

# Real quadratic Borcherds products

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*To Benedict Gross, on his 70th birthday*

**Abstract:** *Rigid meromorphic cocycles* were introduced in [DV21] to formulate a notion of singular moduli for real quadratic fields. The present work further develops their foundations and fleshes out their analogy with meromorphic modular functions with CM divisor by describing a real quadratic analogue of the Borcherds lift mapping certain weakly holomorphic modular forms of weight  $1/2$  to the group of rigid meromorphic cocycles with rational RM divisor.

1	Introduction	1803
2	Generalities on theta cocycles and theta symbols	1808
3	Analytic theta-cocycles	1815
4	Meromorphic theta-symbols	1837
5	Lifting obstructions	1844
6	Generating series	1856
A	The parallel with Mumford’s $p$ -adic theta functions	1860
	References	1863

## 1. Introduction

Let  $p$  be a rational prime and let  $\mathcal{H}_p$  denote Drinfeld’s  $p$ -adic upper half plane, a rigid analytic space whose  $\mathbb{C}_p$ -points are identified with  $\mathbb{C}_p - \mathbb{Q}_p$ . The *Ihara group*  $\Gamma := \mathrm{SL}_2(\mathbb{Z}[1/p])$  acts by Möbius transformations on  $\mathcal{H}_p$  (with non-discrete orbits), and, by translation, on the multiplicative group

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$\mathcal{M}^\times$  of non-zero rigid meromorphic functions on  $\mathcal{H}_p$ . This action preserves the subset  $\mathcal{H}_p^{\text{RM}} \subset \mathcal{H}_p$  of *real multiplication*, or RM points, namely, points of  $\mathcal{H}_p$  that lie in a real quadratic field. A henceforth fixed choice of embeddings of  $\overline{\mathbb{Q}}$  into both  $\mathbb{C}_p$  and  $\mathbb{C}$  gives an inclusion  $\mathcal{H}_p^{\text{RM}} \hookrightarrow \mathbb{R}$  and allows one to view RM points as real numbers, and to make sense of their sign.

A *rigid meromorphic cocycle* is an element of the first cohomology group  $H^1(\Gamma, \mathcal{M}^\times)$ . Such a class is said to be *parabolic* if it admits a (necessarily unique, up to torsion) representative cocycle  $J$  whose restriction to the group  $\Gamma_\infty$  of upper triangular matrices is  $\mathbb{C}_p^\times$ -valued. A parabolic rigid meromorphic cocycle is often conflated with this distinguished representative. In this setting, Theorem 1 of [DV21] asserts that the meromorphic function  $J(S)$ , where  $S \in \Gamma$  is the standard matrix of order 4, can be written as a finite product

$$(1) \quad J(S)(z) = \prod_{\tau \in \Gamma \backslash \mathcal{H}_p^{\text{RM}}} \alpha_\tau(z)^{n_\tau}, \quad n_\tau \in \mathbb{Z},$$

where  $\alpha_\tau(z)$  is a rigid meromorphic function whose zeroes (resp. poles) are concentrated at the negative norm elements in the  $\Gamma$ -orbit of  $\tau$  which are positive (resp. negative):

$$(2) \quad \text{Divisor}(\alpha_\tau) = \sum_{\substack{w \in \Gamma\tau, \\ ww' < 0}} \text{sign}(w) \cdot (w) = \sum_{w \in \Gamma\tau} ((w, w') \cdot (0, \infty)) \cdot (w),$$

where  $((w, w') \cdot (0, \infty))$  denotes the topological intersection pairing between the geodesic  $(w, w')$  joining  $w$  and  $w'$  and the imaginary axis  $(0, \infty) \subset \mathcal{H}_p$ , after choosing an orientation on  $\mathcal{H}$ . While  $\text{Divisor}(\alpha_\tau)$  is an infinite sum of points on  $\mathcal{H}_p$ , its intersection with any affinoid subset of  $\mathcal{H}_p$  is a finite divisor.

The *divisor* of the cocycle  $J$  satisfying (1) is then defined to be

$$(3) \quad \text{Divisor}(J) := \sum_{\tau \in \Gamma \backslash \mathcal{H}_p^{\text{RM}}} n_\tau [\tau] \in \text{Div}(\Gamma \backslash \mathcal{H}_p^{\text{RM}}),$$

where  $\text{Div}(X)$  denotes the group of finite formal integer linear combinations of elements of a set  $X$ . Rigid meromorphic cocycles can be envisaged as real quadratic analogues of meromorphic functions on a modular or Shimura curve whose divisors are concentrated on CM points, such as those arising in the image of Borchers' singular theta lift. One of the goals of this note is to bring this analogy into sharper focus by showing that the principal parts of certain weakly holomorphic modular forms of weight  $1/2$  on  $\Gamma_0(4p)$  encode divisors of rigid meromorphic cocycles.

### 1.1. The main theorem

The *discriminant* of  $\tau \in \mathcal{H}_p^{\text{RM}}$  is the discriminant of the unique primitive integral binary quadratic form of which  $\tau$  is a root. The set  $\mathcal{H}_p^D$  of RM points of discriminant  $D$  is non-empty if and only if  $p$  is inert or ramified in the real quadratic field  $\mathbb{Q}(\sqrt{D})$ . One then has

$$\left(\frac{D}{p}\right) \neq 1, \quad D \equiv 0, 1 \pmod{4}.$$

The set  $\mathcal{H}_p^D$  is stable under the action of  $\text{SL}_2(\mathbb{Z}) \subset \Gamma$  by Möbius transformations, and the quotient  $\text{SL}_2(\mathbb{Z}) \backslash \mathcal{H}_p^D$  is naturally identified with the narrow ideal class group of the order of discriminant  $D$ , with composition given by the classical Gauss composition of binary quadratic forms.

Let  $-d_1$  and  $-d_2$  be a pair of negative discriminants satisfying the following conditions:

1.  $-d_1$  is fundamental, and prime to  $p$ ;
2.  $\text{ord}_p(-d_2) \leq 1$ ;
3.  $\left(\frac{d_1 d_2}{p}\right) \neq 1$ .

Under these assumptions,  $D := d_1 d_2$  is the discriminant of a real quadratic order which is maximal at  $p$ , and  $\mathcal{H}_p^D$  is non-empty. As in [Ko85, p. 238], the pair  $(-d_1, -d_2)$  gives rise to a character  $\omega_{-d_1}$  of the narrow class group  $\text{Cl}^+(D)$  of discriminant  $D$  sending the equivalence class of  $Q = ax^2 + bxy + cy^2$  to

$$\omega_{-d_1}(Q) = \begin{cases} 0 & \text{if } \gcd(a, b, c, d_1) > 1; \\ \left(\frac{-d_1}{r}\right) & \text{if } \gcd(a, b, c, d_1) = 1, \end{cases} \quad \begin{array}{l} \text{with } Q \text{ representing } r, \\ \text{and } \gcd(r, d_1) = 1. \end{array}$$

We associate to the pair  $(-d_1, -d_2)$  of negative discriminants satisfying conditions (1), (2), and (3) above the RM divisors

$$\begin{aligned} \mathbb{D}_{-d_1, -d_2} &:= \sum_{Q \in \text{Cl}^+(D)} \omega_{-d_1}(Q) \cdot [\tau_Q], \\ (4) \quad \mathbb{D}_{-d_1, -d_2}^+ &:= \sum_{Q \in \text{Cl}^+(D)} \omega_{-d_1}(Q) \cdot ([\tau_Q] + [p\tau_Q]), \\ \mathbb{D}_{-d_1, -d_2}^- &:= \sum_{Q \in \text{Cl}^+(D)} \omega_{-d_1}(Q) \cdot ([\tau_Q] - [p\tau_Q]), \end{aligned}$$

where, for  $Q(x, y) = ax^2 + bxy + cy^2$ ,

$$\tau_Q := \frac{-b + \sqrt{D}}{2a},$$

and  $[\tau_Q]$  denotes the class of  $\tau_Q$  in  $\Gamma \backslash \mathcal{H}_p$ . These divisors do not depend on the choice of  $p$ -adic embedding of  $\bar{\mathbb{Q}}$  into  $\bar{\mathbb{Q}}_p$  that was made to define them; i.e., replacing  $\sqrt{D}$  by  $-\sqrt{D}$  leaves  $\mathbb{D}_{-d_1, -d_2}$  unchanged. This follows from the fact that

$$\omega_{-d_1}(-Q) = -\omega_{-d_1}(Q), \quad \tau_{-Q} = \tau'_Q := \frac{-b - \sqrt{D}}{2a},$$

and that interchanging  $\tau_Q$  and  $\tau'_Q$  reverses the orientation of the modular geodesics  $(w, w')$  appearing in (2). Definition (4) is extended to arbitrary discriminants  $-d_1$  by setting

$$(5) \quad \mathbb{D}_{-d_1, -d_2}^+ := \mathbb{D}_{-d_1, -d_2}^+, \quad \mathbb{D}_{-d_1, -d_2}^- := (-1)^m \cdot \mathbb{D}_{-d_1, -d_2}^-, \quad \text{for all } m \geq 1.$$

For any integer  $N \geq 1$  and a weight  $k \in \frac{1}{2}\mathbb{Z}$ , let

$$S_k(N) \subset M_k(N) \subset M_k^!(N)$$

denote the usual spaces of weight  $k$  cusp forms, modular forms, and weakly holomorphic modular forms on  $\Gamma_0(N)$ . It will also be convenient to introduce the subspace  $M_k^{!!}(N) \subset M_k^!(N)$  consisting of the weakly holomorphic modular forms that are regular at all the cusps except  $\infty$ . Any form  $\phi \in M_k^!(N)$  admits a Fourier expansion at the cusp  $\infty$  of the form

$$\phi(q) = \sum_{n \gg -\infty} c_\phi(n) q^n, \quad q := e^{2\pi iz}.$$

The space  $M_k^{!!}(N)$  is an infinite-dimensional complex vector space and admits a basis consisting of modular forms with integer Fourier coefficients.

The main result of this paper is

**Main Theorem.** *Let  $\phi := \sum_{n \gg -\infty} c_\phi(n) q^n \in M_{1/2}^{!!}(4p)$  be a weakly holomorphic modular form with integer Fourier coefficients.*

1. *If  $\left(\frac{-d_1}{p}\right) = 1$ , there exists a rigid meromorphic cocycle  $J_{-d_1, \phi} \in H^1(\Gamma, \mathcal{M}^\times) \otimes \mathbb{Q}$  with*

$$\text{Divisor}(J_{-d_1, \phi}) = \sum_{d_2 \equiv 0, 3 \pmod{4}} c_\phi(-d_2) \cdot \mathbb{D}_{-d_1, -d_2}^+,$$

- where the sum runs over the positive integers  $d_2$  satisfying  $\left(\frac{-d_2}{p}\right) \neq 1$ .
2. If  $\left(\frac{-d_1}{p}\right) = -1$ , there exists a rigid meromorphic cocycle  $J_{-d_1, \phi} \in H^1(\Gamma, \mathcal{M}^\times) \otimes \mathbb{Q}$  with

$$\text{Divisor}(J_{-d_1, \phi}) = \sum_{d_2 \equiv 0, 3 \pmod{4}} c_\phi(-d_2) \cdot \mathbb{D}_{-d_1, -d_2}^-$$

where the sum runs over the positive integers  $d_2$  satisfying  $\left(\frac{-d_2}{p}\right) \neq -1$ .

The rigid meromorphic cocycle  $J_{-d_1, \phi}$  is called a *real quadratic Borcherds lift* attached to the discriminant  $-d_1$  and to the weakly holomorphic modular form  $\phi$ .

### 1.2. Outline

The proof of the main theorem relies on the classification of rigid meromorphic cocycles of [DV21, Theorem 1.23], which rests on the study of the group  $H^1(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times)$  of rigid meromorphic cocycles *modulo scalars*. These objects are called *theta-cocycles* because of their analogy with the  $p$ -adic theta functions arising in the uniformisation theory of Jacobians of Mumford–Schottky curves, discussed in the Appendix.

In §2 general rigid theta-cocycles and their RM values are discussed. The subgroup of analytic cocycles is determined in §3, and it is intimately connected to classical weight two modular forms on  $\Gamma_0(p)$  by the *Schneider–Teitelbaum lift*. It is shown in §4 that the group of rigid *meromorphic* theta-cocycles is of infinite rank; more precisely, that any RM divisor arises as the divisor of a (not necessarily unique) meromorphic theta-cocycle. No essential information is lost in passing from a rigid meromorphic cocycle to its associated theta-cocycle, because the natural map

$$H^1(\Gamma, \mathcal{M}^\times) \longrightarrow H^1(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times)$$

has finite kernel (of exponent dividing 12). However, this map fails to be surjective in general. We analyse the *obstruction* to lifting a rigid meromorphic theta-cocycle with given RM divisor to an actual meromorphic cocycle in §5, and relate it to the *Stark–Heegner point* attached to that divisor. Such a relation suggests that these lifting obstructions can be packaged into modular generating series of weight  $3/2$  for  $\Gamma_0(4p)$ , which is done in §6 by building on the Gross–Kohnen–Zagier theorem for Stark–Heegner points proved in [DT08]. The main theorem then follows from a simple application of Serre duality.

## 2. Generalities on theta cocycles and theta symbols

### 2.1. Rational functions on $\mathcal{H}_p$

Let  $\mathbb{C}_p$  be a completion of an algebraic closure of  $\mathbb{Q}_p$ . Although it is a natural ground field for the theory of rigid analytic spaces, most of our calculations (and all of the interesting ones) take place over the compositum  $C_p$  of all the quadratic extensions of  $\mathbb{Q}_p$ , which is biquadratic when  $p \neq 2$ , and may be viewed as a suitable  $p$ -adic analogue of  $\mathbb{C}$  in the setting of this paper.

Every  $z \in \mathbb{P}_1(\mathbb{C}_p)$  can be represented in projective coordinates by a pair  $[z_1 : z_2]$  where  $(z_1, z_2)$  belongs to the set  $(\mathcal{O}_{\mathbb{C}_p} \times \mathcal{O}_{\mathbb{C}_p})'$  of *primitive vectors* having at least one coordinate in  $\mathcal{O}_{\mathbb{C}_p}^\times$ . If  $(a, b) \in (\mathbb{Z}^2)'$  is a primitive vector, satisfying  $\gcd(a, b) = 1$ , then  $\text{ord}_p(z_1 b - z_2 a) \in \mathbb{Q} \cup \{\infty\}$  depends only on  $z$  and not on its representation. It belongs to  $\mathbb{Q}$  when  $z \notin \mathbb{P}_1(\mathbb{Q}_p)$ , and is a half integer when  $z \in \mathbb{P}_1(C_p) - \mathbb{P}_1(\mathbb{Q}_p)$ .

