langle v, w \rangle$ with a vector $w \in \mathfrak{Q}(\mathbb{Q}_p)$. A simple calculation shows that

$$\langle v, w \rangle = -\det \begin{pmatrix} \tau_1\eta_2 + x_1y_2 & \tau_1\tau_2 + x_1x_2 \\ \eta_1\eta_2 + y_1y_2 & \tau_2\eta_1 + x_2y_1 \end{pmatrix} = (\tau_1y_1 - \eta_1x_1)(\tau_2y_2 - \eta_2x_2)$$

If both $[\tau_1 : \eta_1]$ and $[\tau_2 : \eta_2]$ are different from $[x_1 : y_1]$ and $[x_2 : y_2]$ respectively, then the pairing $\langle v, w \rangle$ is non-zero. For the pairing to be non-zero for all $w \in \mathfrak{Q}(\mathbb{Q}_p)$, $[\tau_i : \eta_i] \notin \mathbb{P}_1(\mathbb{Q}_p)$ for $i = 1, 2$. Therefore, X_p is in bijection with $(\mathbb{P}_1(\mathbb{C}_p) \setminus \mathbb{P}_1(\mathbb{Q}_p)) \times (\mathbb{P}_1(\mathbb{C}_p) \setminus \mathbb{P}_1(\mathbb{Q}_p))$.

On the quadratic space $(M_2(\mathbb{Q}_p), -\det)$, the group $\mathrm{SL}_2(\mathbb{Q}_p) \times \mathrm{SL}_2(\mathbb{Q}_p)$ acts and preserves the quadratic form. Let K be a quadratic field such that the prime p splits in K. Then $\mathrm{SL}_2(K)$ is a subgroup of $\mathrm{SL}_2(\mathbb{Q}_p) \times \mathrm{SL}_2(\mathbb{Q}_p)$ via the two embeddings σ_1, σ_2 of K into $\mathbb{Q}_p$. Therefore we have an action of $\mathrm{SL}_2(K)$ on $M_2(\mathbb{Q}_p)$. In fact we have an action on $\mathcal{N}_p$ and thus we have an action on $\mathcal{H}_p \times \mathcal{H}_p$. Any $\gamma \in \mathrm{SL}_2(K)$ maps $(\tau_1, \tau_2) \in \mathcal{H}_p \times \mathcal{H}_p$ to $(\sigma_1(\gamma)\tau_1, \sigma_2(\gamma)\tau_2)$ where each coordinate is a Möbius transformation by the corresponding matrix.

5.2 Rigid Analytic Structure

We now describe a rigid analytic structure on the p-adic space X_p by writing it as an increasing union of affinoids. Let Λ be a self-dual $\mathbb{Z}_p$-lattice. Put $\Lambda_{\mathcal{O}_{\mathbb{C}_p}} := \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{C}_p}$, and let

$$\Lambda' = \Lambda - p\Lambda, \qquad \Lambda'_{\mathcal{O}_{\mathbb{C}_p}} = \Lambda_{\mathcal{O}_{\mathbb{C}_p}} - \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}} \Lambda_{\mathcal{O}_{\mathbb{C}_p}}$$

denote the sets of primitive vectors in Λ and $\Lambda_{\mathcal{O}_{\mathbb{C}_p}}$ respectively. Each zero-length vector ξ with $[\xi] \in X_p$ can be represented by $v_\xi \in \Lambda'_{\mathcal{O}_{\mathbb{C}_p}}$ which is well-defined up to multiplication by $\mathcal{O}_{\mathbb{C}_p}^\times$. In this manner, we can view each ξ as an element of a zero locus in Λ defined over $\mathcal{O}_{\mathbb{C}_p}$. For each integer $k \geq 0$ and every self-dual lattice $\Lambda \subset V_{\mathbb{Q}_p}$, we define the following affinoid subset

$$X_p^{\leq k} := \{[\xi] \in X_p : \mathrm{ord}_p(\langle v_\xi, w\rangle) \leq k \textit{ for all } w \in (\Lambda')_0\},$$

where the subscript 0 denotes the subset of isotropic vectors in Λ'.

Lemma 5.2. *The $X_p^{\leq k}$ form an increasing union and cover X_p.*

Proof. It follows from the definition that this family of sets forms an increasing union. We need to establish that it is a covering of X_p, which is equivalent to proving $\bigcap_{k\geq 0}(X_p^{\leq k})^c \subseteq X_p^c$. Let $\xi \in V_{\mathbb{C}_p}$ be such that for each $k \geq 0$ there is $w_k \in (\Lambda')_0$ for which $\mathrm{ord}_p(\langle v_\xi, w_k\rangle) > k$. It suffices to establish that there is a convergent subsequence of $\{w_k\}_{k\geq 0}$ such that its limit, say $\omega \in (\Lambda')_0$, satisfies $\langle v_\xi, \omega\rangle = 0$.

It is clear that Λ is closed and bounded under the p-adic metric. Hence, there is a subsequence of $\{w_k\}_{k\geq 0}$ that converges to some $\omega \in \Lambda$. This ω is isotropic because all w_k are isotropic. The choice of w_k implies $\mathrm{ord}_p\langle v_\xi, w_k\rangle > k$ and thus $\langle v_\xi, w_k\rangle \to 0$. By continuity of the function $\langle v_\xi, \cdot\rangle : V_{\mathbb{Q}_p} \to \mathbb{C}_p$, and the uniqueness of the limit, $\langle v_\xi, \omega\rangle = 0$.

□

We can use these definitions to get a coordinate-based description of affinoids on $\mathcal{H}_p \times \mathcal{H}_p$. However, we avoid using $\mathcal{H}_p \times \mathcal{H}_p$ as it just introduces more variables and indices that are not needed for arguments in the remaining parts of this chapter.

5.3 Meromorphic Functions

In this section, we use the affinoid covering to define rigid analytic functions and meromorphic functions on X_p. We provide some examples of rigid meromorphic functions associated to vectors of the underlying vector space V_ℓ.

Definition 5.3. *A rigid analytic function on X_p is a $\mathbb{C}_p$-valued function such that its restriction to each $X_p^{\leq k}$ is a uniform limit of rational functions having poles outside of $X_p^{\leq k}$.*

Definition 5.4. *A rigid meromorphic function is a $\mathbb{C}_p$-valued function such that its restriction to each $X_p^{\leq k}$ is the quotient of two analytic functions. We denote this space by $\mathcal{M}(X_p)$.*

We recall that the divisors of Borcherds products for $O(2,2)$ are linear combinations of Hirzebruch-Zagier divisors. Taking the analogy between the Archimedean and non-Archimedean worlds further, we define special divisors on X_p.

Definition 5.5. *Let $v \in V_{\mathbb{Q}_p}$. A special divisor associated to v is the locus $\Delta_{v,p} := \{[\xi] \in X_p : \langle v, \xi \rangle = 0\}$.*

For any $v \in V_{\mathbb{Q}_p}$, it is clear that there exists a rigid analytic function whose vanishing set (divisor) is $\Delta_{v,p}$. However, we are interested in more exotic functions whose divisors are combinations of infinitely many $\Delta_{v,p}$.

Definition 5.6. *A formal divisor $\sum_{v \in V} a_v \Delta_{v,p}$ is called locally finite if the sum*

$$\sum_{\substack{v \in V \\ \Delta_{v,p} \cap \mathcal{A} \neq \emptyset}} a_v \Delta_{v,p}$$

is finite for any affinoid $\mathcal{A}$. We denote the module of locally finite divisors by $\mathrm{Div}^{\dagger}_{\mathrm{rq}}$.

Lemma 5.7. *Let $v \in \Lambda'$ such that $Q(v) \equiv 0 (mod\ p^k)$. Then there exists $w \in (\Lambda')_0$ such that $v \equiv w (mod\ p^k \Lambda)$.*

Proof. The assumption on v implies that its image in $\Lambda / p^k \Lambda$ is isotropic. By Hensel's lemma, there exists an isotropic vector $w \in \Lambda$ such that $v \equiv w (mod\ p^k \Lambda)$. If w is not primitive, then it implies that v is not primitive as well. Hence, $w \in (\Lambda')_0$. $\square$

The following proposition constructs a rigid meromorphic function that is a simpler version of the functions constituting a rigid meromorphic cocycle. We include this result to illustrate the arguments required to establish convergence of infinite products that we need in Chapter 6.

Proposition 5.8. *Let $\{(v_{1,k}, v_{2,k})\}_{k=0}^{\infty}$ be a sequence of pairs of vectors such that*

1. *$v_{1,k}, v_{2,k} \in \Lambda - p\Lambda$ with $Q(v_{1,k}) \equiv Q(v_{2,k}) \equiv 0 \pmod{p^k}$ for all $k \geq 0$, and*
2. *$v_{1,k} \equiv \alpha_k v_{2,k} \pmod{p^k \Lambda}$ for some sequence of scalars $\alpha_k \in \mathbb{Z}_p^{\times}$.*

Then the product

$$\prod_{k \geq 0} \frac{\langle v_\xi, v_{1,k} \rangle}{\langle v_\xi, v_{2,k} \rangle}$$

converges to an element of $\mathcal{M}(X_p)^{\times} / \mathbb{Z}_p^{\times}$.

Proof. We must establish that for $[\xi] \in X_p^{\leq t}$, the general term of the product scaled by α_k^{-1} approaches 1 as $k \to \infty$. Because $|\alpha_k|_p = 1$, we have

$$\left|\frac{\langle v_\xi, v_{1,k}\rangle}{\alpha_k \langle v_\xi, v_{2,k}\rangle} - 1\right| = \left|\frac{\langle v_\xi, v_{1,k} - \alpha_k v_{2,k}\rangle}{\langle v_\xi, v_{2,k}\rangle}\right|.$$

The second hypothesis implies that $v_{1,k} - \alpha_k v_{2,k} = p^k u_k$ for some $u_k \in \Lambda$. Thus, the numerator is bounded by p^{-k}.

Since the length of $v_{2,k}$ is divisible by p^k and $v_{2,k}$ is primitive, there exists a vector $w \in (\Lambda')_0$ such that $v_{2,k} \equiv w \pmod{p^k \Lambda}$. Therefore, $\langle v_\xi, v_{2,k}\rangle = \langle v_\xi, w\rangle + \langle v_\xi, p^k w_0\rangle$ for some $w_0 \in \Lambda$. For $[\xi] \in X_p^{\leq t}$, we have $\mathrm{ord}_p(\langle w, v_\xi\rangle) \leq t$. Therefore, for $k > t$,

$$\mathrm{ord}_p(\langle v_{2,k}, v_\xi\rangle) = \min\left[\mathrm{ord}_p(\langle w, v_\xi\rangle),\ \mathrm{ord}_p(\langle p^k w_0, v_\xi\rangle)\right] \leq t.$$

This implies $|\langle v_\xi, v_{2,k}\rangle| \geq p^{-t}$. Combining these bounds yields

$$\left|\frac{\langle v_\xi, v_{1,k} - \alpha_k v_{2,k}\rangle}{\langle v_\xi, v_{2,k}\rangle}\right| \leq \frac{p^{-k}}{|\langle v_\xi, v_{2,k}\rangle|} \leq p^{t-k}.$$

As $k \to \infty$, $p^{t-k} \to 0$, meaning $\left|\dfrac{\langle v_\xi, v_{1,k}\rangle}{\alpha_k \langle v_\xi, v_{2,k}\rangle}\right| \to 1$. Thus, the sequence of partial products converges uniformly on $X_p^{\leq t}$ up to the scalars α_k, defining an element of $\mathcal{M}(X_p)^\times/\mathbb{Z}_p^\times$. $\square$

6

Cocycles for Bianchi Groups

In this chapter, we construct cocycles for the group

$$(\Gamma_1^K(N))^p := \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}[\sqrt{-\ell}][1/p]) : \begin{pmatrix} x & y \\ z & w \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} (mod\ N) \right\},$$

with values in $\mathcal{M}(\mathcal{H}_p \times \mathcal{H}_p)/\mathbb{Z}_p^\times$. In the first section, we prove that the group $(\Gamma_1^K(N))^p$ preserves the cosets of the lattice $L^p = L(\ell, N) \otimes \mathbb{Z}[1/p]$ in its dual. The second section details the construction of the cocycles. Let $s, t \in \mathbb{P}_1(K)$ for $K = \mathbb{Q}(\sqrt{-\ell})$. Consider the formal product

$$\Psi_m\{s,t\} := \prod_{\substack{v \in \mathcal{C} \\ Q(v)=m}} \langle v_\xi, v \rangle^{\Delta_{v,\infty}\{s,t\}}, \tag{6.1}$$

where $\mathcal{C}$ is a coset of L^p in $(L^p)^*$. This product can be rewritten as

$$\Psi_m\{s,t\} = \prod_{r\geq 0} \Psi_{m,r}\{s,t\}, \qquad \text{where} \qquad \Psi_{m,r}\{s,t\} := \prod_{\substack{v\in\mathcal{C} \\ \mathrm{ord}_\Lambda(v)=r \\ Q(v)=m}} \langle v_\xi, v\rangle^{\Delta_{v,\infty}\{s,t\}}.$$

