p-adic Modular Forms and Arithmetic

A conference in honor of Haruzo Hida's 60th birthday

Hida's p-adic Rankin L-functions and syntomic regulators of Beilinson-Flach elements

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(Joint with Massimo Bertolini and Victor Rotger)

Also based on earlier work with Bertolini and Kartik Prasanna

Preliminaries

Rankin L-series are attached to a pair

$$f \in S_k(\Gamma_1(N_f), \chi_f), \qquad g \in S_\ell(\Gamma_1(N_g), \chi_g)$$

of cusp forms,

$$f = \sum_{n=1}^{\infty} a_n(f) q^n, \qquad g = \sum_{n=1}^{\infty} a_n(g) q^n.$$

Hecke polynomials ($p \nmid N := \mathrm{lcm}(N_f, N_g)$)

$$x^2 - a_p(f)x + \chi_f(p)p^{k-1} = (x - \alpha_p(f))(x - \beta_p(f)).$$

$$x^2 - a_p(g)x + \chi_g(p)p^{\ell-1} = (x - \alpha_p(g))(x - \beta_p(g)).$$

Rankin L-series, definition

Incomplete Rankin L-series:

$$\begin{aligned} L_N(f \otimes g, s)^{-1} &= \prod_{p \nmid N} (1 - \alpha_p(f)\alpha_p(g)p^{-s})(1 - \alpha_p(f)\beta_p(g)p^{-s}) \\ &\quad \times (1 - \beta_p(f)\alpha_p(g)p^{-s})(1 - \beta_p(f)\beta_p(g)p^{-s}) \end{aligned}$$

This definition, completed by a description of Euler factors at the "bad primes", yields the Rankin L-series

$$L(f \otimes g, s) = L(V_f \otimes V_g, s),$$

where V_f, V_g are the Deligne representations attached to f and g.

Rankin L-series, integral representation

Assume for simplicity that $k = \ell = 2$.

Non-holomorphic Eisenstein series of weight 0:

$$E_\chi(z,s) = \sum_{(m,n)\in N\mathbb{Z}\times\mathbb{Z}}{}' \chi^{-1}(n) y^s |mz+n|^{-2s}.$$

Theorem (Shimura)

Let $\chi := (\chi_f \chi_g)^{-1}$. Then

$$L(f\otimes g, s) = \frac{(4\pi)^s}{\Gamma(s)} \left\langle \bar{f}(z), E_\chi(z, s-1) g(z) \right\rangle_{\Gamma_0(N)}.$$

This is proved using the *Rankin-Selberg method*.

Rankin L-series, properties

The non-holomorphic Eisenstein series have analytic continuation to $s \in \mathbb{C}$ and satisfy a functional equation under $s \leftrightarrow 1 - s$.

Shimura's integral representation for $L(f \otimes g, s)$ leads to its analytic continuation, with a functional equation

$$L(f \otimes g, s) \leftrightarrow L(f \otimes g, 3 - s).$$

Goal of Beilinson's formula: Give a geometric interpretation for $L(f \otimes g, s)$ at the "near central point" $s = 2$.

This geometric interpretation involves the higher Chow groups of $X_0(N) \times X_0(N)$.

Higher Chow groups

Let S=smooth proper surface over a field K.

Definition

The *Higher Chow group* $\mathrm{CH}^2(S,1)$ is the first homology of the *Gersten complex*

$$K_2(K(S)) \xrightarrow{\partial} \oplus_{Z \subset S} K(Z)^\times \xrightarrow{\mathrm{div}} \oplus_{P \in S} \mathbb{Z}.$$

So an element of $\mathrm{CH}^2(S,1)$ is described by a formal linear combination of pairs (Z_j, u_j) where the Z_j are curves in S, and u_j is a rational function on Z_j.