Let  $\mathcal{T}$  be the Bruhat–Tits tree of  $\text{PGL}_2(\mathbb{Q}_p)$ , whose set  $\mathcal{V}(\mathcal{T})$  of vertices is in bijection with the homothety classes of  $\mathbb{Z}_p$ -lattices in  $\mathbb{Q}_p^2$ , two vertices being joined by an (unordered) edge if they admit representative lattices containing each other with index  $p$ . Write  $\mathcal{E}(\mathcal{T})$  for the set of ordered edges of  $\mathcal{T}$ , i.e., ordered pairs of adjacent vertices of  $\mathcal{T}$ , and  $\mathcal{E}_*(\mathcal{T})$  for the set of unordered edges. The action of  $\Gamma$  on  $\mathcal{E}(\mathcal{T})$  partitions this set into two orbits

$$(6) \quad \mathcal{E}(\mathcal{T}) = \mathcal{E}^+(\mathcal{T}) \sqcup \mathcal{E}^-(\mathcal{T}),$$

in such a way that  $\mathcal{E}^+(\mathcal{T})$  contains an edge having  $v_0$  as its source. The edges in  $\mathcal{E}^+(\mathcal{T})$  are referred to as *positively oriented*, and those in  $\mathcal{E}^-(\mathcal{T})$ , as *negatively oriented*. We identify  $\mathcal{T}$  with its combinatorial realisation

$$\mathcal{T} := \mathcal{V}(\mathcal{T}) \cup \mathcal{E}_*(\mathcal{T}).$$

The group  $\text{GL}_2(\mathbb{Q}_p)$  acts on  $\mathcal{T}$  via its left multiplication action on  $\mathbb{Q}_p^2$  viewed as column vectors. The subgroup  $\Gamma$  acts transitively on  $\mathcal{E}_*(\mathcal{T})$  and breaks up  $\mathcal{V}(\mathcal{T})$  into two distinct orbits, in such a way that adjacent vertices lie in distinct  $\Gamma$ -orbits. The vertex  $v_0$  attached to the standard lattice  $\mathbb{Z}_p^2 \subset \mathbb{Q}_p^2$  is called the *standard vertex*. Its stabiliser  $\Gamma_{v_0}$  in  $\Gamma$  is equal to  $\text{SL}_2(\mathbb{Z})$ . More generally, the vertex and edge stabilisers in  $\Gamma$  are  $\text{GL}_2(\mathbb{Z}[1/p])$ -conjugate to  $\text{SL}_2(\mathbb{Z})$  and to the Hecke congruence group  $\Gamma_0(p)$ , respectively. This leads to a description of  $\Gamma$  as an amalgamated product

$$\Gamma = \text{SL}_2(\mathbb{Z}) *_{\Gamma_0(p)} \text{SL}_2(\mathbb{Z})',$$

which is useful in relating the cohomology of  $\Gamma$  with the cohomology of  $\mathrm{SL}_2(\mathbb{Z})$  and  $\Gamma_0(p)$ . If  $\mathcal{G}$  is any finite subgraph of  $\mathcal{T}$ , the stabiliser of  $\mathcal{G}$  in  $\Gamma$  is denoted  $\Gamma_{\mathcal{G}}$ . The groups  $\Gamma_{\mathcal{G}}$  are conjugate to finite index subgroups of  $\mathrm{SL}_2(\mathbb{Z})$  and act discretely on  $\mathcal{H}$ .

Let  $\mathcal{H}_p$  denote Drinfeld’s upper half-plane, a rigid analytic space whose underlying set of  $\mathbb{C}_p$ -points is identified with  $\mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$ . The group  $\mathrm{GL}_2(\mathbb{Q}_p)$  acts on  $\mathcal{H}_p$  by Möbius transformations, and there is a natural  $\mathrm{GL}_2(\mathbb{Q}_p)$ -equivariant *reduction map*

$$\mathrm{red} : \mathcal{H}_p \longrightarrow \mathcal{T}$$

which satisfies the following properties.

- The *standard affinoid region*

$$(7)$$

$$\mathcal{H}_p^\circ := \left\{ z = (z_1 : z_2) \in (\mathcal{O}_{\mathbb{C}_p}^2)' \text{ s.t. } \mathrm{ord}_p(z_1 b - z_2 a) = 0, \quad \forall (a, b) \in (\mathbb{Z}_p^2)' \right\},$$

which is the complement in  $\mathbb{P}_1(\mathbb{C}_p)$  of  $p + 1$  residue discs, maps to the standard vertex  $v_\circ$  under the reduction map.

- The *standard annulus*

$$(8)$$

$$U := \{z \in \mathcal{O}_{\mathbb{C}_p} \text{ such that } 1 < |z| < p\}$$

maps to the edge in  $\mathcal{E}(\mathcal{T})$  attached to the lattice pair  $(\mathbb{Z}_p^2 \supset \mathbb{Z}_p \oplus p\mathbb{Z}_p)$ , which is denoted  $e_\circ$  and called the *standard edge* of  $\mathcal{E}(\mathcal{T})$ . Its stabiliser  $\Gamma_{e_\circ}$  is equal to  $\Gamma_0(p)$ .

These two properties determine the reduction map, because  $\mathcal{H}_p$  is the union of the  $\mathrm{GL}_2(\mathbb{Q}_p)$ -translates of  $\mathcal{H}_p^\circ$  and  $U$ . The set  $\mathcal{H}_p^D$  of RM points of discriminant  $D$  maps to  $\mathcal{V}(\mathcal{T})$  under the reduction map if  $p \nmid D$ , and to  $\mathcal{E}_*(\mathcal{T})$  if  $p$  divides  $D$  exactly.

If  $v$  is a vertex of  $\mathcal{T}$ , let

$$(9) \quad \mathcal{A}_v := \mathrm{red}^{-1}(\{v\}), \quad \mathcal{W}_v := \mathrm{red}^{-1}(\{v, e_1, \dots, e_{p+1}\}),$$

where  $e_1, \dots, e_{p+1}$  are the edges of  $\mathcal{T}$  having  $v$  as an endpoint. The sets  $\mathcal{A}_v$  and  $\mathcal{W}_v$  are called the *standard affinoid* and *standard wide open space* attached to  $v$ , respectively. Likewise, if  $e \in \mathcal{E}_*(\mathcal{T})$  is an unordered edge of  $\mathcal{T}$ , the region

$$(10) \quad \mathcal{W}_e := \mathrm{red}^{-1}(\{e\})$$

is called the *standard annulus* attached to  $e$ . Specifying an orientation of  $e = (v_1, v_2) \in \mathcal{E}(\mathcal{T})$ , i.e., a source vertex  $v_1$  and a target vertex  $v_2$ , determines

an orientation on the associated annulus, by specifying that  $\mathcal{A}_{v_1}$  lies in the interior region bounded by  $\mathcal{W}_e$ , and  $\mathcal{A}_{v_2}$  in the exterior region.

The distance  $d(v, w)$  between two vertices  $v$  and  $w$  of  $\mathcal{T}$  is the number of edges in the shortest path joining them. This notion extends to edges by setting, for all edges  $e = (v, v')$  and all edges  $e' \neq e$ :

$$d(w, e) = d(e, w) = \frac{1}{2}(d(v, w) + d(v', w)), \quad d(e, e') = \frac{1}{2}(d(v, e') + d(v', e')).$$

With this definition, the distance between two vertices or two edges is a positive integer, while the distance between a vertex and an edge is a half of an odd integer. For each integer  $n \geq 0$ , let  $\mathcal{T}^{\leq n}$  and  $\mathcal{T}^{< n}$  denote the subgraphs of  $\mathcal{T}$  consisting of vertices and edges that are at distance  $\leq n$  and  $< n$  respectively from  $v_\circ$ , and let  $\mathcal{H}_p^{\leq n}$  and  $\mathcal{H}_p^{< n}$  denote their inverse images under the reduction map. The collection of  $\mathcal{H}_p^{\leq n}$  and  $\mathcal{H}_p^{< n}$  give coverings of  $\mathcal{H}_p$  by affinoid subsets and wide open subsets, respectively, that are preserved under the action of  $\mathrm{SL}_2(\mathbb{Z})$ . They can be described directly as

$$(11) \quad \mathcal{H}_p^{\leq n} = \{[z_1 : z_2] \in (\mathcal{O}_{\mathbb{C}_p} \times \mathcal{O}_{\mathbb{C}_p})' \text{ with } \mathrm{ord}_p(z_1 b - z_2 a) \leq n \text{ for all } (a, b) \in (\mathbb{Z}^2)'\},$$

and likewise for  $\mathcal{H}_p^{< n}$ .

A rational function  $t$  on  $\mathbb{P}_1(\mathbb{C}_p)$  can be evaluated on divisors on  $\mathbb{P}_1(\mathbb{C}_p)$  in the usual way, by setting

$$t \left( \sum_i n_i \cdot (z_i) \right) := \prod_i t(z_i)^{n_i}.$$

Given a degree zero divisor  $\mathcal{D}$  on  $\mathbb{P}_1(\mathbb{C}_p)$ , there is a unique rational function  $t_{\mathcal{D}}$  with that divisor, up to multiplication by a non-zero scalar. In particular, if  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are degree zero divisors with disjoint supports, the quantity

$$(12) \quad [\mathcal{D}_1; \mathcal{D}_2] := t_{\mathcal{D}_2}(\mathcal{D}_1) \in \mathbb{C}_p^\times$$

does not depend on the choice of  $t_{\mathcal{D}_2}$ . The quantity  $[(z_1) - (z_2); (z_3) - (z_4)]$  is just the familiar cross-ratio of  $(z_1, z_2, z_3, z_4)$ . The symbol  $[\mathcal{D}_1; \mathcal{D}_2]$  is bilinear, symmetric (by Weil reciprocity) and  $\mathrm{GL}_2(\mathbb{C}_p)$ -equivariant. The following lemma controls the  $p$ -adic valuation of  $[\mathcal{D}_1; \mathcal{D}_2]$ .

**Lemma 2.1.** *If  $\mathcal{D}_1$  is a degree zero divisor supported on  $\mathcal{H}_p^{\leq n}$ , and  $\mathcal{D}_2$  is a degree zero divisor supported on  $\mathcal{W}_v$  with  $d(v, v_\circ) = m > n$ , then  $[\mathcal{D}_1; \mathcal{D}_2]$  belongs to  $1 + p^{m-n}\mathcal{O}_{\mathbb{C}_p}$ .*



*Proof.* This is a special case of the following more general result. Let  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are *oriented* subgraphs of  $\mathcal{T}$ , so that it makes sense to speak of their boundary, a degree 0 linear combination of elements of  $\mathcal{V}(\mathcal{T})$ . If

$$\partial\mathcal{G}_1 = \text{red}(\mathcal{D}_1), \quad \partial\mathcal{G}_2 = \text{red}(\mathcal{D}_2),$$

then  $[\mathcal{D}_1; \mathcal{D}_2]$  is a  $p$ -adic unit when  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are disjoint, and it is congruent to 1 modulo  $p^m$ , where  $m$  is the *distance* between  $\mathcal{G}_1$  and  $\mathcal{G}_2$ , defined in the natural way as the minimum distance between vertices and edges of  $\mathcal{G}_1$  and of  $\mathcal{G}_2$ .  $\square$

When  $\mathcal{D}_2$  is supported on elements of  $\mathbb{P}_1(\mathbb{Q})$ , the definition of  $[\mathcal{D}_1, \mathcal{D}_2]$  extends to divisors  $\mathcal{D}_1$  of arbitrary degree, in a way that continues to be equivariant under the action of  $\text{SL}_2(\mathbb{Z})$ . To see this, assume for simplicity that  $\mathcal{D}_2 = (r) - (s)$ , and write  $r = a/b$  and  $s = c/d$  as fractions in lowest terms, adopting the usual convention that  $\infty = 1/0$ . The quantity  $[z; (r) - (s)]$  is then defined by setting

$$[z; (r) - (s)] = \frac{bz - a}{dz - c}.$$

The resulting symbol, extended by bilinearity to  $\text{Div}(\mathbb{P}_1(\mathbb{C}_p)) \times \text{Div}^0(\mathbb{P}_1(\mathbb{Q}))$ , coincides with the earlier one on degree zero divisors and satisfies the weaker equivariance properties

$$(13) \quad \begin{aligned} [\gamma z; (\gamma r) - (\gamma s)] &= [z; (r) - (s)] && \text{for all } \gamma \in \text{SL}_2(\mathbb{Z}), \\ [\gamma z; (\gamma r) - (\gamma s)] &= [z; (r) - (s)] \pmod{p^{\mathbb{Z}}} && \text{for all } \gamma \in \Gamma, \end{aligned}$$

which can be checked directly.

It shall be useful to control the  $p$ -adic valuation of the restriction of the rational function of  $[z; (r) - (s)]$  to the affinoid subsets  $\mathcal{H}_p^{\leq n}$ . For  $r = a/b$  and  $s = c/d$  as above, define

$$\det(r, s) := \pm(ad - bc).$$

**Lemma 2.2.** *Let  $0 \leq n < m$  be integers. For all  $(r, s) \in \mathbb{P}_1(\mathbb{Q})^2$  with  $\text{ord}_p(\det(r, s)) = m$ ,*

1. *the restriction of  $[z; (r) - (s)]$  to  $\mathcal{H}_p^{\leq n}$  takes values in  $v + p^{m-n}\mathcal{O}_{\mathbb{C}_p}$  for some  $v \in \mathbb{Z}_p^\times$ ;*
2. *If  $\mathcal{D}$  is a degree zero divisor supported on  $\mathcal{H}_p^{\leq n}$ , then  $[\mathcal{D}; (r) - (s)]$  belongs to  $1 + p^{m-n}\mathcal{O}_{\mathbb{C}_p}$ .*

*Proof.* If  $r = \frac{a}{b}$  and  $s = \frac{c}{d}$ , the fact that  $p^m$  divides  $ad - bc$  implies that the primitive vectors  $(a, b)$  and  $(c, d)$  in  $\mathbb{Z}^2$  are proportional to each other modulo  $p^m$ . Hence there exists  $v \in \mathbb{Z}_p^\times$  for which  $(a, b) = v \cdot (c, d) + p^m(e, f)$  for some  $(e, f) \in \mathbb{Z}^2$ . It follows that

$$[z; (r) - (s)] = v + p^m \frac{fz - e}{dz - c}.$$

But as  $z$  ranges over  $\mathcal{H}_p^{\leq n}$ , the rational function  $\frac{fz - e}{dz - c}$  takes values in  $p^{-n}\mathcal{O}_{\mathbb{C}_p}$  by (11), and the first statement follows. The second follows directly from the first. □

### 2.2. Theta cocycles and symbols

Because all the examples that are relevant to this work are defined over  $C_p$ , it will be convenient to slightly modify the set up in the introduction, by letting  $\mathcal{A}^\times$  denote the multiplicative group of nowhere vanishing rigid analytic functions on  $\mathcal{H}_p$ , and  $\mathcal{M}^\times$  the group of nonzero rigid meromorphic functions on this domain, which are defined over  $C_p$ . This point of view shall remain in force throughout this article. The groups  $\mathcal{A}^\times$  and  $\mathcal{M}^\times$  continue to be equipped with their standard (weight zero) left action by the group  $\Gamma = \mathrm{SL}_2(\mathbb{Z}[1/p])$ .

**Definition 2.3.** A rigid analytic theta-cocycle is a class in  $H^1(\Gamma, \mathcal{A}^\times/C_p^\times)$ , and a rigid meromorphic theta-cocycle is a class in  $H^1(\Gamma, \mathcal{M}^\times/C_p^\times)$ .

It is useful to dispose of a variant of these objects involving  $\Gamma$ -invariant *modular symbols* instead of one-cocycles. The passage from one to the other is akin to working with the cohomology of a modular curve relative to the cusps instead of the cohomology of the associated open modular curve, and hence the two notions can be thought of as being in duality with each other.

If  $\Omega$  is a  $\Gamma$ -module, the  $\Gamma$ -module of *modular symbols* with values in  $\Omega$  is the set  $\mathrm{MS}(\Omega)$  of functions  $m : \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q}) \rightarrow \Omega$  satisfying

$$\begin{aligned} m\{r, s\} &= -m\{s, r\}, \\ m\{r, t\} &= m\{r, s\} + m\{s, t\}, \end{aligned}$$

for all  $r, s, t \in \mathbb{P}_1(\mathbb{Q})$ . If  $\Pi$  is any subgroup of  $\Gamma$ , let  $\mathrm{MS}^\Pi(\Omega) := H^0(\Pi, \mathrm{MS}(\Omega))$  denote the group of  $\Pi$ -invariant modular symbols, satisfying

$$m\{\gamma r, \gamma s\} = \gamma m\{r, s\} \quad \text{for all } \gamma \in \Pi, \quad r, s \in \mathbb{P}_1(\mathbb{Q}).$$

If  $\Pi$  is a congruence subgroup of  $\mathrm{SL}_2(\mathbb{Z})$ , there is a well-known identification between  $\mathrm{MS}^\Pi(\Omega)$  and the cohomology of the associated modular curve  $X_\Pi$

relative to the cusps:

$$\text{MS}^\Pi(\Omega) = H^1(X_\Pi, \text{cusps}; \Omega).$$

The natural map from the relative cohomology to the cohomology of the complete curve corresponds to the map

$$\text{MS}^\Pi(\Omega) \longrightarrow H^1_{\text{par}}(\Pi, \Omega)$$

sending the  $\Pi$ -invariant  $\Omega$ -valued modular symbol  $m$  to the class on the one-cocycle

$$(14) \quad c(\gamma) := m\{\infty, \gamma\infty\}.$$

This map is defined for any subgroup  $\Pi$  of  $\text{SL}_2(\mathbb{Q})$  such as the group  $\Gamma = \text{SL}_2(\mathbb{Z}[1/p])$  (cf. [DV21, Lemma 1.3]), and it fits into the exact sequence

$$(15) \quad \Omega^\Pi \hookrightarrow \bigoplus_{x \in \Pi \backslash \mathbb{P}_1(\mathbb{Q})} \Omega^{\Pi_x} \longrightarrow \text{MS}^\Pi(\Omega) \xrightarrow{\delta} H^1(\Pi, \Omega) \longrightarrow \bigoplus_{x \in \Pi \backslash \mathbb{P}_1(\mathbb{Q})} H^1(\Pi_x, \Omega),$$

where the direct sums are taken over a system of representatives for the  $\Pi$ -orbits on  $\mathbb{P}_1(\mathbb{Q})$  and  $\Pi_x$  is the stabiliser of  $x$  in  $\Pi$ . The image of  $\delta$  shall be referred to as the *parabolic cohomology* of  $\Pi$  with values in  $\Omega$ , although the reader is cautioned that for  $p$ -arithmetic groups like  $\Pi = \Gamma$ , the cusp stabilisers do not consist solely of parabolic matrices.