These products may not converge because there is no guarantee that the r-th term $\Psi_{m,r}$ approaches 1 as $r\to\infty$. However, the modularity statements in Chapter 4 imply that the sum of the exponents for $\Psi_{m,r}$,

$$\sum_{\substack{v\in\mathcal{C} \\ \mathrm{ord}_\Lambda(v)=r \\ Q(v)=m}} \Delta_{v,\infty}\{s,t\},$$

is the m-th coefficient of a modular form of weight two and a fixed level. Since the space of modular forms of a given weight and level is finite-dimensional, there exist integers $\{c_{m_i}\}_{i=1}^k$ such that

$$\sum_{i=1}^k c_{m_i} \sum_{\substack{v\in\mathcal{C} \\ \mathrm{ord}_\Lambda(v)=r \\ Q(v)=m_i}} \Delta_{v,\infty}\{s,t\} = 0.$$

Therefore, the modified product $\prod_{i=1}^k (\Psi_{m_i,r}\{s,t\})^{c_{m_i}}$ can be rewritten as a product of factors of the form $\langle v_\xi, v\rangle/\langle v_\xi, v'\rangle$. This rewriting of the product, along with the results obtained in Proposition 5.8, allows us to establish convergence up to scalars. Thus, the infinite product $\prod_{i=1}^k (\Psi_{m_i}\{s,t\})^{c_{m_i}}$ is an element of $\mathcal{M}(\mathcal{H}_p\times\mathcal{H}_p)/\mathbb{Z}_p^\times$. Section 6.2 provides complete details. The remainder of the second section is devoted to proving the modular symbol relations.

In the third section, we collect the preceding results and state the main theorem of this chapter. The theorem suggests a tantalizing analogy between Borcherds lifts for $O(2,2)$ and rigid meromorphic cocycles for $O(3,1)$.

6.1 Preliminaries

We recall that V_ℓ is a rational quadratic space generated by $\{e_1, e_2, e_3, e_4\}$ such that

$$Q(e_1) = 1,\ Q(e_2) = \ell,\ Q(e_3) = 0 = Q(e_4), \quad \text{and}$$
$$\langle e_i, e_j \rangle = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2\ell & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The group $\mathrm{SL}_2(K)$ acts on V_ℓ via isometries. As mentioned in Chapter 1, the lattice with which we associate the cocycle is $L^p = L(\ell, N) \otimes \mathbb{Z}[1/p]$, where $L(\ell, N)$ is the integral lattice generated by $\{e_1, e_2, Ne_3, e_4\}$ as defined in Section 3.2. The dual of L^p is given by

$$(L^p)^* = \frac{1}{2}\mathbb{Z}[1/p]e_1 \oplus \frac{1}{2\ell}\mathbb{Z}[1/p]e_2 \oplus \mathbb{Z}[1/p]e_3 \oplus \frac{1}{N}\mathbb{Z}[1/p]e_4.$$

Proposition 6.1. *Let $c, d \in \mathbb{Z}[1/p]$. The group $(\Gamma_1^K(N))^p$ preserves the coset $\left(ce_3 + \frac{d}{N}e_4\right) + L^p$.*

Proof. The proof follows from a straightforward calculation. We omit the details here because the argument in the proof of the first part of Proposition 3.12 applies almost verbatim to this proposition. □

6.2 Convergence of Infinite Products

Let m be a positive rational number whose numerator is coprime to p, and let $\beta = ce_3$ for some integer c coprime to N. For $s, t \in \mathbb{P}_1(K)$, consider the formal product

$$\Psi_{m,\beta}\{s,t\} := \prod_{\substack{v \in \beta + L^p \\ Q(v) = m}} \langle v_\xi, v \rangle^{\Delta_{v,\infty}\{s,t\}}. \tag{6.2}$$

Here, $\Psi_{m,\beta}\{s,t\}$ is a formal product in the variable $[\xi]$, where v_ξ is the primitive element in $\Lambda \otimes \mathbb{C}_p$ corresponding to ξ. For the sake of brevity, we suppress this variable from the notation. Letting $\Lambda := L(\ell, N) \otimes \mathbb{Z}_p$, $\Psi_{m,\beta}\{s,t\}$ can be rewritten as

$$\Psi_{m,\beta}\{s,t\} = \prod_{r \geq 0} \Psi_{m,\beta}(r)\{s,t\}, \qquad \text{where} \qquad \Psi_{m,\beta}(r)\{s,t\} = \prod_{\substack{v \in \beta + L^p \\ \mathrm{ord}_\Lambda(v) = r \\ Q(v) = m}} \langle v_\xi, v \rangle^{\Delta_{v,\infty}\{s,t\}}.$$

Define $\mathcal{V}(\beta, \Lambda, r, m) := \{v \in \beta + L^p : \mathrm{ord}_\Lambda(v) = r,\ Q(v) = m\}$.

Lemma 6.2. *The function $\Psi_{m,\beta}(r)\{s,t\}$ is a finite product of factors of the form $\langle v_\xi, v\rangle$.*

Proof. It suffices to prove that the set $\{v \in \mathcal{V}(\beta, \Lambda, r, m) : \Delta_{v,\infty}\{s,t\} \neq 0\}$ is finite. This set is in bijection with the set

$$\{w \in p^r\beta + L^p : w \in \Lambda - p\Lambda,\ Q(w) = mp^{2r},\ \Delta_{w,\infty}\{s,t\} \neq 0\},$$

via the map $v \mapsto w := p^r v$. By Lemma 4.1, the above set is contained in $(L(\ell, N))^*$. Therefore, it suffices to prove that the larger set $\{w \in (L(\ell, N))^* : Q(w) = mp^{2r}, \Delta_{w,\infty}\{s,t\} \neq 0\}$ is finite.

From (3.9), it is clear that the condition $\Delta_{w,\infty}\{s,t\} \neq 0$ is equivalent to $(w, e_s)(w, e_t) < 0$. Let $w \in (L(\ell, N))^*$ be of length mp^{2r} such that $(w, e_s)(w, e_t) < 0$. We write $w = w_0 + ae_s + be_t$ such that $w_0 \in (\mathbb{Q}e_s \oplus \mathbb{Q}e_t)^\perp$. We thus have $(w, e_s) = -b$, $(w, e_t) = -a$, and $ab < 0$. Furthermore, $(w_0, w_0) - 2ab = mp^{2r}$ which implies $(w_0, w_0) = mp^{2r} + 2ab < mp^{2r}$. Since $\{e_s, e_t\}$ generate a hyperbolic space, (w_0, w_0) is non-negative. Therefore we have $0 \leq (w_0, w_0) < mp^{2r}$. This bound defines a compact domain in a two-dimensional space. Since w_0 is an element of a lattice, there are only finitely many choices for w_0. Moreover, $2ab = (w_0, w_0) - mp^{2r}$. Therefore, there are only finitely many choices for a and b. $\square$

We drop the pair $\{s,t\}$ from $\Delta_{v,\infty}\{s,t\}$ in the remainder of this section, as it plays no special role.

We recall the arguments for the convergence of infinite products from Proposition 5.8. An infinite product of the form $\prod_{r\geq 0} \Psi_{m,\beta}(r)$ converges if the following two conditions are satisfied.

1. Each $\Psi_{m,\beta}(r)$ is a ratio of $\langle v_\xi, v_1\rangle$ and $\langle v_\xi, v_2\rangle$, or a finite product of such ratios, and
2. there exist sequences of pairs $\{v_{1,r}, v_{2,r}\}$, composing ratios in $\Psi_{m,\beta}(r)$, such that $v_{1,r}$ and $v_{2,r}$ become p-adically close as r increases.

The first condition is equivalent to the requirement that the sum of exponents of $\langle v_\xi, \cdot\rangle$ in the product $\Psi_{m,\beta}(r)$, which is $\sum_{v\in\mathcal{V}(\beta,\Lambda,r,m)} \Delta_{v,\infty}$, is equal to zero for every r. However, such sums are coefficients of non-zero modular forms (see Remark (5)) and thus not necessarily zero. Hence, we cannot directly apply the convergence arguments presented in Chapter 5. Our goal in this section is to construct a modified product, composed of $\Psi_{m,\beta}(r)$ for various

m, so that its degree is zero. The term "degree" refers to the sum of the exponents of the factors $\langle v_\xi, \cdot \rangle$.

Lemma 6.3. *Let $S_{\mathrm{prim}}(p^r)$ denote the set of representatives of classes in $\Lambda/p^r\Lambda$ that lie in $\Lambda - p\Lambda$. Let $\mathfrak{L}(p^r)$ denote the set of lines in $\Lambda/p^r\Lambda$ that are generated by some $\overline{\lambda} \in \Lambda/p^r\Lambda$ with $\lambda \in S_{\mathrm{prim}}(p^r)$. For a line $\mathfrak{l} \in \mathfrak{L}(p^r)$, let $\mathrm{Gen}(\mathfrak{l})$ denote the set of generators of $\mathfrak{l}$. The function $\Psi_{m,\beta}(r)$ can be rewritten as*

$$\prod_{\mathfrak{l}\in\mathfrak{L}(p^r)} \prod_{\substack{\lambda\in S_{\mathrm{prim}}(p^r)\\ \overline{\lambda}\in\mathrm{Gen}(\mathfrak{l})}} \prod_{w\in\mathcal{W}(\beta,\lambda,r,m)} \langle v_\xi, p^{-r}w\rangle^{\Delta_{w,\infty}}, \tag{6.3}$$

where $\mathcal{W}(\beta,\lambda,r,m) = \{w \in (p^rL)^* : \pi_r(\overline{w}) = (p^r\overline{\beta}, \overline{\lambda}), Q(w) = mp^{2r}\}$.

Proof. The derivation of (4.2) and (4.3) implies that the product $\Psi_{m,\beta}(r)$ is equal to

$$\prod_{v\in\mathcal{V}(\beta,\Lambda,r,m)} \langle v_\xi, v\rangle^{\Delta_{v,\infty}} = \prod_{\substack{w\in p^r\beta+L^p\\ w\in\Lambda-p\Lambda\\ Q(w)=mp^{2r}}} \langle v_\xi, p^{-r}w\rangle^{\Delta_{p^{-r}w,\infty}}.$$

We have established that the set $\{w \in p^r\beta + L^p : w \in \Lambda - p\Lambda\}$ is equal to

$$\bigsqcup_{\lambda\in S_{\mathrm{prim}}(p^r)} \{w \in (p^rL)^* : \pi_r(\overline{w}) = (p^r\overline{\beta}, \overline{\lambda})\}. \tag{6.4}$$

We also have $\Delta_{w,\infty} = \Delta_{p^{-r}w,\infty}$. Therefore, we can write

$$\Psi_{m,\beta}(r) = \prod_{\lambda\in S_{\mathrm{prim}}(p^r)} \prod_{w\in\mathcal{W}(\beta,\lambda,r,m)} \langle v_\xi, p^{-r}w\rangle^{\Delta_{w,\infty}}.$$

By definition, $S_{\mathrm{prim}}(p^r) = \bigcup_{\mathfrak{l}\in\mathfrak{L}(p^r)}\{\lambda \in S_{\mathrm{prim}}(p^r) : \overline{\lambda} \in \mathrm{Gen}(\mathfrak{l})\}$. Hence, the proof is complete.

□

The degree of the rational function $\Psi_{m,\beta}(r)$ is a sum of Fourier coefficients of certain vector-valued modular forms as described in Proposition 4.6. However, we prefer to formulate our statements in terms of scalar-valued modular forms. Thus we further modify the product in (6.3). Let $\mathfrak{l} \in \mathfrak{L}(p^r)$ and $M = \mathrm{lcm}(4\ell, N)$. Define

$$\Psi_{m,\langle\beta\rangle}(r, \mathfrak{l}) := \prod_{\substack{0<u<M \\ (u,M)=1}} \prod_{\substack{\lambda \in S_{\mathrm{prim}}(p^r) \\ \overline{\lambda} \in \mathrm{Gen}(\mathfrak{l})}} \prod_{w \in \mathcal{W}(u\beta, \lambda, r, m)} \langle v_\xi, p^{-r} w \rangle^{\Delta_{w,\infty}}.$$

The degree of this product is equal to

$$\sum_{\substack{0<u<M \\ (u,M)=1}} \sum_{\substack{\lambda \in S_{\mathrm{prim}}(p^r) \\ \overline{\lambda} \in \mathrm{Gen}(\mathfrak{l})}} \sum_{w \in \mathcal{W}(u\beta, \lambda, r, m)} \Delta_{w,\infty}.$$

By Corollary 4.10, this sum is the m-th coefficient of a modular form of weight two for the group $\Gamma_0(Mp)$ and character $\left(\frac{\cdot}{\ell}\right)$. Because this space of modular forms is finite-dimensional, we can find many linear relations between the Fourier coefficients of these modular forms. The following theorem states that such linear relations give rise to an element in $\mathcal{M}(\mathcal{H}_p \times \mathcal{H}_p)/\mathbb{Z}_p^\times$.

