Stark-Heegner points attached to real quadratic fields

Contributed talk

Conference in honor of Stark's 65th Birthday

Minneapolis, Minnesota

August 2004

Dirichlet's class number formula

Let $\chi : (\mathbf{Z}/f\mathbf{Z})^\times \longrightarrow \pm 1, \quad \chi(-1) = 1$

be an *even* primitive Dirichlet character.

Theorem (Dirichlet)

$$L'(0, \chi) = \log\left(\prod_{a=1}^{f}\left(1 - e^{\frac{2\pi i a}{f}}\right)^{\chi(a)}\right).$$

Note: $u(\chi) := \prod_{a=1}^{f}\left(1 - e^{\frac{2\pi i a}{f}}\right)^{\chi(a)}$ is a unit in $\mathbf{Q}(\sqrt{f})$ (called a *circular unit*).

Stark's conjecture

F= number field,

$\zeta(F, \mathcal{A}, s)$= partial zeta-function attached to the (*narrow*) ideal class $\mathcal{A}$.

Conjecture (Stark) Suppose $\zeta(F, \mathcal{A}, 0) = 0$. Then there exists a unit $u(\mathcal{A})$ of the narrow Hilbert class field of F such that

$$\zeta'(F, \mathcal{A}, 0) = \log |u(\mathcal{A})|.$$

The unit $u(\mathcal{A})$ is called a *Stark unit*.

Note: There is no *independent expression* for $u(\mathcal{A})$.

The Question

Question Can Stark's conjecture be extended to elliptic curves?

E= elliptic curve over $\mathbf{Q}$

$L(E, s)$ = its Hasse-Weil L-function.

Birch and Swinnerton-Dyer Conjecture. If $L(E, 1) = 0$, then there exists $P \in E(\mathbf{Q})$ such that

$$L'(E, 1) = \hat{h}(P) \cdot (\text{ explicit period}).$$

Remark: Like in Stark's conjecture, there is no *independent formula* for P.

Heegner Points

Modular parametrisation attached to E:

$$\Phi : \mathcal{H}/\Gamma_0(N) \longrightarrow E(\mathbf{C}).$$

$K = \mathbf{Q}(\sqrt{-D}) \subset \mathbf{C}$ a *quadratic imaginary field*.

Theorem. If τ belongs to $\mathcal{H} \cap K$, then $\Phi(\tau)$ belongs to $E(K^{\mathrm{ab}})$.

This theorem produces a *systematic* and *well-behaved* collection of algebraic points on E defined over class fields of imaginary quadratic fields.

These points are analogous to circular or elliptic units.

Heegner points

Given $\tau \in \mathcal{H} \cap K$, let

$$F_\tau(x, y) = Ax^2 + Bxy + Cy^2$$

be the primitive binary quadratic form with

$$F_\tau(\tau, 1) = 0, \quad N|A.$$

Define $\text{Disc}(\tau) := \text{Disc}(F_\tau)$.

$$\mathcal{H}^D := \{\tau \text{ s.t. } \text{Disc}(\tau) = D.\}.$$

$H_D =$ ring class field of K attached to D.

Theorem 1. If τ belongs to $\mathcal{H}^D$, then $P_D := \Phi(\tau)$ belongs to $E(H_D)$.

2. (Gross-Zagier)

$$L'(E/K, \mathcal{A}, 1) = \hat{h}(P_D) \cdot (\text{period})$$

The Stark-Heegner conjecture

Stark-Heegner Conjecture (Vague form) Heegner points admit many *variants*, which are to classical Heegner points as Stark units are to circular/elliptic units.

General setting: E defined over F;

$K =$ auxiliary quadratic extension of F;

The Stark-Heegner points should be defined over ring class fields of K.

1. $F = \mathbf{Q}$, $K =$ real quadratic field

2. $F =$ totally real field, $K =$ ATR extension

3. $F =$ imaginary quadratic field.

Key example: real quadratic fields

Set-up: E has conductor $N = pM$, with $p \nmid M$.

$\mathcal{H}_p := \mathbf{C}_p - \mathbf{Q}_p$ (A p-adic analogue of $\mathcal{H}$)

$K =$ real quadratic field, embedded both in $\mathbf{R}$ and $\mathbf{C}_p$.

Motivation for $\mathcal{H}_p$: $\mathcal{H} \cap K = \emptyset$, but $\mathcal{H}_p \cap K$ need not be empty!

Goal: Define a p-adic "modular parametrisation"

$$\Phi : \mathcal{H}_p^D / \Gamma_0(M) \longrightarrow E(H_D),$$

for *positive* discriminants D.

In defining Φ, I follow an approach suggested by **Dasgupta's thesis**.

Hida Theory

$U = p$-adic disc in $\mathbf{Q}_p$ with $2 \in U$;

$\mathcal{A}(U) =$ ring of p-adic analytic functions on U.