**Definition 2.4.** A rigid analytic theta-symbol is a class in  $\text{MS}^\Gamma(\mathcal{A}^\times/C_p^\times)$ , and a rigid meromorphic theta-symbol is a class in  $\text{MS}^\Gamma(\mathcal{M}^\times/C_p^\times)$ .

**Lemma 2.5.** *The invariants  $(\mathcal{A}^\times/C_p^\times)^{\Gamma_\infty} = (\mathcal{M}^\times/C_p^\times)^{\Gamma_\infty}$  are trivial. In particular, the map  $\delta$  of (15) is injective, and every parabolic theta-cocycle is described by a unique theta-symbol.*

*Proof.* Let  $N$  be the parabolic subgroup of  $\Gamma_\infty$ . Any homomorphism in  $H^1(\Gamma_\infty, C_p^\times)$  is trivial on the commutator subgroup of  $\Gamma_\infty$ , which contains  $N^{p^2-1}$ . It follows that any

$$f \in H^0(\Gamma_\infty, \mathcal{M}^\times/C_p^\times)$$

lifts to an element of  $H^0(N^{p^2-1}, \mathcal{M}^\times)$ . But there are no non-constant rigid meromorphic functions invariant under the translation  $z \mapsto z + p^2 - 1$ , and the result follows.  $\square$

**Corollary 2.6.** *A parabolic theta-cocycle admits a unique representative one-cocycle, up to torsion, whose restriction to  $\Gamma_\infty$  is trivial.*

*Proof.* Let  $\Omega := \mathcal{A}^\times/C_p^\times$  or  $\mathcal{M}^\times/C_p^\times$ . A parabolic theta-cocycle  $J$  corresponds to a theta-symbol in  $\text{MS}^\Gamma(\Omega)$  by Lemma 2.5, and the one-cocycle attached to this modular symbol following the recipe of (14) vanishes identically on  $\Gamma_\infty$ .  $\square$

It will be convenient henceforth to identify parabolic theta-cocycles with  $\Gamma$ -invariant modular symbols. In this way, for a parabolic theta-cocycle  $J$  viewed as both a one cocycle and a modular symbol, we have  $J(S) = J\{0, \infty\}^{-1}$ . The description of  $J$  as a theta-symbol often leads to more transparent calculations.

### 2.3. RM values

Recall that the *value* of a rigid meromorphic cocycle  $J$  at an RM point  $\tau$  is defined by setting

$$J[\tau] := J(\gamma_\tau)(\tau),$$

where  $\gamma_\tau$  is a generator of the stabiliser of  $\tau$  in  $\Gamma$ , normalised so that  $\gamma_\tau^n(z)$  tends to  $\tau$  in the *real topology* as  $n$  tends to  $\infty$ , and to  $\bar{\tau}$  as  $n$  tends to  $-\infty$ . The element  $\gamma_\tau$  is called the *automorph* of  $\tau$ . If  $J$  is parabolic, one also has

$$J[\tau] = J\{r, \gamma_\tau r\}(\tau),$$

where  $r \in \mathbb{P}_1(\mathbb{Q})$  is any base point. These RM values are the main subject of [DV21].

It is useful for the considerations of this paper to extend the notion of RM values to theta-cocycles and theta-symbols. If  $v$  is any vertex of  $\mathcal{T}$ , the stabilizer  $\Gamma_v \subset \Gamma$  of  $v$  in  $\Gamma$  is conjugate to  $\text{SL}_2(\mathbb{Z})$  under a matrix in  $\text{GL}_2(\mathbb{Z}[1/p])$ . The second cohomology group  $H^2(\Gamma_v, C_p^\times)$  is therefore trivial, and hence the restriction of a theta-cocycle  $J$  to  $H^1(\Gamma_v, \mathcal{M}^\times/C_p^\times)$  lifts to an element

$$J_v \in H^1(\Gamma_v, \mathcal{M}^\times).$$

This lift is unique up to 12-torsion, because  $H^1(\Gamma_v, C_p^\times) \simeq H^1(\text{SL}_2(\mathbb{Z}), C_p^\times)$  is of order  $\leq 12$ .

To define  $J[\tau]$ , it is convenient to distinguish two cases:

*Case 1.* The RM point  $\tau$  is *unramified*, i.e., it generates a real quadratic field in which the prime  $p$  is inert. In that case,  $\tau$  maps to a vertex  $v$  of the Bruhat–Tits tree under the reduction map, and its automorph  $\gamma_\tau$  belongs to  $\Gamma_v$ . We

can therefore define

$$(16) \quad J[\tau] := J_v(\gamma_\tau)(\tau) \in C_p \cup \{\infty\}.$$

*Case 2.* The RM point  $\tau$  is *ramified*, i.e., it generates a real quadratic field in which  $p$  is ramified. The point  $\tau$  then reduces to an edge  $e = (v_1, v_2)$  of the Bruhat–Tits tree, and the automorph  $\gamma_\tau$  belongs to the edge stabiliser  $\Gamma_e = \Gamma_{v_1} \cap \Gamma_{v_2}$ . We then define

$$(17) \quad J[\tau]^2 := J_{v_1}(\gamma_\tau)(\tau) \times J_{v_2}(\gamma_\tau)(\tau) \in C_p \cup \{\infty\}.$$

This makes the value of  $J[\tau]$  well-defined only up to a sign, a harmless ambiguity. A direct calculation reveals that the value  $J[\tau]$  depends only on the  $\Gamma$ -orbit of  $\tau$ , i.e., that

$$J[\gamma\tau] = \pm J[\tau], \quad \text{for all } \gamma \in \Gamma.$$

When  $J$  lifts to a rigid meromorphic cocycle, the two notions of RM values are of course compatible, and the conjectures of [DV21] predict that  $J[\tau]$  is an algebraic number lying in a compositum of ring class fields of real quadratic fields. When such a lift does not exist, the quantities  $J[\tau]$  are expected to be transcendental in general, but still retain some arithmetic interest, insofar as they are related to Gross–Stark units and Stark–Heegner points, as will be discussed shortly.

### 3. Analytic theta-cocycles

We now turn to the classification of analytic theta-cocycles, and show they form a finitely generated group of rank  $2g + 2$ , where  $g$  is the genus of  $X_0(p)$ . The techniques used to prove this theorem rely on ideas of Stevens and Schneider–Teitelbaum, (cf. [PS11], [Sch84], [Te90]) and occupy §3.1–3.6, from which the classification is deduced in §3.7. This results in a list of three types of generators for the group of theta-cocycles, each with their concomitant idiosyncrasies:

- The boundary theta cocycle,
- The Dedekind–Rademacher cocycle,
- The modular cocycles attached to newforms in  $S_2(\Gamma_0(p))$ .

A discussion of known results about their RM values, which notably include *Gross–Stark units* and *Stark–Heegner points*, can be found in §3.7. These

classes are all proper for the action of the Hecke algebra. In §3.8 the *toric cocycles*, which are attached to a pair of cusps, are introduced. These theta-cocycles behave less straightforwardly with respect to the action of the Hecke algebra, but are completely explicit, and play an important role in what follows.

### 3.1. Rigid analytic functions and boundary distributions

The classification begins with a study of the module of *additive* analytic functions on  $\mathcal{H}_p$  endowed with the weight  $k$  action of  $\Gamma$ , denoted  $\mathcal{A}_k$ . It rests on the fact that  $\mathcal{A}_k$  is isomorphic to a space of locally analytic distributions on the boundary  $\mathbb{P}_1(\mathbb{Q}_p)$  of  $\mathcal{H}_p$ . Assume henceforth that  $k = 2$  for simplicity, although the results described below can certainly be extended to more general positive even weights.

The dual of the space of locally analytic functions on  $\mathbb{P}_1(\mathbb{Q}_p)$ , equipped with the strong topology of uniform convergence on compact open subsets, is called the space of *locally analytic distributions* on  $\mathbb{P}_1(\mathbb{Q}_p)$ , and is denoted  $\mathcal{D}(\mathbb{P}_1(\mathbb{Q}_p))$ . Given  $\mu \in \mathcal{D}(\mathbb{P}_1(\mathbb{Q}_p))$ , the notation

$$\mu(h) =: \int_{\mathbb{P}_1(\mathbb{Q}_p)} h(t) d\mu(t)$$

shall be adopted. More generally, if  $U$  is a compact open subset of  $\mathbb{P}_1(\mathbb{Q}_p)$  and  $1_U$  is its characteristic function, we define

$$\int_U h(t) d\mu(t) := \int_{\mathbb{P}_1(\mathbb{Q}_p)} 1_U(t) h(t) d\mu(t).$$

A distribution  $\mu \in \mathcal{D}(\mathbb{P}_1(\mathbb{Q}_p))$  satisfying  $\mu(1) = 0$ , where  $1$  denotes the constant function  $1$  on  $\mathbb{P}_1(\mathbb{Q}_p)$ , is said to be *of total volume zero*, and the space of such locally analytic distributions is denoted  $\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$ .

The group  $\mathrm{PGL}_2(\mathbb{Q}_p)$  acts naturally on  $\mathcal{D}(\mathbb{P}_1(\mathbb{Q}_p))$  and on  $\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$  via the weight zero action on locally analytic functions on  $\mathbb{P}_1(\mathbb{Q}_p)$ . More precisely,

$$\int_{\mathbb{P}_1(\mathbb{Q}_p)} h(t) d(\mu|\gamma)(t) = \int_{\mathbb{P}_1(\mathbb{Q}_p)} h(\gamma t) d\mu(t), \quad \text{where } \gamma t := \frac{at + b}{ct + d}.$$

To any rigid analytic function  $f \in \mathcal{A}_2$ , we attach a locally analytic distribution  $\mu_f$  on  $\mathbb{P}_1(\mathbb{Q}_p)$  by setting, for all analytic functions  $h(t)$  on a compact open  $U_e \subset \mathbb{P}_1(\mathbb{Q}_p)$  corresponding to an oriented edge  $e \in \mathcal{E}(\mathcal{T})$ ,

$$\int_{U_e} h(t) d\mu_f(t) := \mathrm{res}_e(f(z)h(z)dz).$$

Here,  $\text{res}_e$  denotes the  $p$ -adic annular residue along the oriented annulus  $\mathcal{W}_e$  of (10). The distribution  $\mu_f$  is called the *boundary distribution* attached to  $f$ . It is a direct consequence of the residue theorem that  $\mu_f$  belongs to  $\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$ .

**Proposition 3.1.** *The map  $f \mapsto \mu_f$  induces a topological isomorphism*

$$\text{BD} : \mathcal{A}_2 \xrightarrow{\sim} \mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$$

that is compatible with the  $\text{PGL}_2(\mathbb{Q}_p)$ -actions on both sides.

*Proof.* Setting  $k = 2$  in the statement of Theorem 2.2.1 of [DaTe], the dual of the map denoted  $I_2$  in loc.cit. induces an isomorphism

$$\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p)) \longrightarrow \mathcal{A}_2,$$

in light of the fact that  $\mathcal{A}_2$  is a reflexive Frechet space and hence is identified with its double dual. The “boundary distribution map” BD is just the inverse of this isomorphism. □

The map BD induces an isomorphism on the parabolic cohomology groups, denoted by the same symbol by a slight abuse of notation:

$$(18) \quad \text{BD} : \text{MS}^\Gamma(\mathcal{A}_2) \xrightarrow{\sim} \text{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))).$$

This reduces the problem of understanding  $\text{MS}^\Gamma(\mathcal{A}_2)$  to that of classifying the  $\Gamma$ -invariant modular symbols with values in  $\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$ . An element  $\mu$  of the latter is simply a collection of distributions  $\mu\{r, s\}$  on  $\mathbb{P}_1(\mathbb{Q}_p)$ , indexed by elements  $r, s \in \mathbb{P}_1(\mathbb{Q})$ , that satisfy the usual modular symbol relations

$$\mu\{r, s\} = -\mu\{s, r\}, \quad \mu\{r, s\} + \mu\{s, t\} = \mu\{r, t\},$$

together with the equivariance property

$$(19) \quad \int_{\gamma B} h(t) d\mu\{r, s\}(t) = \int_B h(\gamma t) d\mu\{\gamma^{-1}r, \gamma^{-1}s\}(t), \quad \text{for all } \gamma \in \Gamma.$$

Let  $w_\infty$  and  $w_p$  be elements of determinant  $-1$  and  $p$  respectively in  $\text{GL}_2(\mathbb{Z}[1/p])$ . Their classes modulo  $\Gamma$  generate the normaliser of  $\Gamma$ , and thus they act as involutions on various cohomology groups for  $\Gamma$ . If  $\mu$  is an eigen-symbol for these involutions, then the invariance property of (19) even holds for all  $\gamma \in \text{PGL}_2(\mathbb{Z}[1/p])$ :

$$(20) \quad \int_{\gamma B} h(t) d\mu\{r, s\}(t) = \pm \int_B h(\gamma t) d\mu\{\gamma^{-1}r, \gamma^{-1}s\}(t), \quad \text{for all } \gamma \in \text{PGL}_2(\mathbb{Z}[1/p]),$$

where the sign is given by

$$\operatorname{sgn}_\infty(\mu)^{\operatorname{sgn}(\det(\gamma))} \times \operatorname{sgn}_p(\mu)^{v_p(\det(\gamma))}, \quad \text{with } \begin{cases} \mu | w_\infty &= \operatorname{sgn}_\infty(\mu)\mu, \\ \mu | w_p &= \operatorname{sgn}_p(\mu)\mu. \end{cases}$$

### 3.2. Restriction to $\mathbb{Z}_p$

The compact open subset  $\mathbb{Z}_p \subset \mathbb{P}_1(\mathbb{Q}_p)$  is a ball whose stabiliser in  $\Gamma$  is the usual congruence group  $\Gamma_0(p)$ . The restriction map  $\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p)) \rightarrow \mathcal{D}(\mathbb{Z}_p)$  to the space of distributions on  $\mathbb{Z}_p$  therefore induces a map on modular symbols:

$$\operatorname{res}_{\mathbb{Z}_p} : \operatorname{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))) \rightarrow \operatorname{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p)).$$

The image of this map is called the space of *overconvergent modular symbols* of weight two and level  $p$ .