Theorem 6.4. *Let $\{m_1, \ldots, m_k\}$ be integers such that*

1. *each m_i is coprime to p,*
2. *each m_i is a* proper length *with respect to V_ℓ.*

For a modular form f, denote its m-th Fourier coefficient by $a_f(m)$. Let $\{c_{m_i}\}_{i=1}^k$ be a finite sequence of integers such that

$$\sum_{i=1}^{k} c_{m_i} a_f(m_i) = 0$$

for all $f \in M_2(\Gamma_0(Mp), \left(\frac{\cdot}{\ell}\right))$. Then the product

$$\prod_{r \geq 0} \left(\prod_{\mathfrak{l} \in \mathfrak{L}(p^r)} \prod_{i=1}^{k} \left[\Psi_{m_i,\langle\beta\rangle}(r, \mathfrak{l}) \right]^{c_{m_i}} \right) \tag{6.5}$$

is an element of $\mathcal{M}(\mathcal{H}_p \times \mathcal{H}_p)/\mathbb{Z}_p^\times$.

Proof. From the hypothesis and the discussion preceding the theorem, the degree of the product

$$\prod_{i=1}^{k} \left[\Psi_{m_i,\langle\beta\rangle}(r, \mathfrak{l}) \right]^{c_{m_i}}$$

is zero for each r and $\mathfrak{l} \in \mathfrak{L}(p^r)$. Therefore it can be written as a product of factors of the form $\langle v_\xi, p^{-r}w_1\rangle / \langle v_\xi, p^{-r}w_2\rangle$. We estimate such factors in the next few paragraphs.

Let $\mathfrak{l}$ be a line in $\mathfrak{L}(p^r)$. Let $\lambda_1, \lambda_2 \in \Lambda - p\Lambda$ be such that their respective images $\overline{\lambda_1}, \overline{\lambda_2}$ in $\Lambda/p^r\Lambda$ belong to $\mathrm{Gen}(\mathfrak{l})$. It follows that $\lambda_1 \equiv \alpha\lambda_2 \pmod{p^r\Lambda}$ for some integer α coprime to p.

Recall the isomorphism $\pi_r : D_{p^rL} \to D_{L^p} \oplus D_{p^r\Lambda}$. Let $w_1, w_2 \in (p^rL)^*$ be such that $\pi_r(\overline{w_1}) = (u_1p^r\overline{\beta}, \overline{\lambda}_1)$ and $\pi_r(\overline{w_2}) = (u_2p^r\overline{\beta}, \overline{\lambda}_2)$ for some $0 < u_1, u_2 < M$. Therefore, both $w_1 - \lambda_1$ and $w_2 - \lambda_2$ belong to $p^r\Lambda$. Hence, $w_1 - \alpha w_2 \in p^r\Lambda$ and we obtain the following estimate.

$$\left|\frac{\langle v_\xi, p^{-r}w_1\rangle}{\langle v_\xi, p^{-r}w_2\rangle} - \alpha\right| = \left|\frac{\langle v_\xi, w_1\rangle}{\langle v_\xi, w_2\rangle} - \alpha\right| = \left|\frac{\langle v_\xi, w_1 - \alpha w_2\rangle}{\langle v_\xi, w_2\rangle}\right| \le \frac{p^{-r}}{|\langle v_\xi, w_2\rangle|}.$$

Note that the vector $w_2 \in \Lambda - p\Lambda$ and satisfies $\mathrm{ord}_p(Q(w_2)) = 2r$. For $\xi \in X^{\le t}$ with $t < 2r$, the arguments in the proof of Proposition 5.8 ensure the lower bound $|\langle v_\xi, w_2\rangle| \ge p^{-t}$. Therefore, there exists an element $\alpha_r \in \mathbb{Z}_p^\times$ such that for any $\xi \in X^{\le t}$ with $t < 2r$,

$$\left|\prod_{\mathfrak{l}\in\mathfrak{L}(p^r)} \prod_{i=1}^{k} \left[\Psi_{m_i,\langle\beta\rangle}(r,\mathfrak{l})(\xi)\right]^{c_{m_i}} - \alpha_r\right| \le p^{t-r}.$$

Hence the product in (6.5) converges to an element of $\mathcal{M}(\mathcal{H}_p \times \mathcal{H}_p)/\mathbb{Z}_p^\times$.

$\square$

For each r in (6.5), the products are finite and thus we can interchange them.

Corollary 6.5. *The product*

$$\prod_{r\ge 0}\left(\prod_{\mathfrak{l}\in\mathfrak{L}(p^r)} \prod_{i=1}^{k} \left[\Psi_{m_i,\langle\beta\rangle}(r,\mathfrak{l})\right]^{c_{m_i}}\right)$$

is equal to

$$\prod_{r\ge 0}\left(\prod_{i=1}^{k} \left[\Psi_{m_i,\langle\beta\rangle}(r)\right]^{c_{m_i}}\right)$$

in $\mathcal{M}(\mathcal{H}_p \times \mathcal{H}_p)/\mathbb{Z}_p^\times$, *where* $\Psi_{m,\langle\beta\rangle}(r) := \prod_{\mathfrak{l}\in\mathfrak{L}(p^r)} \Psi_{m,\langle\beta\rangle}(r,\mathfrak{l})$.

6.3 Modular Symbols Valued in Spaces of Functions

In this section, we establish that the mapping

$$\{s,t\} \mapsto \prod_{r\geq 0}\left(\prod_{i=1}^{k}\left[\Psi_{m_i,\langle\beta\rangle}(r)\{s,t\}\right]^{c_{m_i}}\right)$$

is a modular symbol invariant under $(\Gamma_1^K(N))^p$. The verification of 2-term and 3-term relations is straightforward. To establish invariance under $(\Gamma_1^K(N))^p$, we need to prove that for each $\gamma \in (\Gamma_1^K(N))^p$, the equality

$$\prod_{r\geq 0}\prod_{i=1}^{k}(\Psi_{m_i,\langle\beta\rangle}(r)\{\gamma s,\gamma t\}\big|\gamma)^{c_{m_i}} = \prod_{r\geq 0}\prod_{i=1}^{k}\left[\Psi_{m_i,\langle\beta\rangle}(r)\{s,t\}\right]^{c_{m_i}}$$

holds in $\mathcal{M}(\mathcal{H}_p\times\mathcal{H}_p)/\mathbb{Z}_p^\times$. The right action $f|\gamma$ evaluated at $[\xi]\in X_p$ corresponds to evaluating the original function f at $[\gamma\xi]$. In terms of vectors, this replaces the primitive vector v_ξ with a primitive vector generating the line spanned by γv_ξ. For a primitive v_ξ, it is not necessarily the case that γv_ξ is again primitive. However, given any two vectors v_1 and v_2, the ratio $\langle v_\xi, v_1\rangle/\langle v_\xi, v_2\rangle$ is independent of the choice of the generator v_ξ. In the following paragraph, we show that the function $\prod_{i=1}^{k}[\Psi_{m_i,\langle\beta\rangle}(r)\{\gamma s,\gamma t\}]^{c_{m_i}}$ can be written as a product of ratios and thus is independent of the choice of the generator.

The degree of the product $\Psi_{m,\langle\beta\rangle}(r)\{\gamma s,\gamma t\}\big|\gamma$ is

$$\sum_{\substack{0<u<M\\(u,M)=1}}\sum_{\mathfrak{l}\in\mathfrak{L}(p^r)}\sum_{\substack{\lambda\in S_{\mathrm{prim}}(p^r)\\ \overline{\lambda}\in\mathrm{Gen}(\mathfrak{l})}}\sum_{w\in\mathcal{W}(u\beta,\lambda,r,m)}\Delta_{w,\infty}\{\gamma s,\gamma t\}.$$

By Corollary 4.10 and the fact that the space of modular forms is a vector space, the above sum is the m-th coefficient of a modular form of weight two for the group $\Gamma_0(Mp)$ and character $\left(\frac{\cdot}{\ell}\right)$. Since the sequence $\{c_{m_i}\}$ is chosen to annihilate the space of weight two modular forms, the linear combination of the degrees vanishes, meaning the function $\prod_{i=1}^{k}(\Psi_{m_i,\langle\beta\rangle}(r)\{\gamma s,\gamma t\}\big|\gamma)^{c_{m_i}}$ has degree zero, and thus it is independent of the choice of generator. The following lemma simplifies this function by exploiting the fact that γ is an isometry on V_ℓ and preserves cosets of L^p in its dual.

Lemma 6.6. *The product* $\prod_{i=1}^{k}(\Psi_{m_i,\langle\beta\rangle}(r)\{\gamma s,\gamma t\}\big|\gamma)^{c_{m_i}}$ *is equal to*

$$\prod_{i=1}^{k}\prod_{\substack{0<u<M\\(u,M)=1}}\prod_{v\in\mathcal{V}(u\beta,\gamma^{-1}\Lambda,r,m_i)}\langle v_\xi,v\rangle^{c_{m_i}\Delta_{v,\infty}\{s,t\}},$$

where $\mathcal{V}(u\beta,\gamma^{-1}\Lambda,r,m_i)=\{v\in u\beta+L^p:\mathrm{ord}_{\gamma^{-1}\Lambda}(v)=r,Q(v)=m_i\}$*.*

Proof. From the discussion about the choice of generator for ξ and Lemma 6.3, the function $\prod_{i=1}^{k}(\Psi_{m_i,\langle\beta\rangle}(r)\{\gamma s,\gamma t\}\big|\gamma)^{c_{m_i}}$ is equal to

$$\prod_{i=1}^{k}\prod_{\substack{0<u<M\\(u,M)=1}}\prod_{v\in\mathcal{V}(u\beta,\Lambda,r,m_i)}\langle\gamma v_\xi,v\rangle^{c_{m_i}\Delta_{v,\infty}\{\gamma s,\gamma t\}}.$$

From Proposition 3.11, we have $\Delta_{\gamma v,\infty}\{\gamma s,\gamma t\}=\Delta_{v,\infty}\{s,t\}$. Since γ is an isometry, we also have $\langle\gamma v_\xi,\gamma v\rangle=\langle v_\xi,v\rangle$. Therefore, the above product is equal to

$$\prod_{i=1}^{k}\prod_{\substack{0<u<M\\(u,M)=1}}\prod_{\gamma v\in\mathcal{V}(u\beta,\Lambda,r,m_i)}\langle v_\xi,v\rangle^{c_{m_i}\Delta_{v,\infty}\{s,t\}}.$$

Let $\gamma\in(\Gamma_1^K(N))^p$. Then for a vector $v\in V_\ell\otimes\mathbb{Q}_p$, we have $p^r(\gamma v)\in\Lambda-p\Lambda$ if and only if $p^r(v)\in(\gamma^{-1}\Lambda)-p(\gamma^{-1}\Lambda)$. Therefore, $\mathrm{ord}_\Lambda(\gamma v)=\mathrm{ord}_{\gamma^{-1}\Lambda}(v)$. Using the fact that γ is an isometry and γ preserves cosets, we conclude $\gamma^{-1}\mathcal{V}(u\beta,\Lambda,r,m_i)=\mathcal{V}(u\beta,\gamma^{-1}\Lambda,r,m_i)$.

□

We now state a useful proposition, followed by the proof of the invariance property. We postpone the proof of this proposition to a later section.

Proposition 6.7. *For an integer* m*, let*

$$\Psi_{m,\langle\beta\rangle}(\Lambda,r)\{s,t\}:=\prod_{\substack{0<u<M\\(u,M)=1}}\prod_{v\in\mathcal{V}(u\beta,\Lambda,r,m)}\langle v_\xi,v\rangle^{\Delta_{v,\infty}\{s,t\}}.$$

Then for any $\gamma\in(\Gamma_1^K(N))^p$*,*

$$\lim_{R\to\infty}\prod_{r=0}^{R}\prod_{i=1}^{k}\left[\frac{\Psi_{m_i,\langle\beta\rangle}(\Lambda,r)\{s,t\}}{\Psi_{m_i,\langle\beta\rangle}(\gamma\Lambda,r)\{s,t\}}\right]^{c_{m_i}}=1\in\mathcal{M}(\mathcal{H}_p\times\mathcal{H}_p)/\mathbb{Z}_p^\times,$$

for all $s, t \in \mathbb{P}_1(K)$*.*

The above proposition together with the preceding discussion implies the required invariance property.