Beilinson-Flach elements

These are distinguished elements in $\mathrm{CH}^2(S,1)$ arising when

1. $S = X_1(N) \times X_1(N)$ is a product of modular curves;
2. $Z = \Delta \simeq X_1(N)$ is the *diagonal*;
3. $u \in \mathbb{C}(\Delta)^\times$ is a *modular unit.*

Lemma

For all modular units $u \in \mathbb{C}(\Delta)^\times$, there is an element of the form

$$\Delta_u = (\Delta, u) + \sum_i \lambda_i (P_j \times X_1(N), u_i) + \sum_j \eta_j (X_1(N) \times Q_j, v_j)$$

which belongs to $\mathrm{CH}^2(S,1) \otimes \mathbb{Q}$. It is called the Beilinson-Flach element *associated to the pair (Δ, u).*

Modular units

Manin-Drinfeld: the group $\mathcal{O}_{Y_1(N)}^{\times}/\mathbb{C}^{\times}$ has "maximal possible rank", namely $\#(X_1(N) - Y_1(N)) - 1$.

The logarithmic derivative gives a surjective map

$$\mathrm{dlog} : \mathcal{O}_{Y_1(N)}^{\times} \otimes \mathbb{Q} \longrightarrow \mathrm{Eis}_2(\Gamma_1(N), \mathbb{Q})$$

to the space of weight two Eisenstein series with coefficients in $\mathbb{Q}$.

Let $u_\chi \in \mathcal{O}_{Y_1(N)}^{\times} \otimes \mathbb{Q}_\chi$ be the modular unit characterised by

$$\mathrm{dlog}\, u_\chi = E_{2,\chi},$$

$$E_{2,\chi}(z) = 2^{-1} L(\chi, -1) + \sum_{n=1}^{\infty} \sigma_\chi(n) q^n, \quad \sigma_\chi(n) = \sum_{d|n} \chi(d) d.$$

Complex regulators

The complex regulator is the map

$$\mathrm{reg}_{\mathbb{C}} : \mathrm{CH}^2(S,1) \longrightarrow (\mathrm{Fil}^1\, H^2_{\mathrm{dR}}(S/\mathbb{C}))^\vee$$

defined by

$$\mathrm{reg}_{\mathbb{C}}((Z,u))(\omega) = \frac{1}{2\pi i}\int_{Z'} \omega \log|u|,$$

where

- ω is a smooth two-form on S whose associated class in $H^2_{\mathrm{dR}}(S/\mathbb{C})$ belongs to Fil^1;
- Z'=locus in Z where u is regular.

Beilinson's formula

Theorem (Beilinson)

For cusp forms f and g of weight 2 and characters χ_f and χ_g,

$$L(f \otimes g, 2) = C_\chi \times \mathrm{reg}_{\mathbb{C}}(\Delta_{u_\chi})(\bar{\omega}_f \wedge \omega_g),$$

where

$$C_\chi = 16\pi^3 N^{-2} \tau(\chi^{-1}),$$
$$\chi = (\chi_f \chi_g)^{-1}.$$

A p-adic Beilinson formula?

Such a formula should relate:

1. The value at $s = 2$ of certain p-adic L-series attached to f and g;
2. The images of Beilinson-Flach elements under certain *p-adic syntomic regulators*, in the spirit of Coleman-de Shalit, Besser.

Hida's p-adic Rankin L-series

To define $L_p(f \otimes g, s)$, the obvious approach is to interpolate the values

$$L(f \otimes g, \chi, j), \qquad \chi \text{ a Dirichlet character}, \quad j \in \mathbb{Z}.$$

Difficulty: none of these (χ, j) are critical in the sense of Deligne.

Hida's solution: "enlarge" the domain of definition of $L_p(f, g, s)$ by allowing f and g to vary in *p-adic families*.

Hida families

Iwasawa algebra: $\Lambda = \mathbb{Z}_p[[1 + p\mathbb{Z}_p]] \simeq \mathbb{Z}_p[[T]]$:

Weight space: $\Omega := \hom(\Lambda, \mathbb{C}_p) \subset \hom((1 + p\mathbb{Z}_p)^\times, \mathbb{C}_p^\times)$.

The integers form a dense subset of Ω via $k \leftrightarrow (x \mapsto x^k)$.

Classical weights: $\Omega_{\mathrm{cl}} := \mathbb{Z}^{\geq 2} \subset \Omega$.