**Lemma 3.2.** *The map  $\operatorname{res}_{\mathbb{Z}_p}$  is injective.*

*Proof.* The matrix  $\iota_p := \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$  interchanges  $\mathbb{Z}_p$  and its complement  $\mathbb{Z}'_p := \mathbb{P}_1(\mathbb{Q}_p) - \mathbb{Z}_p$ , and normalises  $\Gamma_0(p)$ . It therefore induces mutually inverse isomorphisms

$$\operatorname{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p)) \xleftarrow{\iota_p} \operatorname{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}'_p))$$

for which the diagram

$$\begin{array}{ccc} \operatorname{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))) & \xrightarrow{w_p} & \operatorname{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))) \\ \operatorname{res}_{\mathbb{Z}_p} \downarrow & & \downarrow \operatorname{res}_{\mathbb{Z}'_p} \\ \operatorname{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p)) & \xrightarrow{\iota_p} & \operatorname{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}'_p)) \end{array}$$

commutes. In particular, the involution  $w_p$  interchanges the kernels of  $\operatorname{res}_{\mathbb{Z}_p}$  and of  $\operatorname{res}_{\mathbb{Z}'_p}$ , and it suffices to show that  $\operatorname{res}_{\mathbb{Z}'_p}$  is injective. If  $\mu$  is in the kernel of  $\operatorname{res}_{\mathbb{Z}'_p}$ , then

$$(21) \quad \mu\{r, s\}|_{\mathbb{Z}'_p} = 0, \quad \text{for all } r, s \in \mathbb{P}_1(\mathbb{Q}).$$

The domain  $\mathbb{P}_1(\mathbb{Q}_p)$  admits a decomposition as a disjoint union of  $p + 1$  open balls,

$$(22) \quad \mathbb{P}_1(\mathbb{Q}_p) = B_0 \sqcup B_1 \sqcup \cdots \sqcup B_{p-1} \sqcup \mathbb{Z}'_p,$$



where  $B_j \subset \mathbb{Z}_p$  is the mod  $p$  residue disc of  $-j$ . The group  $\mathrm{SL}_2(\mathbb{Z})$  acts transitively on the collection  $\{B_0, B_1, \dots, \mathbb{Z}'_p\}$ . Let  $\gamma_j \in \mathrm{SL}_2(\mathbb{Z})$  be a matrix satisfying  $\mathbb{Z}'_p = \gamma_j B_j$ . Then for all  $j = 0, \dots, p-1$ , and for all  $r, s \in \mathbb{P}_1(\mathbb{Q})$ ,

$$\mu\{r, s\}|_{B_j} = \mu\{r, s\}|_{\gamma_j^{-1}\mathbb{Z}'_p} = (\mu\{\gamma_j r, \gamma_j s\}|_{\mathbb{Z}'_p})|_{\gamma_j} = 0,$$

where the last equality follows from (21). It now follows from (22) that  $\mu\{r, s\} = 0$  as a distribution on  $\mathbb{P}_1(\mathbb{Q}_p)$ , for all  $r, s \in \mathbb{P}_1(\mathbb{Q})$ . The lemma follows.  $\square$

The space of overconvergent modular symbols is equipped with a Hecke operator  $U_p$ , defined explicitly by

(23)

$$\int_{\mathbb{Z}_p} h(t) d(U_p \mu)\{r, s\}(t) := \sum_{j=0}^{p-1} \int_{\mathbb{Z}_p} h(\alpha_j^{-1} t) d\mu\{\alpha_j r, \alpha_j s\}(t), \quad \text{where } \alpha_j = \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}.$$

The space  $\mathrm{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p)))$  decomposes as a direct sum

$$\mathrm{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))) = \mathrm{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p)))^+ \oplus \mathrm{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p)))^-,$$

where  $\mathrm{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p)))^\epsilon$  denotes, for  $\epsilon \in \{+, -\}$ , the  $\epsilon$ -eigenspace for the action of the involution  $w_p$ . Let  $\mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p))^{U_p=\epsilon}$  denote the space of overconvergent modular symbols on which  $U_p$  acts as multiplication by  $\epsilon$ .

**Proposition 3.3.** *The map  $\mathrm{res}_{\mathbb{Z}_p}$  induces Hecke-equivariant inclusions*

$$\mathrm{res}_{\mathbb{Z}_p} : \mathrm{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p)))^\epsilon \hookrightarrow \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p))^{U_p=\epsilon}.$$

*Proof.* For  $j = 0, 1, \dots, p-1$ , let  $B_j \subset \mathbb{Z}_p$  denote, as in the proof of Lemma 3.2, the residue class of  $-j$  modulo  $p$ , so that

$$\mathbb{Z}_p = B_0 \sqcup B_1 \sqcup \dots \sqcup B_{p-1}, \quad \alpha_j B_j = \mathbb{Z}_p,$$

with  $\alpha_j$  as in (23). By the additivity of the distribution  $\mu\{r, s\} \in \mathcal{D}(\mathbb{Z}_p)$ , we have, for any locally analytic function  $h$  on  $\mathbb{Z}_p$ :

$$\begin{aligned} \int_{\mathbb{Z}_p} h(t) d\mu\{r, s\}(t) &= \sum_{j=0}^{p-1} \int_{B_j} h(t) d\mu\{r, s\}(t) = \sum_{j=0}^{p-1} \int_{\alpha_j^{-1}\mathbb{Z}_p} h(t) d\mu\{r, s\}(t) \\ &= \epsilon \sum_{j=0}^{p-1} \int_{\mathbb{Z}_p} h(\alpha_j^{-1} t) d\mu\{\alpha_j r, \alpha_j s\}(t), \end{aligned}$$

where the last equality follow from (20) in light of the fact that the matrices  $\alpha_j \in \mathrm{PGL}_2(\mathbb{Z}[1/p])$  have determinant  $p$ . The proposition now follows from the definition of the  $U_p$  operator given in (23).  $\square$

Proposition 3.3 shows that  $U_p$  preserves the image of  $\mathrm{res}_{\mathbb{Z}_p}$  and that the minimal polynomial of its restriction to this space divides  $x^2 - 1$ . Composing the map BD of (18) with the restriction map  $\mathrm{res}_{\mathbb{Z}_p}$  thus gives an injection

$$(24) \quad \mathrm{MS}^\Gamma(\mathcal{A}_2) \hookrightarrow \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p))^{U_p^2=1}.$$

### 3.3. Stevens' control theorem

The “total measure” map  $\mathcal{D}(\mathbb{Z}_p) \rightarrow C_p$  that sends  $\mu$  to  $\mu(1)$  induces a “weight two specialisation map”

$$(25) \quad \rho : \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p)) \rightarrow \mathrm{MS}^{\Gamma_0(p)}(C_p)$$

that is compatible with the actions of the Hecke operators on both sides.

**Theorem 3.4** (Stevens). *The weight two specialisation map  $\rho$  induces an isomorphism*

$$(26) \quad \rho : \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p))^{U_p^2=1} \rightarrow \mathrm{MS}^{\Gamma_0(p)}(C_p).$$

*Proof.* The *ordinary subspace* of a Hecke module  $M$  is the direct summand of it on which the  $U_p$  operator acts with slope zero, and is denoted  $M^{\mathrm{ord}}$ . The *control theorem for overconvergent modular symbols* (cf. the case  $k = 0$  of Theorem 1.1 of [PS11]) asserts that  $\rho$  induces an isomorphism

$$\rho^{\mathrm{ord}} : \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p))^{\mathrm{ord}} \rightarrow \mathrm{MS}^{\Gamma_0(p)}(C_p)^{\mathrm{ord}}.$$

But  $\mathrm{MS}^{\Gamma_0(p)}(C_p)$  is isomorphic as a Hecke module to the direct sum of an “Eisenstein line” with two copies of the space of modular forms of weight two on  $\Gamma_0(p)$ . Since all such modular forms are new at  $p$ , it follows that  $U_p^2$  acts as the identity on this space, and that

$$\mathrm{MS}^{\Gamma_0(p)}(C_p)^{\mathrm{ord}} = \mathrm{MS}^{\Gamma_0(p)}(C_p)^{U_p^2=1} = \mathrm{MS}^{\Gamma_0(p)}(C_p).$$

The theorem follows.  $\square$

*Remark 3.5.* The statement of Theorem 3.4 relies crucially on the fact that there are no old-forms of weight two for  $\Gamma_0(p)$ , and would have to be suitably

adapted if  $\Gamma_0(p)$  was replaced by a congruence subgroup obtained by imposing (say) an additional  $\Gamma_1(M)$ -level structure with  $p \nmid M$ . Note however that in general, if  $\Gamma$  is then replaced by the corresponding finite index congruence subgroup  $\Gamma'$  of  $\Gamma$ , [Dar01, part 3 of lemma 1.3] implies that the Hecke algebra acts on  $H_f^1(\Gamma', \mathcal{A}^\times/C_p^\times)$  through its image in the endomorphism ring of the  $p$ -new subspace of the space of weight two modular forms for  $\Gamma_1(M) \cap \Gamma_0(p)$ .

**Corollary 3.6.** *The map*

$$\eta := \rho \circ \text{res}_{\mathbb{Z}_p} \circ \text{BD} : \text{MS}^\Gamma(\mathcal{A}_2) \hookrightarrow \text{MS}^{\Gamma_0(p)}(C_p)$$

*is injective.*

*Proof.* This follows from Propositions 3.1 and 3.3 combined with Theorem 3.4. □

Our goal in what follows is to show that the map  $\eta$  is surjective as well.

### 3.4. The residue map

Let  $\Omega$  be a  $\Gamma$ -module.

**Definition 3.7.** A function  $c : \mathcal{E}(\mathcal{T}) \rightarrow \Omega$  is said to be *harmonic* if it satisfies

$$c(\bar{e}) = -c(e), \text{ for all } e \in \mathcal{E}(\mathcal{T}), \quad \text{and} \quad \sum_{s(e)=v} c(e) = 0, \text{ for all } v \in \mathcal{V}(\mathcal{T}).$$

The space of harmonic functions on the set  $\mathcal{E}(\mathcal{T})$  of oriented edges with values in  $\Omega$  is denoted  $C_{\text{har}}(\Omega)$ . The action of  $\Gamma$  on  $\mathcal{T}$  induces a natural right action of  $\Gamma$  on the space  $C_{\text{har}}(\Omega)$ . In what follows we will be primarily interested in the case where  $\Omega = \mathbb{Z}$  or a bounded subgroup of  $C_p$ , equipped with the trivial action of  $\Gamma$ .

*Remark 3.8.* Elsewhere in the literature (e.g., in [Te90]) it is customary to refer to harmonic functions as *harmonic cocycles*. Since the noun “cocycle” already being used in its more standard form in this article, a more transparent terminology was chosen for Definition 3.7.

Let  $t$  be a local parameter on the oriented annulus  $\mathcal{W}_e$  of (10), identifying  $\mathcal{W}_e$  with the standard annulus in (8). The space  $\mathcal{A}(\mathcal{W}_e)$  of rigid differentials on  $\mathcal{W}_e$  is contained in the module of infinite-tailed series

$$\mathcal{A}(\mathcal{W}_e) = \left\{ \sum_{i \in \mathbb{Z}} a_i t^i \frac{dt}{t}, \quad a_i \in C_p \right\}$$

whose coefficients  $a_i \in C_p$  satisfy suitable  $p$ -adic growth conditions. Let

$$\partial_e : \mathcal{A}(\mathcal{W}_e) \longrightarrow C_p$$

be the  $p$ -adic annular residue sending the differential  $(\sum a_i t^i) dt/t$  to  $a_0$ . One can associate to any  $f \in \mathcal{A}_2$  a harmonic function  $c_f \in C_{\text{har}}(C_p)$  by the rule

$$c_f(e) = \partial_e(f(z)dz), \quad \text{for } e \in \mathcal{E}(\mathcal{T}).$$

The  $\text{PGL}_2(\mathbb{Q}_p)$ -equivariant map

$$\partial : \mathcal{A}_2 \longrightarrow C_{\text{har}}(C_p)$$

sending  $f$  to  $c_f$  is called the *residue map*. The same notation and terminology is used to describe the induced map

$$(27) \quad \partial : \text{MS}^\Gamma(\mathcal{A}_2) \longrightarrow \text{MS}^\Gamma(C_{\text{har}}(C_p))$$

on modular symbols. Let  $e_0$  denote the standard edge of  $\mathcal{E}(\mathcal{T})$ , whose stabiliser is  $\Gamma_0(p)$  and whose associated open ball in  $\mathbb{P}_1(\mathbb{Q}_p)$  is  $\mathbb{Z}_p$ . The evaluation at  $e_0$

$$\text{ev}_{e_0} : C_{\text{har}}(C_p) \longrightarrow C_p$$

is  $\Gamma_0(p)$ -equivariant, and hence induces a map

$$(28) \quad \text{ev}_{e_0} : \text{MS}^\Gamma(C_{\text{har}}(C_p)) \longrightarrow \text{MS}^{\Gamma_0(p)}(C_p).$$

Our strategy to show the surjectivity of  $\eta$  will be to prove the surjectivity of the maps  $\text{ev}_{e_0}$  and  $\partial$  that fit into the following commutative diagram:

$$(29) \quad \begin{array}{ccc} \text{MS}^\Gamma(\mathcal{A}_2) & \xrightarrow{\partial} & \text{MS}^\Gamma(C_{\text{har}}(C_p)) \\ \text{BD} \downarrow & \searrow \eta & \downarrow \text{ev}_{e_0} \\ \text{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))) & & \\ \text{res}_{\mathbb{Z}_p} \downarrow & & \downarrow \\ \text{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p)) & \xrightarrow{\rho} & \text{MS}^{\Gamma_0(p)}(C_p). \end{array}$$

The surjectivity of  $\text{ev}_{e_0}$  is elementary:

**Lemma 3.9.** *The map  $\text{ev}_{e_0} : \text{MS}^\Gamma(C_{\text{har}}(C_p)) \longrightarrow \text{MS}^{\Gamma_0(p)}(C_p)$  is an isomorphism.*

*Proof.* The injectivity of  $ev_{e_0}$  follows from much the same argument as in the proof of the injectivity of the map  $res_{\mathbb{Z}_p}$  given in Lemma 3.2. Namely, an element  $c$  of its kernel satisfies

$$c\{r, s\}(e_0) = 0 \quad \text{for all } r, s \in \mathbb{P}_1(\mathbb{Q}).$$

Recall the subset  $\mathcal{E}^+(\mathcal{T}) \subset \mathcal{E}(\mathcal{T})$  of positively oriented edges described in (6). Since  $\Gamma$  acts transitively on  $\mathcal{E}^+(\mathcal{T})$ , it then follows from the  $\Gamma$ -equivariance of  $c$  that  $c\{r, s\}(e) = 0$  for all  $e \in \mathcal{E}^+(\mathcal{T})$ , and hence, for all  $e \in \mathcal{E}(\mathcal{T})$  by the harmonicity of  $c\{r, s\}$ . To check surjectivity, given  $c_0 \in MS^{\Gamma_0(p)}(C_p)$ , define  $c \in MS^\Gamma(C_{\text{har}}(C_p))$  by setting, for all  $e = \gamma^{-1}e_0 \in \mathcal{E}^+(\mathcal{T})$ ,

$$\begin{aligned} c\{r, s\}(e) &:= c_0\{\gamma r, \gamma s\} & \text{for all } r, s \in \mathbb{P}_1(\mathbb{Q}), \\ c\{r, s\}(\bar{e}) &:= -c\{r, s\}(e) \end{aligned}$$

Although  $\gamma$  is only well-defined up to left multiplication by elements of  $\Gamma_0(p)$ , the  $\Gamma_0(p)$ -invariance of  $c_0$  ensures that the value of  $c\{r, s\}(e)$  does not depend on the choice of  $\gamma$ . Since for any vertex  $v \in \mathcal{V}(\mathcal{T})$  the quantities

$$S_v\{r, s\} := \sum_{s(e)=v} c\{r, s\}(e)$$

define an  $SL_2(\mathbb{Z})$ -invariant modular symbol valued in  $C_p$ , they must be trivial, whence we see that  $c\{r, s\} \in C_{\text{har}}(C_p)$ . It follows from the construction that  $ev_{e_0}(c) = c_0$ . □

### 3.5. The Schneider–Teitelbaum transform

We now show that the residue map  $\partial$  in (27) is surjective. The main ingredient for achieving this is the integral “ $p$ -adic Poisson transform” of Schneider and Teitelbaum which allows one to recover certain elements of  $\mathcal{A}_2$  from their associated boundary distributions.

Let  $C_{\text{har}}^b(C_p) \subset C_{\text{har}}(C_p)$  denote the subspace of *bounded* harmonic functions, i.e., those whose values lie in a bounded subset of  $C_p$ . An element  $c \in C_{\text{har}}^b(C_p)$  can be parlayed into a bounded linear functional  $\mu_c$  on the space of locally constant functions on  $\mathbb{P}_1(\mathbb{Q}_p)$ , by setting

$$\int_{U_e} 1d\mu_c := c(e).$$

The boundedness of  $\mu_c$  implies that it extends uniquely to a *measure* on  $\mathbb{P}_1(\mathbb{Q}_p)$ , i.e., a continuous functional on the space of continuous functions

on  $\mathbb{P}_1(\mathbb{Q}_p)$  endowed with the sup norm. This extension exploits the fact that every continuous function  $h(t)$  on  $\mathbb{P}_1(\mathbb{Q}_p)$  is a uniform limit of locally constant functions to express  $\int_{\mathbb{P}_1(\mathbb{Q}_p)} h(t) d\mu_c(t)$  as a limit of (finite) Riemann sums.