6.3.1 Proof of Proposition 6.7

We first establish two important preliminary results.

Lemma 6.8. *Let* Λ *be as before. For* $\gamma \in (\Gamma_1^K(N))^p$*, there exists an integer* $\kappa(\gamma, \Lambda)$ *such that* $\left|\mathrm{ord}_{\gamma\Lambda}(v) - \mathrm{ord}_{\Lambda}(v)\right| \leq \kappa(\gamma, \Lambda)$ *for all* $v \in \Lambda \otimes \mathbb{Q}_p$*.*

Proof. We scale the lattice $\gamma\Lambda$ by a sufficiently large power of p so that it becomes a submodule of Λ. By the invariant factor theorem, there exists a basis of Λ, $\{f_1, f_2, f_3, f_4\}$, such that the basis of $\gamma\Lambda$ is $\{p^{a_1} f_1, p^{a_2} f_2, p^{a_3} f_3, p^{a_4} f_4\}$ for some integers $a_1 \leq a_2 \leq a_3 \leq a_4$. Let v be a vector in $\Lambda \otimes \mathbb{Q}_p$. We can write $v = v_1 f_1 + v_2 f_2 + v_3 f_3 + v_4 f_4 = v_1' p^{a_1} f_1 + v_2' p^{a_2} f_2 + v_3' p^{a_3} f_3 + v_4' p^{a_4} f_4$. Therefore,

$$\begin{aligned} \mathrm{ord}_{\Lambda}(v) &= -\min_{1 \leq i \leq 4} \{\mathrm{ord}_p(v_i)\}, \qquad \text{and} \\ \mathrm{ord}_{\gamma\Lambda}(v) &= -\min_{1 \leq i \leq 4} \{\mathrm{ord}_p(v_i) - a_i\}. \end{aligned}$$

We note the following inequalities.

$$\begin{aligned} \min_{1 \leq i \leq 4} \{\mathrm{ord}_p(v_i) - a_i\} + a_1 &\leq \min_{1 \leq i \leq 4} \{\mathrm{ord}_p(v_i)\}, \quad \text{and} \\ \min_{1 \leq i \leq 4} \{\mathrm{ord}_p(v_i)\} &\leq \min_{1 \leq i \leq 4} \{\mathrm{ord}_p(v_i) - a_i\} + a_4. \end{aligned}$$

Therefore, $\mathrm{ord}_{\Lambda}(v) + a_1 \leq \mathrm{ord}_{\gamma\Lambda}(v) \leq a_4 + \mathrm{ord}_{\Lambda}(v)$. This implies $|\mathrm{ord}_{\gamma\Lambda}(v) - \mathrm{ord}_{\Lambda}(v)| \leq |a_1| + |a_4|$. □

Lemma 6.9. *Let* r *be an integer greater than* $2\kappa(\gamma, \Lambda)$ *and let* $\mathfrak{l}$ *be a line in* $(\gamma\Lambda)/p^r(\gamma\Lambda)$ *that is generated by a primitive element of* $\gamma\Lambda$*. Then for any* v_1, v_2 *in the set*

$$\{v \in \beta + L^p : \mathrm{ord}_{\gamma\Lambda}(v) = r, \overline{p^r v} \in \mathrm{Gen}(\mathfrak{l})\},$$

we have $\mathrm{ord}_{\Lambda}(v_1) = \mathrm{ord}_{\Lambda}(v_2)$*.*

Proof. Since the images of $p^r v_1$ and $p^r v_2$ in $\gamma\Lambda/p^r\gamma\Lambda$ generate $\mathfrak{l}$, there exists an $\alpha \in \mathbb{Z}_p^\times$ such that $v_1 - \alpha v_2 \in \gamma\Lambda$. Therefore $\mathrm{ord}_{\gamma\Lambda}(v_1 - \alpha v_2) \leq 0$, and we obtain $\mathrm{ord}_{\Lambda}(v_1 - \alpha v_2) \leq \kappa(\gamma, \Lambda)$ by

Lemma 6.8. The assumption on r together with Lemma 6.8 implies $\mathrm{ord}_\Lambda(v_i) \geq r - \kappa(\gamma,\Lambda) > \kappa(\gamma,\Lambda)$ for $i = 1, 2$. Hence, $\mathrm{ord}_\Lambda(v_1) = \mathrm{ord}_\Lambda(\alpha v_2 + (v_1 - \alpha v_2)) = \mathrm{ord}_\Lambda(\alpha v_2) = \mathrm{ord}_\Lambda(v_2)$. $\square$

We drop the pair $\{s,t\}$ from $\Delta_{v,\infty}\{s,t\}$ in the remainder of this section as it plays no special role.

Proof of Proposition 6.7. Fix R such that $R > 3\kappa(\gamma,\Lambda)$ and consider the following expression

$$\prod_{r=0}^{R} \frac{\Psi_{m,\beta}(r,\Lambda)}{\Psi_{m,\beta}(r,\gamma\Lambda)} = \prod_{r=0}^{R} \frac{\prod_{v\in\mathcal{V}(\beta,\Lambda,r,m)} \langle v_\xi, v\rangle^{\Delta_{v,\infty}}}{\prod_{v\in\mathcal{V}(\beta,\gamma\Lambda,r,m)} \langle v_\xi, v\rangle^{\Delta_{v,\infty}}}.$$

This can be rewritten as

$$\prod_{\substack{v\in\beta+L^p\\ \mathrm{ord}_\Lambda(v)\leq R\\ \mathrm{ord}_{\gamma\Lambda}(v)>R\\ Q(v)=m}} \langle v_\xi, v\rangle^{\Delta_{v,\infty}} \times \prod_{\substack{v\in\beta+L^p\\ \mathrm{ord}_\Lambda(v)>R\\ \mathrm{ord}_{\gamma\Lambda}(v)\leq R\\ Q(v)=m}} \langle v_\xi, v\rangle^{-\Delta_{v,\infty}}.$$

In the second product, the condition $\mathrm{ord}_\Lambda(v) > R$ forces $\mathrm{ord}_{\gamma\Lambda}(v) \geq R - \kappa(\gamma,\Lambda)$ by Lemma 6.8. By the same lemma, $\mathrm{ord}_{\gamma\Lambda}(v) \leq R$ forces $\mathrm{ord}_\Lambda(v) \leq R + \kappa(\gamma,\Lambda)$. We can further refine the second product. Let $\mathfrak{L}(p^r,\gamma\Lambda)$ be the set of lines in $(\gamma\Lambda)/p^r(\gamma\Lambda)$ that are generated by primitive elements. Then the second product becomes

$$\prod_{r=R-\kappa(\gamma,\Lambda)}^{R} \prod_{\mathfrak{l}\in\mathfrak{L}(p^r,\gamma\Lambda)} \prod_{k=1}^{\kappa(\gamma,\Lambda)} \prod_{\substack{v\in\mathcal{V}(\beta,\gamma\Lambda,r,m)\\ \overline{p^r v}\in\mathrm{Gen}(\mathfrak{l})\\ \mathrm{ord}_\Lambda(v)=R+k}} \langle v_\xi, v\rangle^{-\Delta_{v,\infty}}.$$

By Lemma 6.9, the set $\{v \in \mathcal{V}(\beta,\gamma\Lambda,r,m) : \overline{p^r v} \in \mathrm{Gen}(\mathfrak{l}), \mathrm{ord}_\Lambda(v) = R + k\}$ is either empty or is equal to the set $\{v \in \mathcal{V}(\beta,\gamma\Lambda,r,m) : \overline{p^r v} \in \mathrm{Gen}(\mathfrak{l})\}$ whenever $r > 2\kappa(\gamma,\Lambda)$. Since $r \geq R - \kappa(\gamma,\Lambda)$ and $R > 3\kappa(\gamma,\Lambda)$, we have $r > 2\kappa(\gamma,\Lambda)$. Hence, the innermost product is either 1 or is equal to

$$\prod_{\substack{v\in\mathcal{V}(\beta,\gamma\Lambda,r,m)\\ \overline{p^r v}\in\mathrm{Gen}(\mathfrak{l})}} \langle v_\xi, v\rangle^{-\Delta_{v,\infty}}.$$

Lemma 6.10. *For a fixed $\mathfrak{l} \in \mathfrak{L}(p^r, \gamma\Lambda)$, the degree of the product*

$$\prod_{\substack{0<u<M \\ (u,M)=1}} \prod_{\substack{v \in \mathcal{V}(u\beta,\gamma\Lambda,r,m) \\ p^r\overline{v} \in \mathrm{Gen}(\mathfrak{l})}} \langle v_\xi, v \rangle^{-\Delta_{v,\infty}}$$

is the m-th coefficient of a modular form of weight 2, level Mp and character $\left(\frac{\cdot}{\ell}\right)$.

Proof. The degree of the product is

$$-\sum_{\substack{0<u<M \\ (u,M)=1}} \sum_{\substack{v \in \mathcal{V}(u\beta,\gamma\Lambda,r,m) \\ p^r\overline{v} \in \mathrm{Gen}(\mathfrak{l})}} \Delta_{v,\infty}.$$

The proof of Lemma 6.3 implies that the set $\mathcal{V}(u\beta, \gamma\Lambda, r, m)$ is in bijection with

$$\bigcup_{\mathfrak{l} \in \mathfrak{L}(p^r,\gamma\Lambda)} \bigcup_{\substack{\lambda \in S_{\mathrm{prim}}(p^r) \\ \overline{\lambda} \in \mathrm{Gen}(\mathfrak{l})}} \mathcal{W}(u\beta, \lambda, r, m).$$

Because $\gamma \in (\Gamma_1^K(N))^p$ acts by isometries, the disjoint union decomposition established in the proof of Lemma 6.3 applies equally to the lattice $\gamma\Lambda$. Restricting this bijection to our fixed line $\mathfrak{l}$, the set of vectors v satisfying $p^r\overline{v} \in \mathrm{Gen}(\mathfrak{l})$ corresponds exactly to the union over $\overline{\lambda} \in \mathrm{Gen}(\mathfrak{l})$ of $\mathcal{W}(u\beta, \lambda, r, m)$. Therefore we have the following equality,

$$\sum_{\substack{0<u<M \\ (u,M)=1}} \sum_{\substack{v \in \mathcal{V}(u\beta,\gamma\Lambda,r,m) \\ p^r\overline{v} \in \mathrm{Gen}(\mathfrak{l})}} \Delta_{v,\infty} = \sum_{\substack{0<u<M \\ (u,M)=1}} \sum_{\substack{\lambda \in S_{\mathrm{prim}}(p^r) \\ \overline{\lambda} \in \mathrm{Gen}(\mathfrak{l})}} \sum_{w \in \mathcal{W}(u\beta,\lambda,r,m)} \Delta_{w,\infty}.$$

By Corollary 4.10, the sum

$$\sum_{\substack{0<u<M \\ (u,M)=1}} \sum_{\substack{\lambda \in S_{\mathrm{prim}}(p^r) \\ \overline{\lambda} \in \mathrm{Gen}(\mathfrak{l})}} \sum_{w \in \mathcal{W}(u\beta,\lambda,r,m)} \Delta_{w,\infty}$$

is the m-th coefficient of a modular form of weight two for the group $\Gamma_0(Mp)$ and character $\left(\frac{\cdot}{\ell}\right)$. □

We now put together all the work and complete the argument. Consider the product

$$\prod_{r=R-\kappa(\gamma,\Lambda)}^{R} \prod_{\mathfrak{l}\in\mathfrak{L}(p^r,\gamma\Lambda)} \left(\prod_{i=1}^{k} \prod_{\substack{0<u<M \\ (u,M)=1}} \prod_{\substack{v\in\mathcal{V}(u\beta,\gamma\Lambda,r,m_i) \\ p^r\overline{v}\in\mathrm{Gen}(\mathfrak{l})}} \langle v_\xi, v\rangle^{\Delta_{v,\infty}a(r)c_{m_i}} \right),$$

where $a(r) = 0$ or 1. The degree of the product

$$\prod_{i=1}^{k} \prod_{\substack{0<u<M \\ (u,M)=1}} \prod_{\substack{v\in\mathcal{V}(u\beta,\gamma\Lambda,r,m_i) \\ p^r\overline{v}\in\mathrm{Gen}(\mathfrak{l})}} \langle v_\xi, v\rangle^{\Delta_{v,\infty}a(r)c_{m_i}}$$

is zero by the choice of c_{m_i}. Therefore it can be written as a ratio of bilinear pairings and we can use the arguments of the previous section to conclude that there is a sequence of constants $\{\alpha_R\}$, such that the ratio

$$\frac{1}{\alpha_R} \times \prod_{i=1}^{k} \prod_{\substack{0<u<M \\ (u,M)=1}} \prod_{r=R-\kappa(\gamma,\Lambda)}^{R} \prod_{\mathfrak{l}\in\mathfrak{L}(p^r,\gamma\Lambda)} \prod_{k=1}^{\kappa(\gamma,\Lambda)} \prod_{\substack{v\in\mathcal{V}(u\beta,\gamma\Lambda,r,m_i) \\ \overline{p^r v}\in\mathrm{Gen}(\mathfrak{l}) \\ \mathrm{ord}_\Lambda(v)=R+k}} \langle v_\xi, v\rangle^{-c_{m_i}\Delta_{v,\infty}}$$

approaches 1 as $R \to \infty$, whenever $v_\xi \in X^{\leq t}$ for a fixed t. Similar arguments can be used for the other product.