If $\tilde{\Lambda}$ is a finite flat extension of Λ, let $\tilde{\mathcal{X}} = \hom(\tilde{\Lambda}, \mathbb{C}_p)$ and let

$$\kappa : \tilde{\mathcal{X}} \longrightarrow \Omega$$

be the natural projection to weight space.

Classical points: $\tilde{\mathcal{X}}_{\mathrm{cl}} := \{x \in \tilde{\mathcal{X}} \text{ such that } \kappa(x) \in \Omega_{\mathrm{cl}}\}$.

Hida families, cont'd

Definition

A *Hida family* of tame level N is a triple $(\Lambda_f, \Omega_f, \underline{f})$, where

1. Λ_f is a finite flat extension of Λ;
2. $\Omega_f \subset \mathcal{X}_f := \hom(\Lambda_f, \mathbb{C}_p)$ is a non-empty open subset (for the p-adic topology);
3. $\underline{f} = \sum_n \mathbf{a}_n q^n \in \Lambda_f[[q]]$ is a formal q-series, such that $\underline{f}(x) := \sum_n x(\mathbf{a}_n) q^n$ is the q series of the *ordinary p-stabilisation* $f_x^{(p)}$ of a normalised eigenform, denoted f_x, of weight $\kappa(x)$ on $\Gamma_1(N)$, for all $x \in \Omega_{f,\mathrm{cl}} := \Omega_f \cap \mathcal{X}_{f,\mathrm{cl}}$.

Hida's theorem

$f =$ normalised eigenform of weight $k \geq 1$ on $\Gamma_1(N)$.

$p \nmid N$ an ordinary prime for f (i.e., $a_p(f)$ is a p-adic unit).

Theorem (Hida)

There exists a Hida family $(\Lambda_f, \Omega_f, \underline{f})$ and a classical point $x_0 \in \Omega_{f,\mathrm{cl}}$ satisfying

$$\kappa(x_0) = k, \qquad f_{x_0} = f.$$

As x varies over $\Omega_{f,\mathrm{cl}}$, the specialisations f_x give rise to a "p-adically coherent" collection of classical newforms on $\Gamma_1(N)$, and one can hope to construct p-adic L-functions by interpolating classical special values attached to these eigenforms.

Hida's p-adic Rankin L-functions

They should interpolate *critical values* of the form

$$\frac{L(f_x \otimes g_y, j)}{\Omega(f_x, g_y, j)} \in \bar{\mathbb{Q}}, \qquad (x, y, j) \in \Omega_{f,\mathrm{cl}} \times \Omega_{g,\mathrm{cl}} \times \mathbb{Z}.$$

Proposition

The special value $L(f_x \otimes g_y, j)$ is critical if and only if either:

- $\kappa(y) \leq j \leq \kappa(x) - 1$; *then* $\Omega(f_x, g_y, j) = *\langle f_x, f_x \rangle$.
- $\kappa(x) \leq j \leq \kappa(y) - 1$; *then* $\Omega(f_x, g_y, j) = *\langle g_y, g_y \rangle$.

Let $\Sigma_f, \Sigma_g \subset \Omega_f \times \Omega_g \times \Omega$ be the two sets of critical points.

Note that they are both dense in the p-adic domain.

Hida's p-adic Rankin L-functions

Theorem (Hida)

*There are two (*a priori quite distinct*) p-adic L-functions,*

$$L_p^f(\underline{f} \otimes \underline{g}), \quad L_p^g(\underline{f} \otimes \underline{g}) : \quad \Omega_f \times \Omega_g \times \Omega \longrightarrow \mathbb{C}_p,$$

interpolating the algebraic parts of $L(f_x \otimes g_y, j)$ for (x, y, j) belonging to Σ_f and Σ_g respectively.