**Proposition 3.10** (Schneider, Teitelbaum). *There is a unique  $\Gamma$ -equivariant splitting of the residue map  $\partial$  on  $C_{\text{har}}^b(C_p)$ , i.e., a map  $\text{ST} : C_{\text{har}}^b(C_p) \rightarrow \mathcal{A}_2$  for which the diagram*

$$(30) \quad \begin{array}{ccc} \mathcal{A}_2 & \xrightarrow{\partial} & C_{\text{har}}(C_p) \longrightarrow 0 \\ & \swarrow \text{---} & \cup \\ & \text{ST} & C_{\text{har}}^b(C_p) \end{array}$$

commutes.

*Proof.* The map  $\text{ST}$  is constructed by integrating a ‘‘Poisson kernel’’ against this measure, as in [Te90], namely, one sets

$$(31) \quad \text{ST}(c)(z) = \int_{\mathbb{P}_1(\mathbb{Q}_p)} \frac{1}{z-t} d\mu_c(t), \quad \text{for all } z \in \mathcal{H}_p.$$

See [Te90] for more details. □

**Lemma 3.11.** *The natural inclusion  $\text{MS}^\Gamma(C_{\text{har}}^b(C_p)) \hookrightarrow \text{MS}^\Gamma(C_{\text{har}}(C_p))$  is an isomorphism.*

*Proof.* Given  $c \in \text{MS}^\Gamma(C_{\text{har}}(C_p))$ , consider its image in  $H^1(\Gamma_0(p), C_p)$  under the map  $\text{ev}_{e_0}$ . Since  $\Gamma_0(p)$  is finitely generated, there is a bounded subset  $\Omega \subset C_p$  for which  $\text{ev}_{e_0}(c) \in H^1(\Gamma_0(p), \Omega)$ . But then the commutativity of the diagram

$$\begin{array}{ccc} \text{MS}^\Gamma(C_{\text{har}}(\Omega)) & \xrightarrow{\text{ev}_{e_0}} & \text{MS}^{\Gamma_0(p)}(\Omega) \\ \downarrow & & \downarrow \\ \text{MS}^\Gamma(C_{\text{har}}(C_p)) & \xrightarrow{\text{ev}_{e_0}} & \text{MS}^{\Gamma_0(p)}(C_p), \end{array}$$

in which the horizontal arrows are isomorphisms by Lemma 3.9, implies that  $c$  belongs to  $\text{MS}^\Gamma(C_{\text{har}}(\Omega)) \subset \text{MS}^\Gamma(C_{\text{har}}(C_p))$ . □

**Corollary 3.12.** *The residue map*

$$\partial : \text{MS}^\Gamma(\mathcal{A}_2) \longrightarrow \text{MS}^\Gamma(C_{\text{har}}(C_p))$$

of (27) is an isomorphism.

*Proof.* The injectivity of  $\partial$  is apparent from the fact that the injective map  $\eta$  in the commutative diagram (29) factors through it. Given  $c \in \text{MS}^\Gamma(C_{\text{har}}(C_p))$ , the harmonic functions  $c\{r, s\}$  belong to  $C_{\text{har}}^b(C_p)$  for all  $r, s \in \mathbb{P}_1(\mathbb{Q})$ , by Lemma 3.11. We may therefore set

$$f\{r, s\} := \text{ST}(c\{r, s\}) \in \mathcal{A}_2.$$

The assignment  $(r, s) \mapsto f\{r, s\}$  is an element of  $\text{MS}^\Gamma(\mathcal{A}_2)$  satisfying  $\partial(f) = c$ , and the result follows.  $\square$

**Theorem 3.13.** *The map  $\eta$  of (29) gives a Hecke-equivariant isomorphism*

$$\eta : \text{MS}^\Gamma(\mathcal{A}_2) \xrightarrow{\sim} \text{MS}^{\Gamma_0(p)}(C_p).$$

*Proof.* This follows immediately from Lemma 3.9 and Corollary 3.12.  $\square$

**Definition 3.14.** The inverse of the isomorphism  $\eta$ , denoted

$$\text{ST} : \text{MS}^{\Gamma_0(p)}(C_p) \xrightarrow{\sim} \text{MS}^\Gamma(\mathcal{A}_2)$$

is called the *Schneider–Teitelbaum lift*.

We close this section by recording the following consequence of the injectivity of the residue map on  $\text{MS}^\Gamma(\mathcal{A}_2)$ :

**Proposition 3.15.** *The spaces  $\text{MS}^\Gamma(\mathcal{A})$  and  $\text{MS}^\Gamma(\mathcal{M})$  of rigid analytic and meromorphic cocycles of weight zero are trivial.*

*Proof.* The image of the map  $d : \text{MS}^\Gamma(\mathcal{A}) \rightarrow \text{MS}^\Gamma(\mathcal{A}_2)$  consists of modular symbols with values in the exact rigid differentials, which have trivial residues, and hence this image is contained in the kernel of the residue map  $\partial$ . Since Corollary 3.12 asserts that  $\partial$  is injective, it follows that, for all  $f \in \text{MS}^\Gamma(\mathcal{A})$  and for all  $r, s \in \mathbb{P}_1(\mathbb{Q})$ , the function  $f\{r, s\}$  is a constant, and hence that  $f$  belongs to  $\text{MS}^\Gamma(\mathbb{C}_p)$ , which is trivial. The triviality of  $\text{MS}^\Gamma(\mathcal{A})$  follows, and that of  $\text{MS}^\Gamma(\mathcal{M})$  is then a consequence of [DV21, Cor. 1.28].  $\square$

### 3.6. The multiplicative Schneider–Teitelbaum lift

The logarithmic derivative gives a natural injection

$$\text{dlog} : \mathcal{A}^\times / C_p^\times \rightarrow \mathcal{A}_2$$

sending the local section  $f$  to  $f'/f$ . It induces a map on the space of  $\Gamma$ -invariant modular symbols:

$$(32) \quad \text{dlog} : \text{MS}^\Gamma(\mathcal{A}^\times/C_p^\times) \longrightarrow \text{MS}^\Gamma(\mathcal{A}_2)$$

The space  $\text{dlog}(\mathcal{A}^\times) \subset \mathcal{A}_2$  is called the space of *rigid differentials of the third kind* on  $\mathcal{H}_p$ , and consists of differentials whose image under  $\partial$  are  $\mathbb{Z}$ -valued harmonic functions on  $\mathcal{E}(\mathcal{T})$ . The image of (32) is likewise called the space of rigid analytic modular symbols of the third kind.

**Proposition 3.16.** *There is a Hecke equivariant morphism*

$$\text{ST}^\times : \text{MS}^{\Gamma_0(p)}(\mathbb{Z}) \longrightarrow \text{MS}^\Gamma(\mathcal{A}^\times/C_p^\times)$$

for which the following diagram commutes:

$$\begin{array}{ccc} \text{MS}^{\Gamma_0(p)}(\mathbb{Z}) & \xrightarrow{\text{ST}^\times} & \text{MS}^\Gamma(\mathcal{A}^\times/C_p^\times) \\ \downarrow & & \downarrow \text{dlog} \\ \text{MS}^{\Gamma_0(p)}(C_p) & \xrightarrow{\text{ST}} & \text{MS}^\Gamma(\mathcal{A}_2) \end{array}$$

The map  $\text{ST}^\times$  is called the *multiplicative Schneider–Teitelbaum lift*. It is constructed, following [Dar01, §3.3], as a multiplicative refinement of (31), as we now describe. First, note that a class

$$\phi \in \text{MS}^{\Gamma_0(p)}(\mathbb{Z})$$

gives rise to a unique element  $\mu_\phi \in \text{MS}^\Gamma(\mathcal{D})$ , where  $\mathcal{D}$  is the module of  $\mathbb{Z}$ -valued distributions on  $\mathbb{P}_1(\mathbb{Q}_p)$ . This element is characterised by the properties

$$\begin{aligned} \mu_\phi\{r, s\}(\mathbb{Z}_p) &= \phi\{r, s\}, \quad \text{for all } r, s \in \mathbb{P}_1(\mathbb{Q}), \\ \mu_\phi\{\gamma r, \gamma s\}(\gamma U) &= \mu_\phi\{r, s\}(U), \quad \text{for all compact open } U \subset \mathbb{P}_1(\mathbb{Q}_p), \text{ and } \gamma \in \Gamma. \end{aligned}$$

The existence and uniqueness of  $\mu_\phi$  exploits the fact that  $\Gamma$  acts almost transitively on the compact open balls in  $\mathbb{P}_1(\mathbb{Q}_p)$  — namely, it has two orbits, which are interchanged under sending a ball to its complement — and that the stabiliser of  $\mathbb{Z}_p$  is  $\Gamma_0(p)$ . Given  $n \geq 1$  and  $a := [a_1 : a_2] \in \mathbb{P}_1(\mathbb{Z}/p^n\mathbb{Z})$ , let

$$B(a, n) := \{(z_1 : z_2) \in \mathbb{P}_1(\mathbb{Q}_p) \quad \text{such that } |a_1 z_2 - a_2 z_1| \leq p^{-n}\} \subset \mathbb{P}_1(\mathbb{Q}_p)$$



be the “mod  $p^n$  residue disc centered at  $a$ ”, and write

$$\mathcal{D}_{\phi,n}\{r, s\} = \sum_{a \in \mathbb{P}_1(\mathbb{Z}/p^n\mathbb{Z})} \mu_\phi\{r, s\}(B(a, n)) \cdot \tilde{a} \in \text{Div}^0(\mathbb{P}_1(\mathbb{Q}_p)),$$

where  $\tilde{a}$  is any sample point in  $B(a, n)$ . Then we have

**Lemma 3.17.** *Let  $J := \text{ST}^\times(\phi)$ . Then*

$$J_{v_\circ}\{r, s\} = \lim_{n \rightarrow \infty} [z; \mathcal{D}_{\phi,n}\{r, s\}].$$

*Proof.* The convergence, modular symbol and  $\text{SL}_2(\mathbb{Z})$ -equivariance properties of the right hand side follow immediately from the similar properties for the divisors  $\mathcal{D}_{\phi,n}\{r, s\}$ , using Lemma 2.2. The lemma follows from a comparison of divisors.  $\square$

This definition of the RM values of theta-symbols may have seemed a bit contrived, but finds a modicum of justification in the following proposition.

**Proposition 3.18.** *If  $J = \text{ST}^\times(\phi)$ , and  $\tau$  is an RM point, then  $J[\tau]$  belongs to  $\mathcal{O}_{C_p}^\times$ .*

*Proof.* If  $\tau$  is unramified and even, assume without loss of generality that  $\tau$  reduces to  $v_\circ$ , so that

$$J[\tau] = J_{v_\circ}\{\infty, \gamma_\tau \infty\}(\tau).$$

But since  $\tau$  belongs to  $\mathcal{H}_p^o$ , the quantities

$$[\tau; \mathcal{D}_{\phi,n}\{\infty, \gamma_\tau \infty\}]$$

are  $p$ -adic units, and therefore so is their limit  $J[\tau]$  as  $n \rightarrow \infty$ . If  $\tau$  is unramified and odd, a similar argument applies after replacing the even vertex  $v_\circ$  by some fixed odd vertex. Finally, when  $\tau$  is ramified, assume without loss of generality that it reduces to the standard edge  $e_\circ = (v_\circ, v_1)$ . A direct calculation shows that

$$\text{ord}_p(J_{v_\circ}\{r, s\}(\tau)) = \frac{1}{2}\phi\{r, s\}, \quad \text{ord}_p(J_{v_1}\{r, s\}(\tau)) = -\frac{1}{2}\phi\{r, s\},$$

and the proposition follows from the definition of  $J[\tau]$  for ramified  $\tau$ .  $\square$

### 3.7. Classification of analytic theta-cocycles

Using the multiplicative Schneider–Teitelbaum lift, one may now classify the space of rigid analytic theta-cocycles  $H^1(\Gamma, \mathcal{A}^\times/C_p^\times) \otimes \mathbb{Q}$ .

There are precisely two conjugacy classes of parabolic subgroups of  $\Gamma_0(p)$ , the group  $P_\infty$  consisting of the upper triangular matrices, stabilising the cusp  $\infty$ , and the group  $P_0$  consisting of lower triangular matrices, stabilising the cusp 0. The  $\Gamma_0(p)$ -module  $\mathcal{F}(\mathbb{P}_1(\mathbb{Q}), \mathbb{Q})$  of  $\mathbb{Q}$ -valued functions on  $\mathbb{P}_1(\mathbb{Q})$  therefore decomposes as a direct sum of the two induced modules

$$\mathcal{F}(\mathbb{P}_1(\mathbb{Q}), \mathbb{Q}) = \text{Ind}_{P_\infty}^{\Gamma_0(p)} \mathbb{Q} \oplus \text{Ind}_{P_0}^{\Gamma_0(p)} \mathbb{Q},$$

and the  $\Gamma_0(p)$ -cohomology of the exact sequence (15) with  $\Omega = \mathbb{Q}$  leads to a short exact sequence

$$(33) \quad 0 \longrightarrow \mathbb{Q} \longrightarrow \text{MS}^{\Gamma_0(p)}(\mathbb{Q}) \longrightarrow H_{\text{par}}^1(\Gamma_0(p), \mathbb{Q}) \longrightarrow 0.$$

The one-dimensional kernel is spanned by the boundary symbol  $m_{\sharp} \in \text{MS}^{\Gamma_0(p)}(\mathbb{Z})$  defined by

$$m_{\sharp}\{r, s\} = \begin{cases} 0 & \text{if } \Gamma_0(p)r = \Gamma_0(p)s, \\ 1 & \text{if } r \in \Gamma_0(p)0, \quad s \in \Gamma_0(p)\infty, \\ -1 & \text{if } r \in \Gamma_0(p)\infty, \quad s \in \Gamma_0(p)0. \end{cases}$$

It accounts for the discrepancy between modular symbols and parabolic cohomology classes. Using the multiplicative Schneider–Teitelbaum transform above, we therefore see that the space of modular theta symbols  $\text{MS}^\Gamma(\mathcal{A}^\times/C_p^\times) \otimes \mathbb{Q}$  is generated by

- The boundary theta cocycle attached to the boundary symbol  $m_{\sharp}$  defined above
- The modular cocycles attached to the cohomology classes of newforms in  $S_2(\Gamma_0(p))$ .

Similar arguments apply to non-parabolic classes. As above, the multiplicative Schneider–Teitelbaum transform provides an isomorphism  $C_{\text{har}}(\mathbb{Z}) \simeq \mathcal{A}^\times/C_p^\times$ , so that we have a short exact sequence of  $\Gamma$ -modules

$$1 \longrightarrow \mathcal{A}^\times/C_p^\times \longrightarrow \text{Ind}_{\Gamma_0(p)}^\Gamma(\mathbb{Z}) \longrightarrow \text{Ind}_{\text{SL}_2(\mathbb{Z})}^\Gamma(\mathbb{Z}) \oplus \text{Ind}_{\text{SL}_2(\mathbb{Z})}^\Gamma(\mathbb{Z}) \longrightarrow 1$$

As before, using Shapiro’s lemma the following short exact sequence is extracted from the long exact sequence on cohomology, where  $m_{\sharp}$  is the boundary symbol, identified with the cohomology class of the cocycle  $\gamma \mapsto m_{\sharp}\{r, \gamma r\}$

for an arbitrarily chosen cusp  $r \in \mathbb{P}_1(\mathbb{Q})$ :

$$0 \longrightarrow \mathbb{Q} \cdot m_{\#} \longrightarrow \mathbb{Q} \otimes H^1(\Gamma, \mathcal{A}^\times/C_p^\times) \longrightarrow H^1(\Gamma_0(p), \mathbb{Q}) \longrightarrow 0.$$

It is classical that the space  $H^1(\Gamma_0(p), \mathbb{Q})$  is of dimension  $2g + 1$ , where  $g$  is the genus of  $X_0(p)$ , and has a  $2g$ -dimensional parabolic subspace. Thus the space of analytic theta-cocycles

$$\mathbb{Q} \otimes H^1(\Gamma, \mathcal{A}^\times/C_p^\times)$$

is of dimension  $2g + 2$ , and the quotient by the parabolic classes is a line, accounted for by the Dedekind–Rademacher cocycle below. We now discuss all these classes in more detail.