□

6.4 Analogy with Borcherds Lift

In this section, we describe the analogy of our results with the Borcherds lift for signature $(2, 2)$.

We first express the main theorem in terms of divisors of the constructed functions. Let m be a *proper length with respect to* $L(\ell, N) \otimes \mathbb{Q}$. Recall that $\beta = ce_3 \in (L(\ell, N))^*$ for some integer c coprime to N, and let $M = \mathrm{lcm}(4\ell, N)$. Following Definition 5.6, we define a modular symbol taking values in $Div^\dagger_{rq}(\mathcal{H}_p \times \mathcal{H}_p)$. For each $s, t \in \mathbb{P}_1(K)$,

$$\mathcal{D}(\beta, m)\{s,t\} := \sum_{\substack{v\in\beta+L^p \\ Q(v)=m}} \Delta_{v,\infty}\{s,t\}\ \Delta_{v,p}, \quad \text{and} \quad \mathcal{D}(\langle\beta\rangle, m)\{s,t\} := \sum_{\substack{0<u<M \\ (u,M)=1}} \mathcal{D}(u\beta, m)\{s,t\}. \tag{6.6}$$

Theorem 6.11. *Let $K = \mathbb{Q}(\sqrt{-\ell})$ be an imaginary quadratic extension in which p splits. Let N be an integer coprime to p and ℓ. Let $\{c_m\}_{m=1}^{m_0}$ be a finite sequence of integers such that*

1. *$c_m = 0$ if p divides m,*

2. *$c_m = 0$ if m is equal to $a^2 + \ell b^2$ for some $a, b \in \mathbb{Q}$, and*

3. *for all $f \in M_2(\Gamma_0(Mp), \left(\frac{\cdot}{\ell}\right))$ we have $\displaystyle\sum_{m=1}^{m_0} c_m \times (m\text{-th coefficient of } f) = 0$.*

Then there exists $\widehat{\Psi} \in \mathrm{MS}((\Gamma_1^K(N))^p, \mathcal{M}(\mathcal{H}_p \times \mathcal{H}_p)^\times/\mathbb{Z}_p^\times)$ such that

$$\mathrm{div}_*(\widehat{\Psi}) = \sum_{m=1}^{m_0} c_m\, \mathcal{D}(\langle\beta\rangle, m) \in \mathrm{MS}((\Gamma_1^K(N))^p, Div_{rq}^\dagger(\mathcal{H}_p \times \mathcal{H}_p)).$$

Remark 6. *The assumptions listed here imply that the collection $\{c_m\}_{m=1}^{m_0}$ belongs to $\mathrm{PP}_0(Mp, \left(\frac{\cdot}{\ell}\right))^\dagger$ (see Definition 1.2).*

We include a brief discussion on how we can view the analogy between our main theorem and the Borcherds lift for $O(2,2)$. Let $F = \mathbb{Q}(\sqrt{p})$ be a quadratic field for a prime p. Let m be a positive integer and define ([4, Definition 2.27])

$$\begin{aligned} T_m := \{(z_1, z_2) \in \mathcal{H} \times \mathcal{H} : az_1z_2 + \lambda z_1 + \lambda' z_2 + b = 0, \ \textit{where} \\ \lambda \in \mathfrak{d}_F^{-1} \ \textit{and}\ a, b \in \mathbb{Z} \ ; \ \lambda\lambda' - ab = m/p\}. \end{aligned}$$

These divisors T_m are known as Hirzebruch-Zagier divisors.

Let $W_0(p, (\frac{p}{\cdot}))$ be the space of weakly holomorphic modular forms of weight 0 for the group $\Gamma_0(p)$ with character $(\frac{p}{\cdot})$. We denote by $W_0^+(p, (\frac{p}{\cdot}))$ the subspace of those $f \in W_0(p, (\frac{p}{\cdot}))$ whose coefficients $c(n)$ satisfy $c(n) = 0$ whenever $(\frac{p}{n}) = -1$.

Theorem 6.12. *(Theorem 3.44 of [4]) Let $f \in W_0^+(p, \left(\frac{p}{\cdot}\right))$ have the Fourier expansion $f = \sum_{m=1}^{m_0} c_m q^{-m} + \sum_{m=0}^{\infty} d_m q^m$. Assume that the principal part coefficients satisfy $c_m \in \mathbb{Z}$. Then there exists a meromorphic Hilbert modular form for the group $\mathrm{SL}_2(\mathcal{O}_F)$ whose divisor is $\displaystyle\sum_{m=1}^{m_0} c_m T_m$.*

Remark 7. *While a full statement of the Borcherds lift predicts both the weight and the exact q-expansion of the resulting form from the input data, we isolate this partial statement to highlight the analogy with our theorem.*

Our theorem describes a lift from $\mathrm{PP}_0\big(Mp, \big(\frac{\cdot}{\ell}\big)\big)^\dagger$, rather than directly from a space of weakly holomorphic modular forms. However, this input space can be viewed as a subset of weakly holomorphic modular forms of weight 0.

Let $\mathrm{WHF}_0\big(Mp, \big(\frac{\cdot}{\ell}\big)\big)^\dagger$ denote the space of weakly holomorphic modular forms of weight 0, level $\Gamma_0(Mp)$, and character $\big(\frac{\cdot}{\ell}\big)$ that are holomorphic at all cusps except ∞. For $\{c_m\}_{m=1}^{m_0} \in \mathrm{PP}_0(Mp, (\frac{\cdot}{\ell}))^\dagger$, we have by definition

$$\sum_{m=1}^{m_0} c_m \times m\text{-th coefficient of } f = 0$$

for every modular form f of weight 2, level $\Gamma_0(Mp)$ with character $\big(\frac{\cdot}{\ell}\big)$. By Serre duality, the space $S_2(\Gamma_0(Mp), (\frac{\cdot}{\ell}))$ is dual to the obstruction space for lifting Laurent polynomials to principal parts of weakly holomorphic forms of weight 0, level $\Gamma_0(Mp)$, and character $\big(\frac{\cdot}{\ell}\big)$. Since $\{c_m\}$ annihilates $S_2(\Gamma_0(Mp), (\frac{\cdot}{\ell}))$, the obstruction vanishes, guaranteeing the existence of an $F \in \mathrm{WHF}_0\big(Mp, \big(\frac{\cdot}{\ell}\big)\big)^\dagger$ with principal part $\sum_{m=1}^{m_0} c_m q^{-m}$ at the cusp ∞. Therefore, $\mathrm{PP}_0(Mp, (\frac{\cdot}{\ell}))^\dagger$ can be viewed as a subset of $\mathrm{WHF}_0(Mp, (\frac{\cdot}{\ell}))^\dagger$. Viewing our theorem as a lift directly from this subset of weakly holomorphic forms suggests a tantalizing analogy between rigid meromorphic cocycles for $O(3,1)$ and the classical Borcherds lift for $O(2,2)$.

A natural question is whether we can construct a lift directly from the entirety of $\mathrm{WHF}_0\big(Mp, (\frac{\cdot}{\ell})\big)^\dagger$, bypassing the additional constraints placed on the coefficients $\{c_m\}_{m=1}^{m_0}$. First, we require $c_m = 0$ whenever $p \mid m$. This is a technical assumption made to simplify the lattice calculations in Chapter 4. We expect that this condition can be removed but would require heavier bookkeeping. Second, we require $c_m = 0$ whenever $m = a^2 + \ell b^2$ for some $a, b \in \mathbb{Q}$. We discuss this in Chapter 8. Third, the sequence must annihilate the full space of modular forms $M_2\big(\Gamma_0(Mp), (\frac{\cdot}{\ell})\big)$, which is a strictly stronger condition than annihilating only the cusp forms S_2. Our proof relies on the fact that the integers $\{c_m\}_{m=1}^{m_0}$ annihilate a certain theta series that is not a cusp form. However, Darmon and Vonk construct an analogous lift directly from a subspace of weakly holomorphic forms of weight $1/2$ that are holomorphic everywhere except at the cusp ∞. Adapting their methods to our setting may provide a pathway to relaxing this full-annihilation requirement in future work.

7

Restriction to Signature $(2,1)$

Throughout our discussion so far, the vector space $V_\ell = \mathbb{Q}e_1 \oplus \mathbb{Q}e_2 \oplus \mathbb{Q}e_3 \oplus \mathbb{Q}e_4$ has been a central object. We associated Archimedean and non-Archimedean symmetric spaces to V_ℓ. Using the Archimedean space and certain integer lattices, we constructed integer-valued modular symbols. Taking inspiration from this approach and using the p-adic space and $\mathbb{Z}[1/p]$-lattices, we constructed modular symbols valued in functions modulo scalars. As mentioned earlier, [5] proposed such constructions for quadratic spaces with arbitrary signatures. The constructions of rigid meromorphic cocycles in [7] and [10] can be viewed as instances of executing the aforementioned strategy for signature $(2,1)$.

In this chapter, we consider a subspace of V_ℓ whose signature is $(2,1)$. Define $W := \mathbb{Q}e_1 \oplus \mathbb{Q}e_3 \oplus \mathbb{Q}e_4$. The p-adic space associated to W can be identified with the diagonal $\{(\tau,\tau) : \tau \in \mathcal{H}_p\}$ inside $\mathcal{H}_p \times \mathcal{H}_p$—the p-adic space associated to V_ℓ. A rigid meromorphic function on $\mathcal{H}_p \times \mathcal{H}_p$ can then be restricted to $\mathcal{H}_p$. This restriction at the level of functions allows us to obtain a restriction from the cocycles of $(\Gamma_1^K(N))^p$ to those of $(\Gamma_1^K(N))^p \cap \mathrm{SL}_2(\mathbb{Z}[1/p])$, as explained in Section 7.3.

We recall that cocycles for Bianchi groups are constructed using the annihilators of the space of modular forms of weight 2. Let $\{m_1, \ldots, m_k\}$ be integers such that each of these is a *proper length with respect to V_ℓ*. Given a suitable collection of integers $\{c_{m_i}\}_{i=1}^k$, there is a cocycle with divisor $\sum_{i=1}^{k} c_{m_i}\mathcal{D}_{m_i}$, where each $\mathcal{D}_{m_i}$ is a cocycle valued in divisors on $\mathcal{H}_p \times \mathcal{H}_p$ associated to the vectors in V_ℓ of length m_i. The diagonal restriction produces a cocycle whose divisor is a combination of $\mathcal{D}_{m_i} \cap \mathcal{H}_p$ for $1 \leq i \leq k$. Similar to $\mathcal{D}_{m_i}$, there are divisors $\mathcal{D}_{n,3}$ on $\mathcal{H}_p$ that are associated to the vectors in W of length n. Therefore, each divisor $\mathcal{D}_{m_i} \cap \mathcal{H}_p$ can be expressed in terms of $\mathcal{D}_{n,3}$. Upon rewriting the divisor $\sum_{i=1}^{k} c_{m_i}(\mathcal{D}_{m_i} \cap \mathcal{H}_p)$ in terms of the divisors $\mathcal{D}_{n,3}$, we observe that the coefficients of the divisors $\mathcal{D}_{n,3}$ match those of the principal part of the formal series obtained by multiplying a suitable theta series with the Laurent polynomial $\sum_{i=1}^{k} c_{m_i} q^{-m_i}$. This observation is precisely stated in Theorem 7.13. The relation between the input of a Bianchi cocycle and that of its diagonal restriction is similar to the one established in [18] between the input of a Borcherds product and its quasi-pullback.

7.1 Projection to the Subspace

Let $\pi \colon V_\ell \to W$ be the projection given by $v \mapsto v - \frac{\langle v, e_2 \rangle}{\langle e_2, e_2 \rangle} e_2$. The group $\mathrm{SL}_2(K)$ acts by isometries on V_ℓ. Its subgroup $\mathrm{SL}_2(\mathbb{Q})$ fixes the vector e_2, and thus $\mathrm{SL}_2(\mathbb{Q})$ acts by isometries on W. A straightforward computation shows that the projection map π is $\mathrm{SL}_2(\mathbb{Q})$-equivariant.