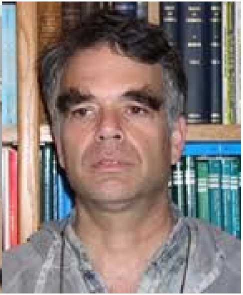
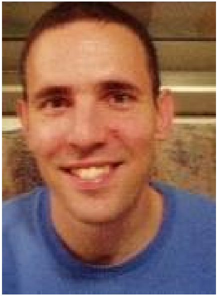
p-adic regulators

$$\begin{array}{ccc} \mathrm{CH}^2(S/\mathbb{Z},1) & \xrightarrow{\mathrm{reg}_{\mathrm{et}}} & H^1_f(\mathbb{Q}, H^2_{\mathrm{et}}(\bar{S},\mathbb{Q}_p)(2)) \\ \downarrow & & \downarrow \\ \mathrm{CH}^2(S/\mathbb{Z}_p,1) & \xrightarrow[\mathrm{reg}_{\mathrm{et}}]{} & H^1_f(\mathbb{Q}_p, H^2_{\mathrm{et}}(\bar{S},\mathbb{Q}_p)(2)) \\ & & \Vert \ \log_p \\ & & \mathrm{Fil}^1\, H^2_{\mathrm{dR}}(S/\mathbb{Q}_p)^\vee \end{array}$$

The dotted arrow is called the *p-adic* regulator and denoted reg_p.

Syntomic regulators

Coleman-de Shalit, Besser: A direct, p-adic analytic description of the p-adic regulator in terms of Coleman's theory of p-adic integration.

The p-adic Beilinson formula: the set-up

$\underline{f}$ = Hida family of tame level N specialising to the weight two cusp form $f \in S_2(\Gamma_0(N), \chi_f)$ at $x_0 \in \Omega_f$.

$\underline{g}$ = Hida family of tame level N specialising to the weight two cusp form $g \in S_2(\Gamma_0(N), \chi_g)$ at $y_0 \in \Omega_g$.

$\chi = (\chi_f \chi_g)^{-1}$.

η_f^{ur} = unique class in $H^1_{\mathrm{dR}}(X_0(N)/\mathbb{C}_p)^f$ which is in the *unit root subspace* for Frobenius and satisfies $\langle \omega_f, \eta_f^{\mathrm{ur}} \rangle = 1$.

The p-adic Beilinson formula

Theorem (Bertolini, Rotger, D)

$$L_p^f(\underline{f}, \underline{g})(x_0, y_0, 2) = \frac{\mathcal{E}(f, g, 2)}{\mathcal{E}(f)\mathcal{E}^*(f)} \times \mathrm{reg}_p(\Delta_{u_\chi})(\eta_f^{\mathrm{ur}} \wedge \omega_g),$$

$$L_p^g(\underline{f}, \underline{g})(x_0, y_0, 2) = \frac{\mathcal{E}(g, f, 2)}{\mathcal{E}(g)\mathcal{E}^*(g)} \times \mathrm{reg}_p(\Delta_{u_\chi})(\omega_f \wedge \eta_g^{\mathrm{ur}}),$$

where

$$\mathcal{E}(f, g, 2) = \begin{array}{l} (1 - \beta_p(f)\alpha_p(g)p^{-2}(1 - \beta_p(f)\beta_p(g)p^{-2}) \\ \times(1 - \beta_p(f)\alpha_p(g)\chi(p)p^{-1})(1 - \beta_p(f)\beta_p(g)\chi(p)p^{-1}) \end{array}$$

$$\mathcal{E}(f) = 1 - \beta_p(f)^2\chi_f^{-1}(p)p^{-2}, \qquad \mathcal{E}^*(f) = 1 - \beta_p(f)^2\chi_f^{-1}(p)p^{-1}.$$

Arithmetic applications: Dasgupta's formula

In his work on the $\mathcal{L}$-invariant for the symmetric square, Dasgupta is led to study $L_p^{\mathrm{Hida}}(\underline{f}, \underline{f})$ when $\underline{f} = \underline{g}$, and its restriction $L_p^{\mathrm{Hida}}(\underline{f}, \underline{f})(x, x, j)$ to the diagonal in $\Omega_f \times \Omega_f$.

This restriction has *no critical values*.

The "Artin formalism" for p-adic L-functions suggests that it should factor into a product of

1. the Coates-Schmidt p-adic L-function $L_p^{\mathrm{CS}}(\mathrm{Sym}^2(\underline{f}))(x, j)$, which does have critical points;
2. the Kubota-Leopoldt p-adic L-function $L_p^{\mathrm{KL}}(\chi_f, j + 1 - \kappa(x))$.