**3.7.1. The boundary theta-symbol** The simplest example of a theta-symbol is the so-called *boundary theta-symbol* described by fixing a base point  $\eta \in \mathbb{P}_1(C_p)$  and setting

$$(34) \quad J_{\#}\{r, s\}(z) := [(z) - (\eta); (r) - (s)].$$

The boundary theta-symbol  $J_{\#}$  is the multiplicative Schneider–Teitelbaum lift of the boundary symbol  $m_{\#}$ . When viewed as a parabolic theta-cocycle, it lies in the kernel of the residue map to  $H^1(\Gamma_0(p), \mathbb{Z})$ . Note that it is easily seen to lift to an element of  $H^1(\Gamma, \mathcal{A}^\times)$  defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (cz + d)^{-1}$$

Because  $J_{\#}$  is very simple, taking values in the group of rational functions on  $\mathbb{P}_1$ , its RM values are not very interesting. If  $F(x, s) = Ax^2 + Bxy + Cy^2$  is a binary quadratic form of discriminant  $D = B^2 - 4AC$ , then its root is  $\tau_F = (-B + \sqrt{D})/2A$ , while its stabiliser is generated by

$$\gamma_F = \begin{pmatrix} u - Bv & -2Cv \\ 2Av & u + Bv \end{pmatrix}, \quad u^2 - Dv^2 = 1,$$

where  $u + v\sqrt{D}$  is a fundamental solution to Pell’s equation. A straightforward calculation shows that for any  $r \in \mathbb{P}_1(\mathbb{Q})$  we have

$$J_{\#}[\tau_F] = J_{\#}\{r, \gamma_{\tau}r\}(\tau_F) = u \pm v\sqrt{D} \pmod{\mathbb{Z}[1/p]^\times}.$$

It follows that the cocycle  $J_{\sharp}$  has algebraic RM values, albeit somewhat uninteresting ones, since they always belong to the field of “real multiplication”. To obtain more interesting class invariants we now consider the RM values of analytic theta-cocycles arising from the multiplicative Schneider–Teitelbaum lifts of richer elements of  $\text{MS}^{\Gamma_0(p)}(\mathbb{Z})$ .

**3.7.2. Modular theta-symbols** Let  $f \in S_2(\Gamma_0(p))$  be a normalised cuspidal newform with Fourier coefficients in a field  $K_f \subset \mathbb{R}$ , and let  $\omega_f := 2\pi i f(z) dz$  be the associated regular differential on  $X_0(p)$ . The real analytic differentials

$$\omega_f^+ := \frac{1}{2}(\omega_f + \bar{\omega}_f), \quad \omega_f^- := \frac{1}{2}(\omega_f - \bar{\omega}_f)$$

give rise to modular symbols  $\varphi_f^+$  and  $\varphi_f^- \in \text{MS}^{\Gamma_0(p)}(K_f)$ , defined by

$$(35) \quad \varphi_f^+ \{r, s\} := (\Omega_f^+)^{-1} \int_r^s \omega_f^+, \quad \varphi_f^- \{r, s\} = (\Omega_f^-)^{-1} \int_r^s \omega_f^-,$$

where  $\Omega_f^+$  and  $\Omega_f^-$  are so-called *real and imaginary periods* attached to  $f$ , which can be chosen in such a way that  $\varphi_f^\pm$  is  $K_f$ -valued. This only determines these modular symbols, and the periods  $\Omega_f^\pm$ , up to multiplication by  $K_f^\times$ , and we always have

$$\Omega_f^+ \Omega_f^- = \Omega_f := \langle f, f \rangle = \langle \omega_f^+, \omega_f^- \rangle \pmod{K_f^\times}.$$

We can further insist that  $\varphi_f^\pm$  takes values in  $\mathcal{O}_{K_f}$ , and maps surjectively to  $\mathbb{Z}$  when  $f$  has integer Fourier coefficients, which then determines  $\Omega_f^-$  and  $\Omega_f^+$  up to a sign.

The multiplicative Schneider–Teitelbaum lifts of these morphisms are denoted

$$J_f^+, J_f^- \in \text{MS}^\Gamma(\mathcal{A}^\times / C_p^\times) \otimes \mathcal{O}_{K_f}.$$

They are called the *modular theta-symbols* attached to  $f$ . Recall again that we can exploit the natural injection

$$\text{MS}^\Gamma(\mathcal{A}^\times / C_p^\times) \longrightarrow \text{H}^1(\Gamma, \mathcal{A}^\times / C_p^\times)$$

to view  $J_f^\pm$  as elements of the latter group when this is convenient. Assume now for simplicity that  $f$  has rational Fourier coefficients and hence corresponds to an elliptic curve  $E_f$  of conductor  $p$ , and let  $\Phi_{\text{Tate}} : C_p^\times \longrightarrow E_f(C_p)$  be the Tate  $p$ -adic uniformisation of  $E_f$ . One of the main conjectures of [Dar01] is

**Conjecture 3.19.** *After eventually replacing the theta-symbols  $J_f^\pm$  by suitable powers, the local points  $\Phi_{\text{Tate}}(J_f^\pm[\tau]) \in E(C_p)$  are defined over  $H_\tau$ , for all  $\tau \in \mathcal{H}_p^{\text{RM},\circ}$ .*

The algebraicity of Stark–Heegner points attached to genus characters was proved in [BD09], and later extended in [Mo11]. The general case remains open.

**3.7.3. The Dedekind–Rademacher theta-cocycle** Let

$$E_2^{(p)}(z) = \frac{p-1}{12} + 2 \sum_{n \geq 1} \sigma_1^{(p)}(n) e^{2\pi i n z}, \quad \text{where } \sigma_1^{(p)}(n) = \sum_{p \nmid d|n} d$$

be the weight two Eisenstein series on  $\Gamma_0(p)$ , and let  $\omega_{\text{Eis}} := 2\pi i E_2^{(p)}(z) dz$  be the associated regular differential on the open modular curve  $Y_0(p)$ . The periods of  $\omega_{\text{Eis}}$  are encoded in the *Dedekind–Rademacher homomorphism*  $\varphi_{\text{DR}} : \Gamma_0(p) \rightarrow \mathbb{Z}$  defined by

$$(36) \quad \varphi_{\text{DR}} := (2\pi i)^{-1} \int_{z_0}^{\gamma z_0} \omega_{\text{Eis}}.$$

The *Dedekind–Rademacher cocycle*  $J_{\text{DR}} \in H^1(\Gamma, \mathcal{A}^\times / C_p^\times)$  was defined in [DPV2, § 1]. Its image under the residue map is equal to  $\varphi_{\text{DR}}$ . It is a prototypical instance of a non-parabolic theta-cocycle, and in fact generates (up to finite index) the cokernel of the natural inclusion

$$\text{MS}^\Gamma(\mathcal{A}^\times / C_p^\times) \rightarrow H^1(\Gamma, \mathcal{A}^\times / C_p^\times).$$

Concerning the RM values of  $J_{\text{DR}}$ , one has the following

**Theorem 3.20.** *For all  $\tau \in \mathcal{H}_p^{\text{RM},\circ}$ , the value  $J_{\text{DR}}[\tau]$  is a  $p$ -unit in the class field  $H_\tau$  attached to  $\tau$ , and generates it if the order attached to  $\tau$  does not admit a unit of norm  $-1$ .*

This algebraicity result follows from the proof of the  $p$ -adic Gross–Stark conjecture given in [DDP11] and from the proof of Gross’s “tower of norms” conjecture in [DK]. An independent and somewhat more direct approach based on the antiparallel deformations of  $p$ -irregular Hilbert modular Eisenstein series and modeled on the “analytic proof” of the factorisation of norms of singular moduli [GZ85] is described in [DPV2].

### 3.8. Toric theta-cocycles

If  $(r, s) \in \mathbb{P}_1(\mathbb{Q})^2$  is an ordered pair, its stabiliser in  $\mathrm{GL}_2(\mathbb{Q})$  is a split torus whose  $\mathbb{Z}[1/p]$ -points are a group of rank one, generated up to finite index by a loxodromic transformation with eigenvalues of the form  $p^{\pm e}$ . This section associates to such an ordered pair a rigid analytic theta-cocycle, referred to as a *toric cocycle*. Unlike the Hecke eigensymbols described in previous subsections,  $J_{r,s}$  is not an eigenvector for the action of the Hecke operators, but admits an explicit description which shall be useful in later calculations.

If  $\xi_1$  and  $\xi_2$  are two points of the extended upper-half plane  $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}_1(\mathbb{Q})$ , the symbol  $[\xi_1, \xi_2]$  is used to denote the hyperbolic geodesic segment on  $\mathcal{H}$  going from  $\xi_1$  to  $\xi_2$ . The intersection of two (open or closed) geodesic segments on  $\mathcal{H}$  is defined in the natural way. A point  $\xi \in \mathcal{H}^*$  is said to be  $(r, s)$ -admissible if it does not lie on any geodesic in  $\Gamma[r, s]$ . Clearly, all  $(r, s)$ -admissible  $\xi$  belong to  $\mathcal{H}$ , and the set of  $(r, s)$ -admissible base points is preserved by the action of  $\Gamma$ . Since the non-admissible points are contained in a countable union of sets of measure zero, the existence of admissible base points is clear.

Let  $\Sigma_{r,s}$  denote the  $\Gamma$ -orbit of the pair  $(r, s)$ , and let  $\Sigma_{r,s}^{(m)} \subset \Sigma_{r,s}$  be the subset of pairs  $(u, v)$  with  $\mathrm{ord}_p(\det(u, v)) = m$ . It is not hard to see that  $\Sigma_{r,s}^{(m)}$  is non-empty for all  $m \geq 0$  and that

$$\Sigma_{r,s} = \bigcup_{m=0}^{\infty} \Sigma_{r,s}^{(m)}.$$

Fix a base point  $\eta \in \mathcal{H}_p$  and an  $(r, s)$ -admissible base point  $\xi \in \mathcal{H}$ , and set

$$J_{r,s}(\gamma)(z) = \prod_{(u,v) \in \Sigma_{r,s}} [(z) - (\eta); (u) - (v)]^{[u,v] \cdot [\xi, \gamma \xi]} := \prod_{m=0}^{\infty} J_{r,s}^{(m)}(\gamma)(z),$$

where

$$J_{r,s}^{(m)}(\gamma)(z) := \prod_{(u,v) \in \Sigma_{r,s}^{(m)}} [(z) - (\eta); (u) - (v)]^{[u,v] \cdot [\xi, \gamma \xi]}.$$

**Lemma 3.21.** *For each  $\gamma \in \Gamma$ , the infinite product defining  $J_{r,s}(\gamma)$  converges to a rigid analytic function on  $\mathcal{H}_p$  and its image in  $\mathcal{A}^\times / C_p^\times$  satisfies a cocycle condition modulo scalars, namely*

$$J_{r,s}(\gamma_1 \gamma_2) = J_{r,s}(\gamma_1) \times \gamma_1 \cdot J_{r,s}(\gamma_2).$$

*Proof.* Observe first that  $\Gamma_\circ := \mathrm{SL}_2(\mathbb{Z})$  acts on the set  $\Sigma_{r,s}^{(m)}$  by Möbius transformations, and that there are finitely many orbits for this action:

$$\Sigma_{r,s}^{(m)} = \Gamma_\circ \cdot (r_1, s_1) \sqcup \Gamma_\circ \cdot (r_2, s_2) \sqcup \cdots \sqcup \Gamma_\circ \cdot (r_\ell, m_\ell).$$

But the cardinality of the set

$$\{(u, v) \in \Gamma_\circ(r_j, s_j) \text{ such that } [u, v] \cdot [\xi, \gamma\xi] = \pm 1\}$$

represents the number of intersection points between the images of the geodesics  $[r_j, s_j]$  and  $[\xi, \gamma\xi]$  in the quotient  $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H}$ . Since this number is finite, it follows that the product defining  $J_{r,s}^{(m)}(\gamma)(z)$  has finitely many factors that are  $\neq 1$ , and hence is a rational function of  $z$ . To prove convergence of

$$J_{r,s}(\gamma)(z) := \prod_{m=0}^{\infty} J_{r,s}^{(m)}(\gamma)(z)$$

as a rigid meromorphic function of  $z \in \mathcal{H}_p^{\leq n}$ , it suffices to show that the restriction of  $J_{r,s}^{(m)}(\gamma)$  to  $\mathcal{H}_p^{\leq n}$  converges uniformly to 1 as  $m \rightarrow \infty$ . But this follows directly from Lemma 2.2. We have hence showed that the infinite product defining  $J_{r,s}(z)$  converges absolutely and uniformly on affinoid subsets of  $\mathcal{H}_p$ . To verify the cocycle condition for  $J_{r,s}$  modulo scalars, observe that

$$\begin{aligned} J_{r,s}(\gamma_1\gamma_2)(z) &= \prod_{(u,v) \in \Sigma_{r,s}} [(z - (\eta)); (u) - (v)]^{[u,v] \cdot [\xi, \gamma_1\gamma_2\xi]} \\ &= \prod_{(u,v) \in \Sigma_{r,s}} [(z - (\eta)); (u) - (v)]^{[u,v] \cdot [\xi, \gamma_1\xi]} \\ &\quad \times \prod_{(u,v) \in \Sigma_{r,s}} [(z - (\eta)); (u) - (v)]^{[u,v] \cdot [\gamma_1\xi, \gamma_1\gamma_2\xi]} \\ (37) \qquad \qquad &= J_{r,s}(\gamma_1)(z) \times \Pi, \end{aligned}$$

where

$$\Pi := \prod_{(u,v) \in \Sigma_{r,s}} [(z - (\eta)); (u) - (v)]^{[u,v] \cdot [\gamma_1\xi, \gamma_1\gamma_2\xi]}.$$

Since  $\Sigma_{r,s}$  is stable under  $\Gamma$ , one can replace  $(u, v)$  by  $(\gamma_1 u, \gamma_1 v)$  in the infinite product defining  $\Pi$  to obtain

$$\Pi = \prod_{(u,v) \in \Sigma_{r,s}} [(z - (\eta)); (\gamma_1 u) - (\gamma_1 v)]^{[\gamma_1 u, \gamma_1 v] \cdot [\gamma_1\xi, \gamma_1\gamma_2\xi]}$$

$$\begin{aligned}
 &= \prod_{(u,v) \in \Sigma_{r,s}} [(\gamma_1^{-1}z) - (\gamma_1^{-1}\eta); (u) - (v)]^{[u,v] \cdot [\xi, \gamma_2 \xi]} \\
 (38) \quad &= J_{r,s}(\gamma_2)(\gamma_1^{-1}z) \pmod{C_p^\times},
 \end{aligned}$$

where the  $\Gamma$ -invariance properties of the cross-ratio and of the intersection product have been invoked to obtain the penultimate identity above. The cocycle property follows from (37) and (38).  $\square$

**Lemma 3.22.** *The class of  $J_{r,s}$  in  $H^1(\Gamma, \mathcal{A}^\times/C_p^\times)$  does not depend on the choice of base point  $\eta \in \mathcal{H}_p$  and of admissible base point  $\xi \in \mathcal{H}$  that were made to define it.*

*Proof.* Changing the base point  $\eta$  to  $\eta'$  merely multiplies the functions  $J_{r,s}(\gamma)$  by a non-zero scalar, and hence does not affect the cocycle  $J_{r,s} \in Z^1(\Gamma, \mathcal{A}^\times/C_p^\times)$ . As for replacing  $\xi$  by  $\xi'$  in the definition of  $J_{r,s}$ , a direct calculation reveals that the associated cocycles differ by the coboundary  $dF$ , where

$$F(z) = \prod_{(u,v) \in \Sigma_{r,s}} [(z) - (\eta); (u) - (v)]^{[u,v] \cdot [\xi, \xi']} \in \mathcal{A}^\times.$$

$\square$

We now ask whether the theta-cocycle  $J_{r,s} \in H^1(\Gamma, \mathcal{A}^\times/C_p^\times)$  lifts to a genuine analytic cocycle

$$\tilde{J}_{r,s} \stackrel{?}{\in} H^1(\Gamma, \mathcal{A}^\times).$$

As mentioned before, the cocycle  $J_{r,s}$  need not admit such a lift, but its restriction to  $\Gamma_\circ := \text{SL}_2(\mathbb{Z})$  does, by the essential triviality of  $H^2(\text{SL}_2(\mathbb{Z}), C_p^\times)$ . To describe such a lift, we first restrict  $J_{r,s}$  to  $H^1(\Gamma_\circ, \mathcal{A}^\times/C_p^\times)$  and construct an explicit lift of it to a class

$$J_{r,s}^\circ \in H^1(\Gamma_\circ, \mathcal{A}^\times).$$

The first part of Lemma 2.2 suggests that replacing  $[(z) - (\eta); (r) - (s)]$  by  $[z; (r) - (s)]$  in the definition of  $J_{r,s}$  leads to an infinite product which need not converge in general. However, we have

**Proposition 3.23.** *For all  $\gamma \in \Gamma_\circ$ , the infinite product*

$$J_{r,s}^\circ(\gamma)(z) := \prod_{m=0}^\infty J_{r,s}^{\circ,(m)}(\gamma)(z),$$



where

$$J_{r,s}^{\circ,(m)}(\gamma)(z) := \prod_{(u,v) \in \Sigma_{r,s}^{(m)}} ([z; (u) - (v)])^{[u,v] \cdot [\xi, \gamma \xi]}$$

converges to a rigid analytic function on  $\mathcal{H}_p$ , up to 12-th roots of unity, and gives rise to an element of  $H^1(\Gamma_{\circ}, \mathcal{A}^{\times} / \mu_{12})$ .