Verification of action on e_2. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Q})$. We use the image of e_2, which is $M_2 = \left(\begin{smallmatrix} \sqrt{-\ell} & 0 \\ 0 & \sqrt{-\ell} \end{smallmatrix}\right)$, from the model of V_ℓ. Thus,

$$\gamma M_2 \overline{\gamma}^{-1} = M_2 \gamma \overline{\gamma}^{-1} = M_2 \gamma \gamma^{-1} = M_2.$$

7.2 Restriction: Archimedean Story

We recall the discussion on the Archimedean symmetric spaces from Chapter 3. The vector space $V_\ell \otimes \mathbb{R}$ has signature $(3,1)$ and we consider the space of lines of negative length. This space is a manifold of dimension three and is identified with $\mathcal{H}^{(3)} = \mathbb{C} \times \mathbb{R}_{>0}$. Using the model of V_ℓ, we see that the identification is obtained by sending each matrix $\left(\begin{smallmatrix} x+\sqrt{-\ell}y & b \\ b' & -x+\sqrt{-\ell}y \end{smallmatrix}\right)$ of determinant 1 to the point $(x+iy, b)$ on $\mathcal{H}^{(3)}$. Viewing W inside V_ℓ, we see that the vectors in W that are of length -1 are of the form $\left(\begin{smallmatrix} x & b \\ b' & -x \end{smallmatrix}\right)$ with $x^2 + bb' = -1$. Thus, the Archimedean symmetric space associated to W is $\{(x, b) : x \in \mathbb{R}, b \in \mathbb{R}_{>0}\}$. The symmetric space can be seen as the Poincaré upper half plane $\mathcal{H}$ embedded inside the upper half-space $\mathcal{H}^{(3)}$.

For the case of signature $(3,1)$, integer-valued modular symbols are constructed using the intersections of geodesics, or equivalently, the integrals of 1-forms on $\mathcal{H}^{(3)}$. For signature $(2,1)$, we can also construct integer-valued modular symbols by describing the 1-forms on $\mathcal{H}$.

For $w \in W$ and $s, t \in \mathbb{P}_1(\mathbb{Q})$, we define

$$\Delta_{w,\infty,3}\{s,t\} := e^{2\pi(w,w)} \int_{B(H\{s,t\})} \eta(\tau, w),$$

where $\eta(\cdot,\cdot)$ is the differential form described in Section 3.1. Let e_s, e_t denote isotropic vectors corresponding to s, t. By performing the calculations identical to those in Section 3.1.2, we get

$$\Delta_{w,\infty,3}\{s,t\} = \begin{cases} -\mathrm{sgn}((w,e_t)) & (w,e_s)(w,e_t) < 0 \\ -\frac{1}{2}\mathrm{sgn}((w,e_t)) & (w,e_s) = 0, (w,e_t) \neq 0 \\ \frac{1}{2}\mathrm{sgn}((w,e_s)) & (w,e_t) = 0, (w,e_s) \neq 0 \\ 0 & (w,e_s)(w,e_t) > 0. \end{cases} \tag{7.1}$$

The function $\Delta_{w,\infty,3}\{s,t\}$ also satisfies the two-term and three-term relations. Moreover, $\Delta_{\gamma w,\infty,3}\{\gamma s, \gamma t\} = \Delta_{w,\infty,3}\{s,t\}$ for any $\gamma \in \mathrm{SL}_2(\mathbb{Q})$. The proofs of these facts are identical to those in Proposition 3.11.

Note. From now on, we use $\Delta_{v,\infty,4}\{\cdot,\cdot\}$ instead of the notation $\Delta_{v,\infty}\{\cdot,\cdot\}$ to denote intersections in $\mathcal{H}^{(3)}$. This allows us to distinguish between intersections in $\mathcal{H}$ and $\mathcal{H}^{(3)}$.

The intersection number $\Delta_{v,\infty,4}\{s,t\}$ is equal to $\Delta_{\pi(v),\infty,3}\{s,t\}$ for $v \in V_\ell$ and $s,t \in \mathbb{P}_1(\mathbb{Q})$. This equality can be verified using (3.8) and (7.1). Let m be a *proper length with respect to* V_ℓ. Recall that for $\beta = ce_3$, where c is an integer coprime to N, the sum

$$Z(m, \beta + L(\ell,N))\{\cdot,\cdot\} = \sum_{\substack{v \in \beta + L(\ell,N) \\ Q(v) = m}} \Delta_{v,\infty,4}\{\cdot,\cdot\}$$

is a modular symbol invariant under $\Gamma_1^K(N)$. Any $v \in \beta + L(\ell,N)$ is of the form $ae_1 + be_2 + (c + Nc')e_3 + de_4$, where all $a, b, c', d \in \mathbb{Z}$. The vector v can be written as $be_2 + \pi(v)$, and thus $\Delta_{v,\infty,4}\{s,t\} = \Delta_{\pi(v),\infty,3}\{s,t\}$ for $s,t \in \mathbb{P}_1(\mathbb{Q})$. Define $L_W(N) := L(\ell,N) \cap W$. We then have

$$Z(m, \beta + L(\ell,N))\{s,t\} = \sum_{b \in \mathbb{Z}} \sum_{\substack{w \in \beta + L_W(N) \\ Q(w) = m - \ell b^2}} \Delta_{w,\infty,3}\{s,t\}. \tag{7.2}$$

The inner sum is zero whenever $b^2 > m/\ell$ as established by the following lemma.

Lemma 7.1. *Let $w \in W$ with $Q(w) < 0$. Then $\Delta_{w,\infty,3}\{s,t\} = 0$ for any pair $s,t \in \mathbb{P}_1(\mathbb{Q})$.*

Proof. Let $w = \alpha_1 e_{s,t} + \alpha_2 e_s + \alpha_3 e_t$ where $e_{s,t}$ is orthogonal to e_s and e_t. Therefore $Q(w) = \alpha_1^2 Q(e_{s,t}) + \alpha_2\alpha_3(e_s, e_t)$. Since $(e_s, e_t) < 0$ and $Q(e_{s,t}) > 0$, $Q(w) < 0$ implies $\alpha_2\alpha_3 > 0$, which is equivalent to $(w, e_s)(w, e_t) > 0$. By (7.1), $\Delta_{w,\infty,3}\{s,t\} = 0$. □

Let m' be a rational number that is not a perfect square in $\mathbb{Q}$. For $s,t \in \mathbb{P}_1(\mathbb{Q})$, define

$$Z(m', \beta + L_W(N))\{s,t\} := \sum_{\substack{w \in \beta + L_W(N) \\ Q(w) = m'}} \Delta_{w,\infty,3}\{s,t\}.$$

This sum is finite by our assumption on m', as can be verified following the proof of Lemma 6.2. In (7.2), the rational m cannot be written as $a^2 + \ell b^2$ for any $a, b \in \mathbb{Q}$. Therefore, $m - \ell b^2$ is not a square in $\mathbb{Q}$ for any $b \in \mathbb{Q}$. Hence,

$$Z(m, \beta + L(\ell, N))\{s,t\} = \sum_{b \in \mathbb{Z}} Z(m - \ell b^2, \beta + L_W(N))\{s,t\}.$$

Remark 8. *The condition that m cannot be written as $x^2 + \ell y^2$ implies that $m - \ell b^2$ is never the square of a rational number. The notion of* proper length *captures the cases where intersections do not occur at the boundary. Therefore the analogous notion of* proper length with respect to W *implies the non-square condition.*

We end this section by describing the specific subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ for which the sum $Z(m, \beta + L_W(N))\{\cdot,\cdot\}$ defines a Γ-invariant modular symbol.

Lemma 7.2. *The sum $Z(m, \beta + L_W(N))\{\cdot,\cdot\}$ defines a $\Gamma_1^{\mathbb{Q}}(N)$-invariant modular symbol, where*

$$\Gamma_1^{\mathbb{Q}}(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

Remark 9. *The group here is the well-known congruence group $\Gamma_1(N)$. However, we use a superscript $\mathbb{Q}$ to mirror the Bianchi group notation: $\Gamma_1^K(N)$.*

Proof. The two-term and three-term relations are clear. It remains only to verify the $\Gamma_1^{\mathbb{Q}}(N)$-

invariance. Note that $\Gamma_1^{\mathbb{Q}}(N) = \mathrm{SL}_2(\mathbb{Q}) \cap \Gamma_1^K(N)$. For any $\gamma \in \Gamma_1^{\mathbb{Q}}(N)$,

$$\begin{aligned}
Z(m, \beta + L_W(N))\{s,t\}\big|\gamma &= \sum_{\substack{w \in \beta + L_W(N) \\ Q(w)=m}} \Delta_{w,\infty,3}\{\gamma s, \gamma t\}\big|\gamma \\
&= \sum_{\substack{w \in \beta + L_W(N) \\ Q(w)=m}} \Delta_{w,\infty,3}\{\gamma s, \gamma t\} \\
&= \sum_{\substack{w \in \beta + L_W(N) \\ Q(w)=m}} \Delta_{\gamma^{-1}w,\infty,3}\{s, t\} \\
&= \sum_{\substack{w \in \gamma\beta + \gamma L_W(N) \\ Q(\gamma^{-1}w)=m}} \Delta_{w,\infty,3}\{s, t\} \\
&= \sum_{\substack{w \in \beta + L_W(N) \\ Q(w)=m}} \Delta_{w,\infty,3}\{s, t\}.
\end{aligned}$$

We have used that the group $\Gamma_1^{\mathbb{Q}}(N)$ preserves the cosets of $L_W(N)$ in its dual. This is established by the same computation as carried out in the proof of Proposition 6.1. □

7.3 Restriction: The Non-Archimedean Story

The p-adic symmetric space $X_{p,4}$ attached to $V_\ell \otimes \mathbb{Q}_p$ is constructed, as explained in Section 5.1, by considering isotropic lines in $V_\ell \otimes \mathbb{C}_p$ that are not orthogonal to any isotropic line in $V_\ell \otimes \mathbb{Q}_p$. We use the same recipe to associate a p-adic symmetric space to $W \otimes \mathbb{Q}_p$. One approach is to follow the calculations of Section 5.1. However, we use the inclusion $W \hookrightarrow V_\ell$.

Proposition 7.3. *The p-adic symmetric space associated to W is $\mathcal{H}_p$.*

Proof. Let $w \in W \otimes \mathbb{C}_p$ be an isotropic vector. We determine the conditions under which such a vector is orthogonal to some isotropic vector in $W \otimes \mathbb{Q}_p$. If w is orthogonal to some isotropic vector in $W \otimes \mathbb{Q}_p \subset V_\ell \otimes \mathbb{Q}_p$, then $w \in V_\ell \otimes \mathbb{Q}_p$. Thus $w \in W \otimes \mathbb{Q}_p$. Since any isotropic vector of $W \otimes \mathbb{Q}_p$ is orthogonal to itself, the line generated by an isotropic vector w is in the symmetric space if and only if $w \notin W \otimes \mathbb{Q}_p$.

Any vector v in $V_\ell \otimes \mathbb{C}_p$ is an element of $W \otimes \mathbb{C}_p$ if and only if v is orthogonal to e_2. Therefore, the p-adic symmetric space attached to W is precisely the set of isotropic lines in $V_\ell \otimes \mathbb{C}_p - V_\ell \otimes \mathbb{Q}_p$ that are orthogonal to e_2. This is equivalent to saying that the p-adic symmetric space attached to W is $\Delta_{e_2,p,4}$. Identifying $X_{p,4}$ with $\mathcal{H}_p \times \mathcal{H}_p$, the points in

$\mathcal{H}_p \times \mathcal{H}_p$ that correspond to lines orthogonal to e_2 constitute the diagonal $\{(\tau, \tau) : \tau \in \mathcal{H}_p\}$. Therefore, the symmetric space attached to W is identified with $\mathcal{H}_p$. $\square$

Definition 7.4. *Any meromorphic function on $\mathcal{H}_p \times \mathcal{H}_p$ yields a function on $\mathcal{H}_p$ by restricting to the diagonal $\{(\tau, \tau) : \tau \in \mathcal{H}_p\} \subset \mathcal{H}_p \times \mathcal{H}_p$. We use* DiagRes *to denote the restriction map from $\mathcal{M}(\mathcal{H}_p \times \mathcal{H}_p)^\times$ to $\mathcal{M}(\mathcal{H}_p)^\times$.*

Definition 7.5. *The inclusion $\mathcal{H}_p \hookrightarrow \mathcal{H}_p \times \mathcal{H}_p$ induces a map from $Div^\dagger_{rq}(\mathcal{H}_p \times \mathcal{H}_p)$ to $Div^\dagger_{rq}(\mathcal{H}_p)$ defined by*

$$\begin{aligned}\Pi_*\colon Div^\dagger_{rq}(\mathcal{H}_p \times \mathcal{H}_p) &\to Div^\dagger_{rq}(\mathcal{H}_p)\\ \Delta_{v,p,4} &\mapsto \Delta_{\pi(v),p,3} = \Delta_{v,p,4} \cap \Delta_{e_2,p,4} = \{(\tau_1, \tau_2) \in \Delta_{v,p,4} : \tau_1 = \tau_2\},\end{aligned}$$

where $\Delta_{v,p,4}$ is the same as $\Delta_{v,p}$ defined in Definition 5.5. We use a slightly different notation because we now work with two distinct p-adic spaces simultaneously.