Dasgupta's formula

Theorem (Dasgupta)

$$L_p^{\mathrm{Hida}}(\underline{f}, \underline{f})(x, x, j) = L_p^{\mathrm{CS}}(\mathrm{Sym}^2(\underline{f}))(x, j) \times L_p^{\mathrm{KL}}(\chi_f, j + 1 - \kappa(x)).$$

Theorem (Gross)

Let χ be an even Dirichlet character, K an imaginary quadratic field in which p splits.

$$L_p^{\mathrm{Katz}}(\chi_{|K}, s) = L_p^{\mathrm{KL}}(\chi \epsilon_K \omega, s) L_p^{\mathrm{KL}}(\chi^{-1}, 1 - s).$$

The role of elliptic units in Gross' proof is played by Beilinson-Flach elements (and associated units) in Dasgupta's argument.

For more, see Samit's lecture tomorrow!

Euler systems of "Garrett-Rankin-Selberg type"

There is a strong parallel between:

1. Beilinson-Kato elements in $\mathrm{CH}^2(X_1(N), 2)$, or in $K_2(X_1(N)) \otimes \mathbb{Q}$, formed from pairs of modular units;
2. Beilinson-Flach elements in $\mathrm{CH}^2(X_1(N)^2, 1)$, or in $K_1(X_1(N) \times X_1(N)) \otimes \mathbb{Q}$, formed from modular units supported on the diagonal;
3. Gross-Kudla Schoen diagonal cycles in $\mathrm{CH}^2(X_1(N)^3)_0$ formed from the principal diagonal in the triple product of modular curves.

The first two can be viewed as "degenerate cases" of the last.

p-adic formulae

1. (Kato-Brunault-Gealy, M. Niklas, Bertolini-D):

$$L_p^{\mathrm{MS}}(f, \chi_1, 2) L_p^{\mathrm{MS}}(f, \chi_2, 1) \leftrightarrow \mathrm{reg}_p\{u_{\chi_1}, u_{\chi_1,\chi_2}\}(\eta_f^{\mathrm{ur}});$$

L_p^{MS} = Mazur-Swinnerton-Dyer L-function.

2. (Bertolini-Rotger-D)

$$L_p^{f,\mathrm{Hida}}(f \otimes g, 2) \leftrightarrow \mathrm{reg}_p(\Delta_\chi)(\eta_f^{\mathrm{ur}} \wedge \omega_g);$$

$L_p^{f,Hida}$ = Hida's Rankin p-adic L-function;

3. (Rotger-D)

$$L_p^{f,\mathrm{HT}}(\underline{f} \otimes \underline{g} \otimes \underline{h}, 2) \leftrightarrow \mathrm{AJ}_p(\Delta_{GKS})(\eta_f^{\mathrm{ur}} \wedge \omega_g \wedge \omega_h).$$

$L_p^{f,\mathrm{HT}}$ = Harris-Tilouine's triple product p-adic L-function.

Complex formulae

All of the formulae of the previous slide admit complex analogues:

- The first two are due to Beilinson;
- The last, which relates *heights* of diagonal cycles to central critical derivatives of Garrett-Rankin triple product L-series, is due to Gross-Kudla and Wei-Zhang-Zhang. (But here the analogy is less immediate.)

On the importance of p-adic formulae

p-adic formulae enjoy the following advantages over their complex analogues:

1. the p-adic regulators and Abel-Jacobi maps factor through their counterparts in p-adic étale cohomology, which yield arithmetically interesting *global cohomology classes* with p-adic coefficients.
2. The p-adic formulae can be subjected to *variation in p-adic families*, yielding global classes with values in p-adic representations for which the geometric construction ceases to be available.