Hida. There exists a unique q-expansion

$$f_\infty = \sum_{n=1}^{\infty} \underline{a}_n q^n, \quad \underline{a}_n \in \mathcal{A}(U),$$

such that $\forall k \geq 2$, $k \in \mathbf{Z}$, $k \equiv 2 \pmod{p-1}$,

$$f_k := \sum_{n=1}^{\infty} \underline{a}_n(k) q^n$$

is an eigenform of weight k on $\Gamma_0(N)$, and

$$f_2 = f_E.$$

For $k > 2$, f_k arises from a newform of level M, which we denote by $f_k^\dagger$.

Heegner points for real quadratic fields

Definition. If $\tau \in \mathcal{H}_p/\Gamma_0(M)$, let $\gamma_\tau \in \Gamma_0(M)$ be a generator for $\mathrm{Stab}_{\Gamma_0(M)}(\tau)$.

Choose $r \in \mathbf{P}_1(\mathbf{Q})$, and consider the "Shimura period" attached to τ and $f_k^\dagger$:

$$J_\tau^\dagger(k) := \Omega_E^{-1} \int_r^{\gamma_\tau r} (z-\tau)^{k-2} f_k^\dagger(z) dz.$$

This does not depend on r.

Proposition. There exist $\lambda_k \in \mathbf{C}^\times$ such that $\lambda_2 = 1$ and

$$J_\tau(k) := \lambda_k^{-1}(a_p(k)^2 - 1) J_\tau^\dagger(k)$$

takes values in $\bar{\mathbf{Q}} \subset \mathbf{C}_p$ and extends to a p-adic anaytic function of $k \in U$.

The definition of Φ

Note: $J_\tau(2) = 0$. We define:

$$\log_E \Phi(\tau) := \frac{d}{dk} J_\tau(k)|_{k=2}.$$

There are more precise formulae giving $\Phi(\tau)$ itself, and not just its formal group logarithm.

Conjecture 1. If τ belongs to $\mathcal{H}^D$, then $P_D := \Phi(\tau)$ belongs to $E(H_D)$.

2. ("Gross-Zagier")

$$L'(E/K, \mathcal{A}, 1) = \widehat{h}(P_D) \cdot (\text{period})$$

Numerical examples

Magma `shp` package (Green, Pollack)

`www.math.mcgill.ca/darmon/programs/shp`

$$E = X_0(11) : y^2 + y = x^3 - x^2 - 10x - 20.$$

> HP,P,hD := stark_heegner_points(E,8,Qp);

The discriminant D = 8 has class number 1

Computing point attached to quadratic form (1,2,-1)

Stark-Heegner point (over Cp) =

$$(-2088624084707821, 1566468063530870w + 2088624084707825) + O(11^{15})$$

This point is close to $[9/2, 1/8(7s-4), 1]$

$(9/2 : 1/8(7s-4) : 1)$ is a global point on E(K).

A second example

$E = 37A : y^2 + y = x^3 - x, \quad D = 1297.$

> ,,hD := stark_heegner_points(E,1297,Qp);

The discriminant D = 1297 has class number 11

1 Computing point for quadratic form (1,35,-18)

2 Computing point for quadratic form (-4,33,13)

3 Computing point for quadratic form (16,9,-19)

4 Computing point for quadratic form (-6,25,28)

5 Computing point for quadratic form (-8,23,24)

6 Computing point for quadratic form (2,35,-9)

7 Computing point for quadratic form (9,35,-2)

8 Computing point for quadratic form (12,31,-7)

9 Computing point for quadratic form (-3,31,28)

10 Computing point for quadratic form (12,25,-14)

11 Computing point for quadratic form (14,17,-18)

Sum of the Stark-Heegner points (over Cp) =

$(0 : -1 : 1)) + (37^{100})$

This p-adic point is close to $[0, -1, 1]$

$(0 : -1 : 1)$ is indeed a global point on E(K).

Polynomial hD satisfied by the x-ccordinates:

$$\begin{aligned} 961x^{11} &- 4035x^{10} - 3868x^9 + 19376x^8 + 13229x^7 \\ &- 27966x^6 - 21675x^5 + 11403x^4 + 11859x^3 \\ &+ 1391x^2 - 369x - 37 \end{aligned}$$

> G := GaloisGroup(hD);

Permutation group G acting on a set of cardinality 11

(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)

(1, 10)(2, 9)(3, 8)(4, 7)(5, 6)

> #G;

22

Connection with Stark's conjecture

A preview of Dasgupta's lecture

Replace f_E by a modular unit α (or rather, its *logarithmic derivative*, an Eisenstein series F_2 of weight 2).

Replace f_k by the corresponding family F_k of Eisenstein series of varying weight.

Dasgupta recovers certain p-adic *Gross-Stark units* in narrow ring class fields of real quadratic fields.