Lemma 7.6. *The maps* DiagRes *and* Π_* *are* $\mathrm{SL}_2(\mathbb{Q})$*-equivariant.*

Proof. Let $\phi \in \mathcal{M}(\mathcal{H}_p \times \mathcal{H}_p)^\times$ be a meromorphic function, and let $\gamma \in \mathrm{SL}_2(\mathbb{Q})$. Then we have

$$\begin{aligned}\mathrm{DiagRes}(\phi\big|\gamma)(\tau) = (\phi\big|\gamma)(\tau, \tau) &= \phi(\gamma\tau, \overline{\gamma}\tau)\\ &= \phi(\gamma\tau, \gamma\tau)\\ &= \mathrm{DiagRes}(\phi)(\gamma\tau) = (\mathrm{DiagRes}(\phi)\big|\gamma)(\tau).\end{aligned}$$

Therefore, $\mathrm{DiagRes}(\phi\big|\gamma) = \mathrm{DiagRes}(\phi)\big|\gamma$. Similarly, let $\mathcal{D} = \sum_{v \in V_\ell} a_v \Delta_{v,p,4} \in Div^\dagger_{rq}(\mathcal{H}_p \times \mathcal{H}_p)$ be a divisor. Then for $\gamma \in \mathrm{SL}_2(\mathbb{Q})$,

$$\begin{aligned}\Pi_*\left(\mathcal{D}\big|\gamma\right) = \Pi_*\left(\sum_{v \in V_\ell} a_v \Delta_{v,p,4}\big|\gamma\right) &= \Pi_*\left(\sum_{v \in V_\ell} a_v \Delta_{\gamma^{-1}v,p,4}\right)\\ &= \sum_{v \in V_\ell} a_v \Delta_{\pi(\gamma^{-1}v),p,3}\\ &= \sum_{v \in V_\ell} a_v \Delta_{\gamma^{-1}\pi(v),p,3} = \sum_{v \in V_\ell} a_v \Delta_{\pi(v),p,3}\big|\gamma = \Pi_*(\mathcal{D})\big|\gamma.\end{aligned}$$

$\square$

Lemma 7.7. *Let $\mathcal{M}^{\times}_{\mathrm{rq},4}$ (resp. $\mathcal{M}^{\times}_{\mathrm{rq},3}$) denote the multiplicative group of rigid meromorphic functions on $\mathcal{H}_p \times \mathcal{H}_p$ (resp. $\mathcal{H}_p$) whose divisors are in $Div^{\dagger}_{rq}(\mathcal{H}_p \times \mathcal{H}_p)$ (resp. $Div^{\dagger}_{rq}(\mathcal{H}_p)$). Then the following diagram is commutative.*

$$\begin{array}{ccc}
\mathcal{M}^{\times}_{\mathrm{rq},4} & \xrightarrow{\mathrm{DiagRes}} & \mathcal{M}^{\times}_{\mathrm{rq},3} \\
\Big\downarrow{\scriptstyle \mathrm{div}} & & \Big\downarrow{\scriptstyle \mathrm{div}} \\
Div^{\dagger}_{rq}(\mathcal{H}_p \times \mathcal{H}_p) & \xrightarrow{\Pi_*} & Div^{\dagger}_{rq}(\mathcal{H}_p).
\end{array}$$

Proof. Let $\phi \in \mathcal{M}^{\times}_{\mathrm{rq},4}$. Then we need to prove $\mathrm{div}(\mathrm{DiagRes}(\phi)) = \Pi_*(\mathrm{div}(\phi))$. Since $\mathrm{DiagRes}(\phi)$ is the restriction $\phi\big|\Delta_{e_2,p,4}$, we have $\mathrm{div}(\mathrm{DiagRes}(\phi)) = \mathrm{div}(\phi) \cap \Delta_{e_2,p,4}$. By definition, $\Pi_*(\mathrm{div}(\phi)) = \mathrm{div}(\phi) \cap \Delta_{e_2,p,4}$ and thus the lemma is proved. $\square$

A $(\Gamma_1^K(N))^p$-invariant modular symbol is a map from $\mathbb{P}_1(K) \times \mathbb{P}_1(K)$ valued in $\mathcal{M}^{\times}_{\mathrm{rq},4}$. Restricting the domain to $\mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q})$ and applying DiagRes gives a map to $\mathcal{M}^{\times}_{\mathrm{rq},3}$. Similarly, starting with a $Div^{\dagger}_{rq}(\mathcal{H}_p \times \mathcal{H}_p)$-valued modular symbol, we can use restriction combined with Π_* to obtain a function from $\mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q})$ to $Div^{\dagger}_{rq}(\mathcal{H}_p)$. The following lemma establishes that both resulting functions are modular symbols invariant under $(\Gamma_1^{\mathbb{Q}}(N))^p := (\Gamma_1^K(N))^p \cap \mathrm{SL}_2(\mathbb{Q})$.

Lemma 7.8. *Let Ω_3 be a $(\Gamma_1^{\mathbb{Q}}(N))^p$-module and Ω_4 be a $(\Gamma_1^K(N))^p$-module such that there is an $\mathrm{SL}_2(\mathbb{Q})$-equivariant homomorphism $\Psi\colon \Omega_4 \to \Omega_3$. Let $\mathfrak{m}^K\{\cdot,\cdot\}$ be a $(\Gamma_1^K(N))^p$-invariant modular symbol valued in Ω_4. Then the map*

$$\begin{aligned}
\mathfrak{m}^{\mathbb{Q}}\colon \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q}) &\to \Omega_3 \\
\{s,t\} &\mapsto \Psi(\mathfrak{m}^K\{s,t\})
\end{aligned}$$

is a $(\Gamma_1^{\mathbb{Q}}(N))^p$-invariant modular symbol.

Proof. The two-term and three-term relations follow since Ψ is a homomorphism. Let $\gamma \in (\Gamma_1^{\mathbb{Q}}(N))^p$. Then

$$\mathfrak{m}^{\mathbb{Q}}\{\gamma s, \gamma t\}\big|\gamma = \Psi(\mathfrak{m}^K\{\gamma s, \gamma t\})\big|\gamma = \Psi(\mathfrak{m}^K\{s,t\}) = \mathfrak{m}^{\mathbb{Q}}\{s,t\}.$$

$\square$

Consequently, we have the following commutative diagram.

$$\begin{tikzcd}
\mathrm{MS}((\Gamma_1^K(N))^p, \mathcal{M}_{\mathrm{rq},4}^\times) \arrow[r, "\mathrm{DiagRes}"] \arrow[d, "\mathrm{div}"] & \mathrm{MS}((\Gamma_1^{\mathbb{Q}}(N))^p, \mathcal{M}_{\mathrm{rq},3}^\times) \arrow[d, "\mathrm{div}"] \\
\mathrm{MS}((\Gamma_1^K(N))^p, Div_{rq}^\dagger(\mathcal{H}_p \times \mathcal{H}_p)) \arrow[r, "\Pi_*"] & \mathrm{MS}((\Gamma_1^{\mathbb{Q}}(N))^p, Div_{rq}^\dagger(\mathcal{H}_p)).
\end{tikzcd}$$

We also get a commutative diagram for the module of functions modulo scalars.

$$\begin{tikzcd}
\mathrm{MS}((\Gamma_1^K(N))^p, \mathcal{M}_{\mathrm{rq},4}^\times/\mathbb{Z}_p^\times) \arrow[r, "\mathrm{DiagRes}"] \arrow[d, "\mathrm{div}"] & \mathrm{MS}((\Gamma_1^{\mathbb{Q}}(N))^p, \mathcal{M}_{\mathrm{rq},3}^\times/\mathbb{Z}_p^\times) \arrow[d, "\mathrm{div}"] \\
\mathrm{MS}((\Gamma_1^K(N))^p, Div_{rq}^\dagger(\mathcal{H}_p \times \mathcal{H}_p)) \arrow[r, "\Pi_*"] & \mathrm{MS}((\Gamma_1^{\mathbb{Q}}(N))^p, Div_{rq}^\dagger(\mathcal{H}_p)).
\end{tikzcd}$$

By Theorem 6.11, there exists a modular symbol valued in $\mathcal{M}_{\mathrm{rq},4}^\times/\mathbb{Z}_p^\times$ whose divisor is a certain linear combination of the divisor-valued modular symbols $\mathcal{D}(\beta, m)$ defined in (6.6). We use the map DiagRes to obtain a modular symbol valued in $\mathcal{M}_{\mathrm{rq},3}^\times/\mathbb{Z}_p^\times$ whose corresponding divisor-valued modular symbol is then a combination of $\Pi_*(\mathcal{D}(\beta, m))$. In the remainder of this section, we compute $\Pi_*(\mathcal{D}(\beta, m))$ in terms of vectors of W. From now on, we use the notation $\mathcal{D}_{\beta,m,4}$ instead of $\mathcal{D}(\beta, m)$. Recall that we have,

$$\mathcal{D}_{\beta,m,4}\{s,t\} = \sum_{\substack{v \in \beta + L^p \\ Q(v)=m}} \Delta_{v,\infty,4}\{s,t\}\, \Delta_{v,p,4}.$$

Applying the map $\Pi_*(\cdot)$, we get

$$\Pi_*(\mathcal{D}_{\beta,m,4})\{s,t\} = \sum_{\substack{v \in \beta + L^p \\ Q(v)=m}} \Delta_{v,\infty,4}\{s,t\}\, \Delta_{\pi(v),p,3}.$$

Since $\Delta_{v,\infty,4}\{s,t\} = \Delta_{\pi(v),\infty,3}\{s,t\}$ for any $s, t \in \mathbb{P}_1(\mathbb{Q})$, we rewrite the above equation as

$$\Pi_*(\mathcal{D}_{\beta,m,4})\{s,t\} = \sum_{\substack{v \in \beta + L^p \\ Q(v)=m}} \Delta_{\pi(v),\infty,3}\{s,t\}\, \Delta_{\pi(v),p,3}.$$

Every vector $v \in \beta + L^p$ uniquely decomposes as $v = \pi(v) + be_2$, where $b \in \mathbb{Z}[1/p]$. Define $L_W^p := L^p \cap W$. Thus,

$$\Pi_*(\mathcal{D}_{\beta,m,4})\{s,t\} = \sum_{b \in \mathbb{Z}[1/p]} \sum_{\substack{w \in \beta + L_W^p \\ Q(w)=m-b^2\ell}} \Delta_{w,\infty,3}\{s,t\}\, \Delta_{w,p,3}. \tag{7.3}$$

The divisor $\Pi_*(\mathcal{D}_{\beta,m,4})\{s,t\}$ is locally finite on $\mathcal{H}_p$ because $\mathcal{D}_{\beta,m,4}\{s,t\}$ is a locally finite divisor on $\mathcal{H}_p \times \mathcal{H}_p$. The above sum can be refined by applying the properties of the special divisors $\Delta_{v,p,4}$. Recall from our discussion of Archimedean spaces that for any $w \in \beta + L_W^p$, we have $\Delta_{w,\infty,3}\{s,t\} = 0$ for all $s,t \in \mathbb{P}_1(\mathbb{Q})$ whenever $m - b^2\ell < 0$. Therefore, we must have $b^2 \leq m/\ell$. Moreover, the following lemma shows that b must be an integer.

Lemma 7.9. *Let $w \in W \otimes \mathbb{Q}_p$ with $Q(w) = m - \ell b^2$. If $b \in \mathbb{Z}[1/p] \setminus \mathbb{Z}$, then $\Delta_{w,p,3} = \emptyset$.*

Proof. We first consider the case where the length $Q(w)$ is a square in $\mathbb{Q}_p$. Let $Q(w) = \alpha^2$ for some $\alpha \in \mathbb{Q}_p$. Since the space $W \otimes \mathbb{Q}_p$ is of rank three and discriminant -1, we have a basis of $W \otimes \mathbb{Q}_p$ given by $\{w, \ell_1, \ell_2\}$ where ℓ_1, ℓ_2 are isotropic vectors with $(\ell_1, \ell_2) = -1$. The vectors orthogonal to $w \in W \otimes \mathbb{C}_p$ are of the form $\alpha_1\ell_1$ and $\alpha_2\ell_2$ for $\alpha_1, \alpha_2 \in \mathbb{C}_p$. However, these are orthogonal to ℓ_1, ℓ_2 respectively. In $W \otimes \mathbb{C}_p$, the only isotropic vectors orthogonal to w are scalar multiples of either ℓ_1 or ℓ_2. However, these are orthogonal to ℓ_1 and ℓ_2, respectively. Consequently, $\Delta_{w,p,3} = \emptyset$.