Beilinson-Kato classes

Beilinson elements: $\{u_\chi, u_{\chi_1,\chi_2}\} \in K_2(X_1(N))(\mathbb{Q}_{\chi_1}) \otimes F$,

$$\mathrm{dlog}\, u_\chi = E_2(1,\chi), \qquad \mathrm{dlog}\, u_{\chi_1,\chi_2} = E_2(\chi_1,\chi_2).$$

étale regulator:

$$\begin{aligned}\mathrm{reg}_{\mathrm{et}} : K_2(X_1(N))(\mathbb{Q}_{\chi_1}) &\longrightarrow H^2_{\mathrm{et}}(X_1(N)_{\mathbb{Q}_{\chi_1}}, \mathbb{Q}_p(2)) \\ &\longrightarrow H^1(\mathbb{Q}_{\chi_1}, H^1_{\mathrm{et}}(\overline{X_1(N)}, \mathbb{Q}_p(2))).\end{aligned}$$

Beilinson-Kato class:

$$\begin{aligned}\kappa(f, E_2(1,\chi), E_2(\chi_1,\chi_2)) \quad := \quad & \mathrm{reg}_{\mathrm{et}}(\{u_\chi, u_{\chi_1,\chi_2}\})^f \in H^1(\mathbb{Q}_{\chi_1}, V_f(2)) \\ & \overset{\mathrm{res}}{\leftarrow} H^1(\mathbb{Q}, V_f(2)(\chi_1^{-1}))).\end{aligned}$$

Beilinson-Flach classes

étale regulator:

$$\begin{aligned}\mathrm{reg}_{\mathrm{et}}: K_1(X_1(N)^2) &\longrightarrow H^3_{\mathrm{et}}(X_1(N)^2, \mathbb{Q}_p(2)) \\ &\longrightarrow H^1(\mathbb{Q}, H^2_{\mathrm{et}}(\overline{X_1(N)}^2, \mathbb{Q}_p(2))). \\ &\longrightarrow H^1(\mathbb{Q}, H^1_{\mathrm{et}}(\overline{X_1(N)}, \mathbb{Q}_p)^{\otimes 2}(2))\end{aligned}$$

Beilinson-Flach class:

$$\kappa(f, g, E_2(\chi)) := \mathrm{reg}_{\mathrm{et}}(\Delta_\chi)^{f,g} \in H^1(\mathbb{Q}, V_f \otimes V_g(2)).$$

Gross-Kudla-Schoen diagonal classes

étale Abel-Jacobi map:

$$\begin{aligned} \mathrm{AJ}_{\mathrm{et}} : \mathrm{CH}^2(X_1(N)^3)_0 &\longrightarrow H^4_{\mathrm{et}}(X_1(N)^3, \mathbb{Q}_p(2))_0 \\ &\longrightarrow H^1(\mathbb{Q}, H^3_{\mathrm{et}}(\overline{X_1(N)}^3, \mathbb{Q}_p(2))) \\ &\longrightarrow H^1(\mathbb{Q}, H^1_{\mathrm{et}}(\overline{X_1(N)}, \mathbb{Q}_p)^{\otimes 3}(2)) \end{aligned}$$

Gross-Kudla Schoen class:

$$\kappa(f, g, h) := \mathrm{AJ}_{\mathrm{et}}(\Delta)^{f,g,h} \in H^1(\mathbb{Q}, V_f \otimes V_g \otimes V_h(2)).$$

A p-adic family of global classes

Theorem (Rotger-D)

Let $\underline{f}$, $\underline{g}$, $\underline{h}$ be three Hida families. There is a Λ-adic cohomology class

$$\kappa(f, \underline{g}, \underline{h}) \in H^1(\mathbb{Q}, V_f \otimes (\underline{V}_g \otimes_\Lambda \underline{V}_h)_1),$$

where $\underline{V}_g, \underline{V}_h$ = Hida's Λ-adic representations attached to $\underline{f}$ and $\underline{g}$, satisfying, for all "weight two" points $(y, z) \in \Omega_g \times \Omega_h$,

$$\log_p \kappa(f, g_y, h_z)(\eta_f^{\mathrm{ur}} \wedge \omega_{g_y} \wedge \omega_{h_z}) \leftrightarrow L_p^{f,\mathrm{HT}}(f, \underline{g}, \underline{h})(y, z, 2).$$

This Λ-adic class generalises Kato's class, which one recovers when $\underline{g}$ and $\underline{h}$ are families of Eisenstein series.