*Proof.* For integers  $m > n \geq 0$ , consider the restriction of  $J_{r,s}^{\circ,(m)}(\gamma)(z)$  to the affinoid  $\mathcal{H}_p^{\leq n}$ . By Lemma 2.2, this restriction is constant modulo  $p^{m-n}$  and hence its mod  $p^{m-n}$  reduction defines a cocycle in  $H^1(\Gamma_{\circ}, (\mathbb{Z}/p^{m-n}\mathbb{Z})^{\times})$ . Since the abelianisation of  $\Gamma_{\circ}$  is of order 12, it follows that

$$J_{r,s}^{\circ,(m)}(\gamma)(z)|_{\mathcal{H}_p^{\leq n}} \in \mu_{12} \pmod{p^{m-n}}.$$

The convergence of the infinite product defining  $J_{r,s}^{\circ}$  (up to 12-th roots of unity) follows. The fact that it is a cocycle for  $\Gamma_{\circ}$  follows from the  $\Gamma_{\circ}$ -equivariance of the expression  $[z; (u) - (v)]$ .  $\square$

This proposition gives some information about the RM values of the theta-cocycle  $J_{r,s}$ :

**Corollary 3.24.** *For all RM points  $\tau \in \mathcal{H}_p$ , the value  $J_{r,s}[\tau]$  belongs to  $\mathcal{O}_{C_p}^{\times}$ .*

*Proof.* If  $\tau$  is an unramified point, we can assume without loss of generality, by translating it by an appropriate element of  $GL_2(\mathbb{Z}[1/p])$ , that it belongs to the standard affinoid  $\mathcal{H}_p^{\circ}$ . The definition of this affinoid implies that

$$[\tau; (r) - (s)] \in \mathcal{O}_{C_p}^{\times},$$

and hence that  $J_{r,s}^{\circ,(m)}(\gamma_{\tau})(\tau)$  is a  $p$ -adic unit, for all  $m$ . The corollary follows in this case. The proof for ramified  $\tau$  proceeds along similar lines.  $\square$

Let us now compute the image of the theta-cocycle  $J_{r,s}$  in  $H^1(\Gamma_0(p), \mathbb{Z})$  under the residue map. Let  $\Sigma_{r,s}^{(00)}$  be the subset of  $\Sigma_{r,s}^{(0)}$  consisting of pairs  $(u, v)$  satisfying

$$u \notin \mathbb{Z}_p, \quad v \in \mathbb{Z}_p.$$

This subset is nonempty, stable under the action of  $\Gamma_0(p)$ , and breaks up as a finite union of  $\Gamma_0(p)$ -orbits:

$$\Sigma_{r,s}^{(00)} = \Gamma_0(p)(r_1, s_1) \cup \dots \cup \Gamma_0(p)(r_t, s_t).$$

Let  $[r_j, s_j]_{X_0(p)}$  be the image of the geodesic path  $[r_j, s_j]$  on  $\mathcal{H}^*$  in the relative homology of  $X_0(p)$  relative to the cusps. It does not depend on the

choice of orbit representatives, and their sum depends only on  $(r, s)$ . Let  $\varphi_{r,s} : \Gamma_0(p) \rightarrow \mathbb{Z}$  be the homomorphism defined by

$$\varphi_{r,s}(\gamma) = ([r_1, s_1]_{X_0(p)} + \cdots + [r_t, s_t]_{X_0(p)}) \cdot \gamma,$$

where  $\cdot$  denotes the intersection pairing

$$H_1(X_0(p); \{0, \infty\}, \mathbb{Z}) \times H_1(Y_0(p), \mathbb{Z}) \rightarrow \mathbb{Z}.$$

**Proposition 3.25.** *The image of  $J_{r,s}$  under the residue map is equal to  $2\varphi_{r,s}$ .*

*Proof.* Recall the standard annulus  $U$  of (8) having  $\Gamma_0(p)$  as its stabiliser in  $\Gamma$ . The residue map takes a cocycle  $J \in H^1(\Gamma, \mathcal{A}^\times / C_p^\times)$  to the homomorphism

$$\phi_J : \Gamma_0(p) \rightarrow \mathbb{Z}, \quad \phi_J(\gamma) := \text{res}_U(\text{dlog} J(\gamma)),$$

where  $\text{res}_U$  is the  $p$ -adic annular residue attached to  $U$ . Consider the infinite product expression of Proposition 3.23 for  $J_{r,s}$  and observe that the terms  $\text{dlog} J_{r,s}^{(m)}(\gamma)$  for  $m \geq 1$  contribute nothing to the annular residue at  $U$ : indeed, two cusps  $u, v$  for which  $p \mid \det(u, v)$  belong to the same component of the complement of  $U$ , and hence

$$\text{res}_U(\text{dlog}[z; (u) - (v)]) = 0$$

for such pairs. On the other hand, if  $(u, v)$  belongs to  $\Sigma_{r,s}^{(0)}$ , then

$$\text{res}_U(\text{dlog}[z; (u) - (v)]) = \begin{cases} 1 & \text{if } u \notin \mathbb{Z}_p, v \in \mathbb{Z}_p, \\ -1 & \text{if } u \in \mathbb{Z}_p, v \notin \mathbb{Z}_p, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, any pair  $(u, v)$  for which the residue of  $\text{dlog}[z; (u) - (v)]$  is equal to 1 (resp.  $-1$ ) is of the form  $(\alpha r_j, \alpha s_j)$ , (resp.  $(\alpha s_j, \alpha r_j)$ ) for some  $\alpha \in \Gamma_0(p)$ . It follows that

$$\begin{aligned} \text{res}_U(\text{dlog} J_{r,s}(\gamma)) &= \sum_{j=1}^t \sum_{\alpha \in \Gamma_0(p)} (+1)[\alpha r_j, \alpha s_j] \cdot [\xi, \gamma \xi] \\ &\quad + \sum_{\alpha \in \Gamma_0(p)} (-1)[\alpha s_j, \alpha r_j] \cdot [\xi, \gamma \xi] \\ &= 2 \sum_{j=1}^t \sum_{\alpha \in \Gamma_0(p)} [\alpha r_j, \alpha s_j] \cdot [\xi, \gamma \xi]. \end{aligned}$$

The sum in the last expression is the intersection product of the relative homology class attached to  $[r_1, s_1] + \dots + [r_j, s_j]$  with the class of  $\gamma$  in  $H_1(Y_0(p), \mathbb{Z})$ . The proposition follows.  $\square$

### 4. Meromorphic theta-symbols

Following [DV21], this section recalls the construction of certain *meromorphic* theta-symbols

$$J_\tau \in \text{MS}^\Gamma(\mathcal{M}^\times / C_p^\times)$$

which are indexed by orbits of RM points  $\tau \in \Gamma \backslash \mathcal{H}_p^{\text{RM}}$  and generate, along with the space of analytic theta symbols  $\text{MS}^\Gamma(\mathcal{A}^\times / C_p^\times)$ , the group of meromorphic theta-symbols up to torsion.

The RM values of meromorphic theta symbols appear to be of great arithmetic significance, and yield in particular the singular moduli for real quadratic fields that were introduced in [DV21]. In the current paper, we investigate instead their *toric values*, defined in §5, which are of central importance to the Gross–Kohnen–Zagier theorem obtained in §6.

#### 4.1. Divisors

Let  $\tau \in \mathcal{H}_p(C_p)$  be an RM point of discriminant  $D$ , and let  $F$  be the real quadratic field that it generates. The element  $\tau$  is the root of a unique (up to sign) primitive integral binary quadratic form, whose discriminant is of the form  $Dp^{2n}$  with  $n \geq 0$ . Our fixed complex and  $p$ -adic embedding of  $\mathbb{Q}$  allows us to view  $F$  simultaneously as a subfield of  $\mathbb{R}$  and of  $C_p$ .

For  $w \in F$ , let  $(w, w')$  denote the geodesic on  $\mathcal{H}$  joining  $w$  to its Galois conjugate  $w'$ , and let  $\Pi \subset \Gamma$  be an arithmetic subgroup. The geodesic  $(w, w')$  can be written as an infinite union

$$(w, w') = \cup_{j=-\infty}^\infty [\gamma_w^j \xi, \gamma_w^{j+1} \xi],$$

where  $\gamma_w$  is a generator of the stabiliser of  $w$  in  $\Pi$  having  $w'$  as an attractive fixed point and  $w$  as a repulsive fixed point, and  $\xi$  is an arbitrary base point on  $(w, w')$ . Each geodesic in this decomposition maps to the same basic closed geodesic of  $\Pi \backslash \mathcal{H}$ . Likewise, if  $r, s$  are elements of  $\mathbb{P}_1(\mathbb{Q})$ , recall that  $[r, s]$  denotes the hyperbolic geodesic on  $\mathcal{H}$  joining these two elements, which maps to a compact (but not necessarily closed) geodesic on any quotient  $\Pi \backslash \mathcal{H}$ . The

geodesics  $(w, w')$  and  $[r, s]$  always intersect properly, and we set

$$(39) \quad \delta_{r,s}(w) := (w, w') \cdot [r, s] = \begin{cases} 1 & \text{if the two geodesics intersect positively;} \\ -1 & \text{if they intersect negatively;} \\ 0 & \text{otherwise.} \end{cases}$$

The infinite formal sum

$$(40) \quad \Delta_\tau\{r, s\} := \sum_{w \in \Gamma\tau} \delta_{r,s}(w) \cdot (w)$$

defines a  $\Gamma$ -invariant modular symbol with values in the  $\Gamma$ -module  $\mathbb{Z}\langle\Gamma\tau\rangle$  of (possibly infinite)  $\mathbb{Z}$ -linear combinations of points of  $\Gamma\tau$ . Set

$$\Delta_\tau^{\leq n}\{r, s\} := \sum_{w \in (\Gamma\tau) \cap \mathcal{H}_p^{\leq n}} \delta_{r,s}(w) \cdot (w),$$

and define  $\Delta_\tau^{\leq n}\{r, s\}$  analogously, replacing  $\mathcal{H}_p^{\leq n}$  by  $\mathcal{H}_p^{< n}$ .

**Lemma 4.1.** *The expressions  $\Delta_\tau^{\leq n}\{r, s\}$  and  $\Delta_\tau^{< n}\{r, s\}$  are degree zero divisors on  $\mathcal{H}_p$ .*

*Proof.* This is explained in [DV21, Lemma 1.21], but the argument is reproduced here with the current notations for the convenience of the reader. The RM points  $w$  in the support of  $\Delta_\tau^{\leq n}\{r, s\}$  consist of zeroes of binary quadratic forms of discriminant  $Dp^{2m}$  with

$$1/2\text{ord}_p(D) + m \leq n$$

for which  $\delta_{r,s}(w) \neq 0$ , i.e., for which the geodesic  $(w, w')$  intersects  $[r, s]$  non-trivially. There are finitely many binary quadratic forms of a given discriminant satisfying this condition, and hence  $\Delta_\tau^{\leq n}\{r, s\}$  belongs to  $\text{Div}(\mathcal{H}_p^{\text{RM}})$  for all  $(r, s) \in \mathbb{P}_1(\mathbb{Q})^2$ . Since  $\mathcal{H}_p^{\leq n}$  is preserved by the action of  $\text{SL}_2(\mathbb{Z})$ , the function

$$(r, s) \mapsto \deg \Delta_\tau^{\leq n}\{r, s\}$$

defines an element in  $\text{MS}^{\text{SL}_2(\mathbb{Z})}(\mathbb{Z})$ . Since this space of modular symbols is trivial, it follows that  $\deg \Delta_\tau^{\leq n}\{r, s\}$  is identically zero for all  $n, \tau$  and  $(r, s)$ . The proof for  $\Delta_\tau^{< n}\{r, s\}$  is the same.  $\square$

If  $v$  is a vertex of  $\mathcal{T}$ , recall the wide open subset  $\mathcal{W}_v$  of  $\mathcal{H}_p$  associated to  $v$ , consisting of the points whose image under the reduction map is either  $v$

or an edge containing  $v$ . The same reasoning as in the proof of Lemma 4.1 shows that the expression

$$\Delta_\tau^v\{r, s\} := \sum_{w \in (\Gamma\tau) \cap \mathcal{W}_v} \delta_{r,s}(w) \cdot (w)$$

is a degree zero divisor. If  $\tau$  is unramified, then for all  $n \geq 0$  we have

$$\Delta_\tau^{\leq n}\{r, s\} = \sum_{v \in \mathcal{V}(\mathcal{T}^{\leq n})} \Delta_\tau^v\{r, s\}.$$

When  $\tau$  is a ramified RM point, and hence reduces to an edge of  $\mathcal{T}$ , the situation is a bit more subtle. Namely, we can write  $\Delta_\tau^{\leq n}\{r, s\} = \Delta_\tau^{\leq n}\{r, s\}$  as

$$\Delta_\tau^{\leq n}\{r, s\} = \sum_{v \in \mathcal{V}(\mathcal{T}^{< n})} \Delta_\tau^v\{r, s\},$$

with the righthand sum ranging only over the *even* vertices of  $\mathcal{T}$  when  $n$  is odd, and over the odd vertices of  $\mathcal{T}$ , when  $n$  is even.

### 4.2. Definitions and convergence

To define the rigid meromorphic theta symbols  $J_\tau$ , one starts from the association  $(r, s) \mapsto \Delta_\tau^{\leq n}\{r, s\}$  which yields, by Lemma 4.1, an element of the group of  $\text{SL}_2(\mathbb{Z})$ -invariant modular symbols valued in divisors:

$$\text{MS}^{\text{SL}_2(\mathbb{Z})}(\text{Div}^0(\mathcal{H}_p^{\leq n})).$$

These modular symbols are now upgraded to meromorphic theta symbols. Since the combinatorial structures are different, the cases where  $\tau$  is unramified or ramified are treated separately.

Suppose first that the RM point  $\tau$  in  $\mathcal{H}_p$  is *unramified*. Recall the symbols  $[\mathcal{D}_1, \mathcal{D}_2]$  defined in (12) for a pair of divisors  $\mathcal{D}_1, \mathcal{D}_2$ . Fix an auxiliary base point  $\eta \in \mathcal{H}_p$ .