We now establish the statement in the lemma. Let $w \in W \otimes \mathbb{Q}_p$ with $Q(w) = m - \ell b^2$. For $b \in \mathbb{Z}[1/p] \setminus \mathbb{Z}$, we write $b = b_0 p^{-n}$ for some $b_0 \in \mathbb{Z}$ and $n \geq 1$. We then have

$$m - \ell b_0^2 p^{-2n} = \frac{p^{2n}m - \ell b_0^2}{p^{2n}}.$$

Hensel's lemma guarantees that the numerator is a square in $\mathbb{Q}_p$ for $n \geq 1$. Therefore, $\Delta_{w,p,3}$ is a trivial divisor for w with length $Q(w) = m - \ell b_0^2 p^{-2n}$, whenever $n \geq 1$. Thus the only contribution comes when $b \in \mathbb{Z}$. $\square$

The lemma and the discussion above it imply that there are only finitely many values of b that contribute to the sum (7.3).

Let m be any rational which is not a square and $\beta \in (L_W^p)^*$. For $s,t \in \mathbb{P}_1(\mathbb{Q})$, we define the divisor

$$\mathcal{D}_{\beta,m,3}\{s,t\} := \sum_{\substack{w \in \beta + L_W(N) \\ Q(w)=m}} \Delta_{w,\infty,3}\{s,t\}\, \Delta_{w,p,3}.$$

Therefore, $\Pi_*(\mathcal{D}_{\beta,m,4}) = \sum_{b \in \mathbb{Z}} \mathcal{D}_{\beta,m-\ell b^2,3}$.

7.4 p-adic Analogue of Quasi-Pullback of Borcherds Products

Let $M = \mathrm{lcm}(4\ell, N)$, where ℓ and N are as in the previous sections. In this section, we work with a specific space of cusp forms that we define below.

Definition 7.10. *We denote by $S_{3/2}\big(\Gamma_0(Mp), \big(\frac{\cdot}{\ell}\big)\big)$ the space of cusp forms of weight $3/2$ for the group $\Gamma_0(Mp)$ with character $\big(\frac{\cdot}{\ell}\big)$. A function $g\colon \mathcal{H} \to \mathbb{C}$ belongs to this space if it satisfies the following conditions:*

1. *g is holomorphic on $\mathcal{H}$ and vanishes at all cusps of $\Gamma_0(Mp)$,*
2. *the function $h(\tau) := g(\tau)\theta(\ell\tau)$ is a modular form of weight 2, level Mp with character $\big(\frac{\cdot}{\ell}\big)$.*

Definition 7.11. *We define the set $\mathrm{PP}_{1/2}\big(Mp, \big(\frac{\cdot}{\ell}\big)\big)^\dagger$ to be the set of finite collections of integers, $\{d_n\}_{n=1}^{m_0}$ for an arbitrary positive integer m_0, such that*

1. *$d_n = 0$ if n is a square, and*
2. *for each cusp form $g \in S_{3/2}\big(\Gamma_0(Mp), \big(\frac{\cdot}{\ell}\big)\big)$,*

$$\sum_{n=1}^{m_0} d_n \times \{n\text{-}th\ coefficient\ of\ g(\tau)\} = 0.$$

Definition 7.12. *Let $\delta = \{c_m\}_{m=1}^{m_0}$ be an element in $\mathrm{PP}_0\big(Mp, \big(\frac{\cdot}{\ell}\big)\big)^\dagger$ for $M = \mathrm{lcm}(4\ell, N)$. Define*

$$\begin{aligned} \Theta_{\ell,*}\colon \mathrm{PP}_0\big(Mp, \big(\tfrac{\cdot}{\ell}\big)\big)^\dagger &\to \mathrm{PP}_{1/2}\big(Mp, \big(\tfrac{\cdot}{\ell}\big)\big)^\dagger \\ \{c_m\}_{m=1}^{m_0} &\mapsto \{d_n\}_{n=1}^{m_0}, \end{aligned}$$

where

$$d_n := \sum_{\substack{j\in\mathbb{Z} \\ \ell j^2 \le m_0 - n}} c_{n+\ell j^2}.$$

The well-definedness of the map $\Theta_{\ell,*}$ is established in Lemma 7.14. To state our second main theorem, we slightly modify our notation. We use $\mathcal{D}_{\langle\beta\rangle,m,4}$ to denote the divisor

$\sum_{\substack{0<u<M\\(u,M)=1}} \mathcal{D}_{u\beta,m,4}$ and $\mathcal{D}_{\langle\beta\rangle,m,3}$ to denote the divisor $\sum_{\substack{0<u<M\\(u,M)=1}} \mathcal{D}_{u\beta,m,3}$.

Theorem 7.13. *Let $\delta = \{c_m\}_{m=1}^{m_0}$ be an element in $\mathrm{PP}_0\big(Mp, \left(\frac{\cdot}{\ell}\right)\big)^\dagger$ (see Definition 1.2). Let Ψ_δ be the corresponding modular symbol constructed in Theorem 6.11. The divisor-valued modular symbol corresponding to* $\mathrm{DiagRes}(\Psi_\delta)$ *is*

$$\sum_{n=1}^{m_0} d_n \mathcal{D}_{\langle\beta\rangle,n,3},$$

where $\{d_n\}_{n=1}^{m_0} = \Theta_{\ell,*}(\{c_m\}_{m=1}^{m_0})$.

Proof. The divisor of Ψ_δ is $\sum_{m=1}^{m_0} c_m \mathcal{D}_{\langle\beta\rangle,m,4}$. The divisor of $\mathrm{DiagRes}(\Psi_\delta)$ is

$$\sum_{m=1}^{m_0} c_m \Pi_*(\mathcal{D}_{\langle\beta\rangle,m,4}) = \sum_{m=1}^{m_0} c_m \sum_{j\in\mathbb{Z}} \mathcal{D}_{\langle\beta\rangle,m-\ell j^2,3} = \sum_{n=1}^{m_0} \left(\sum_{\ell j^2 \le m_0 - n} c_{n+\ell j^2} \right) \mathcal{D}_{\langle\beta\rangle,n,3}.$$

□

Remark 10. *This theorem provides an explicit formula for the divisor of the diagonal restriction of a given Bianchi cocycle. The formula suggests that there is a similar relation of multiplying by a θ-series as known in the case of quasi-pullback of Borcherds lifts from $O(2,2)$ to $O(2,1)$. As explained in Chapter 1, this theorem strengthens the analogy between classical Borcherds lifts and rigid meromorphic cocycles for orthogonal groups.*

Lemma 7.14. *The map $\Theta_{\ell,*} \colon \mathrm{PP}_0\big(Mp, \left(\frac{\cdot}{\ell}\right)\big)^\dagger \to \mathrm{PP}_{1/2}\big(Mp, \left(\frac{\cdot}{\ell}\right)\big)^\dagger$ is well-defined.*

Proof. The claim follows from a direct computation. We first note that if n is a square then $c_{n+\ell j^2}$ is zero for each j by definition.

Let $g \in S_{3/2}(\Gamma_0(Mp), \left(\frac{\cdot}{\ell}\right))$. Let the q-expansion of g be

$$g(\tau) = \sum_{n=1}^{\infty} a_g(n) q^n.$$

Multiplying $g(\tau)$ by the theta series $\theta(\ell\tau) = \sum_{j\in\mathbb{Z}} q^{\ell j^2}$ yields a modular form of weight 2 for the group $\Gamma_0(Mp)$, namely

$$g(\tau)\theta(\ell\tau) = \sum_{m=1}^{\infty} \left(\sum_{\ell j^2 \le m} a_g(m - \ell j^2) \right) q^m \in M_2(\Gamma_0(Mp), \left(\tfrac{\cdot}{\ell}\right)).$$

Let $\delta = \{c_m\}_{m=1}^{m_0} \in \mathrm{PP}_0\big(Mp, \big(\frac{\cdot}{\ell}\big)\big)^\dagger$. By definition of $\mathrm{PP}_0\big(Mp, \big(\frac{\cdot}{\ell}\big)\big)^\dagger$, the sum

$$\sum_{m=1}^{m_0} c_m \times \left(\sum_{\ell j^2 \le m} a_g(m - \ell j^2) \right) = 0.$$

By substituting $n = m - \ell j^2$ and exchanging the order of summation, the left-hand side can be rearranged as

$$\sum_{n=1}^{m_0} \big(\sum_{\ell j^2 \le m_0 - n} c_{n+\ell j^2} \big) a_g(n) = \sum_{n=1}^{m_0} d_n a_g(n).$$

Consequently, $\sum_{n=1}^{m_0} d_n a_g(n) = 0$. Hence, the sequence $\{d_n\}_{n=1}^{m_0}$ lies in $\mathrm{PP}_{1/2}\big(Mp, \big(\frac{\cdot}{\ell}\big)\big)^\dagger$. $\square$

8

Future Directions

The main theorems presented in this thesis rely on several simplifying hypotheses. We believe that investigating and eventually removing these hypotheses will deepen our understanding of rigid meromorphic cocycles and strengthen the proposed analogy.

8.1 Proper Lengths

The notion of *proper length* is central to the construction of rigid meromorphic cocycles presented in this thesis. However, the current framework does not extend to constructing rigid cocycles associated with rational numbers m that are not *proper lengths*. In this section, we discuss the underlying obstructions and outline potential alternative approaches.

Recall that the Fourier expansion of theta series as described in Equation (3.7) is given by

$$\sum_{i=1}^{r}\left[\sum_{k=0}^{m-1}\left(\sum_{\substack{X_1\in h_i'+L'\\ Q(X_1)=k}}\ \sum_{\substack{X_2\in h_i''+L''\\ Q(X_2)=m-k}} -\mathrm{sgn}((X_2,e_s))\right)+\sum_{\substack{X_1\in h_i'+L'\\ Q(X_1)=m}}\Theta_0(h_i''+L'')\right]. \tag{8.1}$$

There are two types of coefficients in the above expansion. In this thesis, we restrict our attention to the coefficients that do not involve zeta values. These coefficients can be expressed as intersections of geodesics, allowing us to construct integer-valued modular symbols. The remaining Fourier coefficients also admit a topological interpretation, as detailed by Funke and Millson [9]. Briefly, this approach involves attaching a Borel-Serre component to each cusp and describing cycles associated with vectors whose length is the norm of an element in K. However, these cycles are not attached to individual vectors, but rather to a collection $\{v + \alpha e\}$, where e is an isotropic vector corresponding to the given cusp. The divisor of a rigid cocycle is a sum over vectors, where each vector contributes an intersection pairing and a well-defined divisor on the associated p-adic space. To utilize the interpretation given in [9], one must formulate new candidates for divisors associated with the collection $\{v + \alpha e\}$.

8.2 The case of an Inert Prime

Throughout this thesis, we have assumed that the prime p splits in $K = \mathbb{Q}(\sqrt{-\ell})$. This assumption allows us to view $\mathrm{SL}_2(K) \subset O(3,1)$ analogously to the Hilbert modular group. We then identified the associated p-adic space with $\mathcal{H}_p \times \mathcal{H}_p$ as described in Chapter 5.

A natural future direction is to investigate the setting where p is inert. In this case, the vector space $V_\ell \otimes \mathbb{Q}_p$ acts as a p-adic analogue of a real quadratic space with signature $(3,1)$. As before, we need to determine a suitable p-adic symmetric space. Following the methodology of Section 5.1, one could explicitly realize this space using the following non-split $\mathbb{Q}_p$-vector space of matrices,

$$W_p := \left\{ \begin{pmatrix} a & b \\ c & -a^\sigma \end{pmatrix} : a \in K, \text{ and } b, c \in \mathbb{Q}_p \right\}.$$

The quadratic form Q on W_p is given by the negative of the determinant. Using this model we find that the rigid space X_p is in bijection with $(\mathbb{P}_1(\mathbb{C}_p) \setminus \mathbb{P}_1(\mathbb{Q}_{p^2})) \times (\mathbb{P}_1(\mathbb{C}_p) \setminus \mathbb{P}_1(\mathbb{Q}_{p^2}))$. Interestingly, this space is discussed in the work of Hassan [12] in the context of evaluating rigid meromorphic cocycles of higher weights. To develop the theory for the inert prime setting, it remains to construct rigid meromorphic cocycles for this specific setting and investigate their properties. Future work could focus on computational aspects, such as gathering evidence for algebraicity, establishing connections to explicit class field theory, or proving lifting theorems.

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