Kato's reciprocity law

Kato's idea: Specialise the Λ-adic cohomology class $\kappa(f, \underline{E}(\chi), \underline{E}(\chi_1, \chi_2))$ to Eisenstein series *of weight one*.

$$\kappa_{\mathrm{kato}}(f, \chi_1, \chi_2) := \kappa(f, E_1(1, \chi), E_1(\chi_1, \chi_2)).$$

Theorem (Kato)

The class $\kappa_{\mathrm{Kato}}(f, \chi_1, \chi_2)$ is cristalline if and only if $L(f, \chi_1, 1)L(f, \chi_2, 1) = 0$.

Corollary

Let E be an elliptic curve over $\mathbb{Q}$ and χ a Dirichlet character. If $L(E, \chi_1, 1) \neq 0$, then $\hom(\mathbb{C}(\chi), E(\bar{\mathbb{Q}}) \otimes \mathbb{C}) = 0$.

Reciprocity law for diagonal cycles

One can likewise consider the specialisations of $\kappa(f, \underline{g}, \underline{h})$ when $\underline{g}$ and $\underline{h}$ are evaluated at points of *weight one*.

Theorem (Rotger-D)

Let $(y, z) \in \Omega_g \times \Omega_h$ be points with $\mathrm{wt}(y) = \mathrm{wt}(z) = 1$. The class $\kappa(f, g_y, h_z)$ is cristalline if and only if $L(f \otimes g_y \otimes h_z, 1) = 0$.

Corollary

Let E be an elliptic curve over $\mathbb{Q}$ and ρ_1, ρ_2 odd irreducible two-dimensional Galois representations. If $L(E, \rho_1 \otimes \rho_2, 1) \neq 0$, then $\hom(\rho_1 \otimes \rho_2, E(\bar{\mathbb{Q}}) \otimes \mathbb{C}) = 0$.

Reciprocity laws for Beilinson-Flach elements

When $\underline{g}$ is cuspidal and only $\underline{h}$ is a family of Eisenstein series, the class $\kappa(f, \underline{g}, \underline{E})$ constructed from families of Beilinson Flach elements should satisfy similar reciprocity laws (details are still to be worked out).

BSD application (Bertolini, Rotger, in progress):

$$L(E, \rho, 1) \neq 0 \Rightarrow \mathrm{hom}(\rho, E(\bar{\mathbb{Q}}) \otimes \mathbb{C}) = 0.$$

The work of Loeffler-Zerbes

In their article

"Iwasawa Theory and p-adic L-functions over $\mathbb{Z}_p^2$-extensions",

David Loeffler and Sarah Zerbes construct a generalisation of Perrin-Riou's "big dual exponential map" for the two-variable $\mathbb{Z}_p$-extension of an imaginary quadratic field K:

$$\mathrm{Log}_{V,K} : H^1_{\mathrm{Iw}}(K, V) := (\varprojlim H^1(K_n, T))_{\mathbb{Q}_p} \longrightarrow \mathbb{D}_{\mathrm{cris}}(V) \otimes \tilde{\Lambda}_K .$$

They then conjecture, following Perrin-Riou, a construction of the two-variable p-adic L-function attached to V/K as the image under $\mathrm{Log}_{V,K}$ of a suitable *norm-compatible system* of global classes.

The work of Lei-Loeffler-Zerbes

Goal: Construct this conjectured global class using the Beilinson-Flach family $\kappa(f, \underline{g}, \underline{E})$, when $\underline{g}$ is a family of *theta-series* attached to K.



A rough classification of Euler systems

The Euler systems that have been most studied so far fall into two broad categories:

1. The Euler system of *Heegner points*, and its "degenerate cases", elliptic units and circular units. (Cf. work with Bertolini, Prasanna, and in Francesc Castella's ongoing PhD thesis.) Cycles on $U(2) \times U(1)$.

2. Euler systems of Garrett-Rankin-Selberg type: *diagonal cycles* and the "degenerate settings" of the Beilinson-Flach and Beilinson-Kato elements. Cycles on $SO(4) \times SO(3)$.

3. Other settings? p-adic families of cycles on $U(n) \times U(n-1)$?

Le mot de la fin

In further developments of the theory of Euler systems, the notion of p-adic deformations of automorphic forms and their associated Galois representations pioneered by Hida is clearly destined to play a central role.


Happy Birthday!