**Lemma 4.2.** *Assume that  $\tau$  is unramified. The rational functions*

$$J_\tau^{\leq n}\{r, s\}(z) := [(z) - (\eta); \Delta_\tau^{\leq n}\{r, s\}]$$

*converge uniformly on affinoid subsets of  $\mathcal{H}_p$  to a rigid analytic function  $J_\tau\{r, s\}(z)$  on  $\mathcal{H}_p$ .*

*Proof.* Let  $\mathcal{A} \subset \mathcal{H}_p^{\leq N}$  be some affinoid subset of  $\mathcal{H}_p$ , and assume without loss of generality (by enlarging  $\mathcal{A}$  if necessary) that  $\eta \in \mathcal{A}$ . For all  $N < n < m$ ,

$$(41) \quad J_\tau^{\leq m}\{r, s\}(z) \div J_\tau^{\leq n}\{r, s\}(z) = [(z) - (\eta); \Delta_\tau^{\leq m}\{r, s\} - \Delta_\tau^{\leq n}\{r, s\}] \\ = \prod_{n < d(v, v_0) \leq m} [(z) - (\eta); \Delta_\tau^v\{r, s\}].$$

By Lemma 2.1, each factor in this product belongs to  $1 + p^{n-N}\mathcal{O}_{C_p}$  as  $z$  ranges over  $\mathcal{H}_p^{\leq N} \supset \mathcal{A}$ . It follows that the sequence  $(J_\tau^{\leq n}\{r, s\})_{n \geq 1}$  of partial products is uniformly Cauchy when restricted to  $\mathcal{A}$ , and hence converges to a rigid meromorphic function on this domain.  $\square$

The proof of Lemma 4.2 yields the following corollary giving a more flexible expression for the rigid meromorphic function  $J_\tau\{r, s\}$  when  $\tau$  is unramified. We recall that a *rational affinoid* is an affinoid subset of  $\mathcal{H}_p$  which can be expressed as the inverse image under the reduction map of a (finite) subgraph of  $\mathcal{T}$ . For instance, any  $\Gamma$ -translate of  $\mathcal{H}_p^{\leq n}$  is a rational affinoid.

**Corollary 4.3.** *Assume that  $\tau$  is unramified. Suppose that  $\mathcal{H}_p^{[n]}$  is an increasing sequence of rational affinoids satisfying  $\mathcal{H}_p = \cup_{n \geq 1} \mathcal{H}_p^{[n]}$ , and let*

$$\Delta_\tau^{[n]}\{r, s\} := \sum_{w \in (\Gamma\tau) \cap \mathcal{H}_p^{[n]}} \delta_{r,s}(w) \cdot (w).$$

*Then  $\Delta_\tau^{[n]}\{r, s\}$  is a degree zero divisor and*

$$J_\tau\{r, s\} = \lim_{n \rightarrow \infty} [(z) - (\eta); \Delta_\tau^{[n]}\{r, s\}].$$

Suppose now that the RM point  $\tau$  in  $\mathcal{H}_p$  is *ramified*. In this case, Lemma 4.2 and Corollary 4.3 cease to be true, since the sequence  $(J_\tau^{\leq n}\{r, s\})_{n \geq 0}$  need not converge uniformly on affinoid subsets. The next lemma shows that it is made up of two a priori distinct convergent subsequences

$$\{J_\tau^{\leq n}\{r, s\}(z)\}_{n \text{ even}}, \quad \{J_\tau^{\leq n}\{r, s\}(z)\}_{n \text{ odd}}.$$

**Lemma 4.4.** *Assume that  $\tau$  is a ramified RM point. The rational functions*

$$J_\tau^{\leq n}\{r, s\}(z) := [(z) - (\eta); \Delta_\tau^{\leq n}\{r, s\}]$$

*as  $n$  ranges over the odd (resp. even) integers converge uniformly on affinoid subsets of  $\mathcal{H}_p$  to rigid analytic functions  $J_\tau^+\{r, s\}(z)$  (resp.  $J_\tau^-\{r, s\}(z)$ ) on  $\mathcal{H}_p$ .*

*Proof.* The uniform convergence of each subsequence follows from the identity, generalising (41)

$$(42) \quad J_{\tau}^{\leq n+2}\{r, s\}(z) \div J_{\tau}^{\leq n}\{r, s\}(z) = \prod_{d(v, v_{\circ})=n+1} [(z) - (\eta); \Delta_{\tau}^v\{r, s\}].$$

□

One can give a more intrinsic expression for the rigid meromorphic functions  $J_{\tau}^{\pm}\{r, s\}$  in terms of more general affinoid coverings of  $\mathcal{H}_p$ . Namely, a closed, connected, finite subgraph  $\mathcal{G}$  of  $\mathcal{T}$  is said to be *full* if every vertex  $v$  of  $\mathcal{G}$  is either of degree 1 or  $p + 1$ . The vertices that are of degree 1 are called the *boundary vertices* of  $\mathcal{G}$ . The graph  $\mathcal{G}$  is then said to be *even* (resp. *odd*) if every boundary vertex of  $\mathcal{G}$  is odd (resp. even). An affinoid region in  $\mathcal{H}$  is said to be even or odd if it is the inverse image under the reduction map  $\text{red}$  of a full subgraph of  $\mathcal{T}$  of the same parity. The set of even and odd affinoid subsets is preserved by the action of  $\Gamma$ , while elements of  $\text{GL}_2(\mathbb{Z}[1/p])$  of determinant  $p$  interchange these two types of affinoids. When  $\tau$  is ramified, and  $\mathcal{A}$  is an even (resp. odd) affinoid subset, the divisor  $\Delta_{\tau}^{\mathcal{A}}\{r, s\}$  can be uniquely expressed as a sum of the divisors  $\Delta_{\tau}^v\{r, s\}$  as  $v$  ranges over the even (resp. odd) vertices of  $\mathcal{T}$  that lie in  $\text{red}(\mathcal{A})$ .

The following lemma is the counterpart of Corollary 4.3 when  $\tau$  is ramified:

**Lemma 4.5.** *Assume that  $\tau$  is a ramified RM point. Let  $\mathcal{H}_p^{[n]}$  be any increasing sequence of even rational affinoids satisfying  $\mathcal{H}_p = \cup_{n \geq 1} \mathcal{H}_p^{[n]}$ , and let*

$$\Delta_{\tau}^{[n]}\{r, s\} := \sum_{w \in (\Gamma\tau) \cap \mathcal{H}_p^{[n]}} \delta_{r,s}(w) \cdot (w).$$

*Then  $\Delta_{\tau}^{[n]}\{r, s\}$  is a degree zero divisor and*

$$J_{\tau}^{+}\{r, s\} = \lim_{n \rightarrow \infty} [(z) - (\eta); \Delta_{\tau}^{[n]}\{r, s\}].$$

*A similar statement holds for  $J_{\tau}^{-}$ , with even affinoids replaced by odd ones.*

The proof of this lemma proceeds along the same lines as that of Cor. 4.3, and is therefore omitted.

### 4.3. Basic properties

A number of formal and foundational properties on the collections of rigid meromorphic functions  $J_{\tau}\{r, s\}$  will now be established.

**Lemma 4.6.** *Suppose  $\tau$  is unramified. The functions  $J_\tau\{r, s\}(z)$  satisfy the following properties:*

- (a) *The assignment  $(r, s) \mapsto J_\tau\{r, s\}$  is a modular symbol with values in  $\mathcal{M}^\times$ .*
- (b) *The rigid meromorphic function  $J_\tau\{r, s\}$  is independent, up to multiplication by a non-zero scalar, of the choice of base point  $\eta$ .*
- (c) *The modular symbol  $J_\tau$  satisfies the  $\Gamma$ -invariance property*

$$J_\tau\{\gamma r, \gamma s\}(\gamma z) = J_\tau\{r, s\}(z), \pmod{C_p^\times}, \quad \text{for all } \gamma \in \Gamma.$$

*Proof.* Property (a) of  $J_\tau\{r, s\}$  follows from the similar property of the divisors  $\Delta_\tau^{\leq n}\{r, s\}$ , while (b) follows from the fact that the functions  $J_\tau^\eta\{r, s\}$  and  $J_\tau^{\eta'}\{r, s\}$  defined using different base points  $\eta$  and  $\eta'$  satisfy

$$J_\tau^\eta\{r, s\} = \lambda J_\tau^{\eta'}\{r, s\}, \quad \text{where } \lambda = \lim_{n \rightarrow \infty} [(\eta') - (\eta); \Delta_\tau^{\leq n}\{r, s\}].$$

Property (c) holds because

$$(43) \quad J_\tau\{\gamma r, \gamma s\}(\gamma z) = \lim_{n \rightarrow \infty} [(\gamma z) - (\eta); \Delta_\tau^{\leq n}\{\gamma r, \gamma s\}]$$

$$(44) \quad = \lim_{n \rightarrow \infty} [\gamma((z) - (\gamma^{-1}\eta)); \gamma \Delta_\tau^{\mathcal{A}_n}\{r, s\}],$$

where  $\mathcal{A}_n := \gamma^{-1}(\mathcal{H}_p^{\leq n})$  and

$$\Delta_\tau^{\mathcal{A}_n}\{r, s\} := \sum_{w \in (\Gamma\tau) \cap \mathcal{A}_n} \delta_{r,s}(w)(w).$$

The  $\Gamma$ -equivariance of the symbol  $[\mathcal{D}_1; \mathcal{D}_2]$  implies that

$$J_\tau\{\gamma r, \gamma s\}(\gamma z) = \lim_{n \rightarrow \infty} [(z) - (\gamma^{-1}\eta); \Delta_\tau^{\mathcal{A}_n}\{r, s\}].$$

Property (c) now follows from Corollary 4.3, in light of the fact that  $\mathcal{H}_p = \cup_{n \geq 1} \mathcal{A}_n$ , and from part (b) of Lemma 4.6. □

Likewise, one establishes with the same proofs the following ramified counterpart:

**Lemma 4.7.** *Suppose  $\tau \in \mathcal{H}_p$  is a ramified RM point. The elements  $J_\tau^+$  and  $J_\tau^-$  belong to  $\text{MS}^\Gamma(\mathcal{M}^\times/C_p^\times)$ .*

*Proof.* The argument proceeds exactly as in the proof of Lemma 4.6. □



It is instructive to examine the difference between the two rigid meromorphic theta-symbols  $J_\tau^+$  and  $J_\tau^-$  attached to a ramified RM point  $\tau$ . To this end, we associate to  $\tau$  a  $\Gamma_0(p)$ -invariant modular symbol  $m_\tau \in \text{MS}^{\Gamma_0(p)}(\mathbb{Z})$  by setting

$$m_\tau\{r, s\} := \text{deg}(\Delta_\tau^U\{r, s\}) := \sum_{w \in \Gamma\tau \cap U} \delta_{r,s}(w),$$

where  $U$  is the standard annulus whose stabiliser in  $\Gamma$  is equal to  $\Gamma_0(p)$ . The following lemma is an easy consequence of the definitions.

**Lemma 4.8.** *For all ramified RM points  $\tau \in \mathcal{H}_p$ , the difference between  $J_\tau^+$  and  $J_\tau^-$  is the rigid analytic theta-symbol given by*

$$J_\tau^+ \div J_\tau^- = \text{ST}^\times(m_\tau).$$

When  $\tau$  is ramified, it is natural to define the theta-symbol  $J_\tau$  by averaging over  $J_\tau^+$  and  $J_\tau^-$ .

**Definition 4.9.** Let  $\tau$  be a ramified RM point in  $\mathcal{H}_p$ . The theta-symbol attached to  $\tau$  is defined by

$$J_\tau\{r, s\} := (J_\tau^+\{r, s\} \times J_\tau^-\{r, s\})^{1/2} = J_\tau^\pm\{r, s\} \times \text{ST}^\times(m_\tau)^{\mp 1/2}.$$

Lemmas 4.6 and 4.7 imply that  $J_\tau$  is a  $\Gamma$ -invariant modular symbol with values in  $\mathcal{M}^\times/C_p^\times$ . It is called the *meromorphic theta-symbol* associated to  $\tau \in \Gamma \backslash \mathcal{H}_p^{\text{RM}}$ .

**Proposition 4.10.** *Up to elements of  $\text{MS}^\Gamma(\mathcal{A}^\times/C_p^\times)$ , every class in  $\text{MS}^\Gamma(\mathcal{M}^\times/C_p^\times)$  can be expressed as a finite product of cocycles of the form  $J_\tau$ , as  $\tau$  ranges over  $\Gamma \backslash \mathcal{H}_p^{\text{RM}}$ .*

*Proof.* It is shown in [DV21] that the full collection of theta symbols  $J_\tau$ , as  $\tau$  ranges over the distinct  $\Gamma$ -orbits of both unramified and ramified RM points, generates  $\text{MS}^\Gamma(\mathcal{M}^\times/C_p^\times)$  up to analytic theta-symbols. More precisely, [DV21, Theorem 1.24] makes the corresponding additive assertion for classes in  $\text{MS}^\Gamma(\mathcal{A}_2)$ , which are related to classes in  $\text{MS}^\Gamma(\mathcal{A}^\times/C_p^\times)$  by the logarithmic derivative map

$$\text{dlog} : \text{MS}^\Gamma(\mathcal{A}^\times/C_p^\times) \longrightarrow \text{MS}^\Gamma(\mathcal{A}_2).$$

□

### 5. Lifting obstructions

#### 5.1. Definition and basic properties

Applying the exact functor  $\Omega \mapsto \text{MS}(\Omega)$  to the short exact sequences

$$\begin{aligned} 1 &\longrightarrow C_p^\times \longrightarrow \mathcal{A}^\times \longrightarrow \mathcal{A}^\times/C_p^\times \longrightarrow 1, \\ 1 &\longrightarrow C_p^\times \longrightarrow \mathcal{M}^\times \longrightarrow \mathcal{M}^\times/C_p^\times \longrightarrow 1 \end{aligned}$$

and then taking their  $\Gamma$ -cohomology leads to the commutative diagram in which the rows are exact up to finite subgroups, and the vertical maps are injective:

$$(45) \quad \begin{array}{ccccccc} & & & \text{MS}^\Gamma(\mathcal{A}^\times/C_p^\times) & \xlongequal{\text{Obs}} & \Pi & \\ & & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \text{MS}^\Gamma(\mathcal{M}^\times) & \longrightarrow & \text{MS}^\Gamma(\mathcal{M}^\times/C_p^\times) & \xrightarrow{\text{Obs}} & T(C_p), \end{array}$$

where  $T$  is the split torus over  $C_p$  whose character group is identified with the free  $\mathbb{Z}$ -module

$$X(T) := \text{hom}(\text{H}^1(\Gamma, \text{MS}(\mathbb{Z})), \mathbb{Z}),$$

(equipped with the trivial action of  $\text{Gal}(\bar{C}_p/C_p)$ ), and  $\Pi$  is the natural image of  $\text{MS}^\Gamma(\mathcal{A}^\times/C_p^\times)$  in  $T(C_p)$  under the connecting homomorphism.

**Definition 5.1.** If  $J$  belongs to  $\text{MS}^\Gamma(\mathcal{M}^\times/C_p^\times)$ , its image under Obs is called the *parabolic lifting obstruction* attached to  $J$ .

The terminology is justified by the fact that a power of the theta-symbol  $J$  lifts to a genuine rigid meromorphic modular symbol precisely when its lifting obstruction vanishes.

The next lemma analyses the rightmost column of diagram (45).

**Lemma 5.2.** *There is a natural injection with torsion cokernel*

$$\vartheta : \text{MS}^{\Gamma_0(p)}(\mathbb{Z}) \longrightarrow \text{hom}(X(T), \mathbb{Z}).$$

*In particular, the character group  $X(T)$  is isomorphic to  $\mathbb{Z}^{2g+1}$ , where  $g$  is the genus of the modular curve  $X_0(p)$ . The subgroup  $\Pi$  is a lattice in  $T(C_p)$  (i.e., a discrete subgroup of  $T(C_p)$  of rank  $2g + 1$ ), whose tensor product with  $\mathbb{Q}$  is isomorphic to  $X(T) \otimes \mathbb{Q}$  as a Hecke module.*

*Proof.* The asserted injection  $\vartheta$  is the first in the long exact cohomology sequence

$$\mathrm{MS}^{\Gamma_0(p)}(\mathbb{Z}) \xrightarrow{\vartheta} \mathrm{H}^1(\Gamma, \mathrm{MS}(\mathbb{Z})) \longrightarrow \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), \mathrm{MS}(\mathbb{Z}))^2 \longrightarrow \mathrm{H}^1(\Gamma_0(p), \mathrm{MS}(\mathbb{Z})),$$

which arises from the second exact sequence of [Se80, II. §2.8, Prop. 13] applied to the  $\Gamma$ -module  $\mathrm{MS}(\mathbb{Z})$  and to the action of  $\Gamma$  on the Bruhat–Tits tree of  $\mathrm{PGL}_2(\mathbb{Q}_p)$ , whose edge and vertex stabilisers are isomorphic to  $\Gamma_0(p)$  and  $\mathrm{SL}_2(\mathbb{Z})$  respectively, and whose fundamental region consists of a single closed edge. The kernel is a quotient of  $\mathrm{H}^0(\mathrm{SL}_2(\mathbb{Z}), \mathrm{MS}(\mathbb{Z}))^2$ , which is trivial, while the cokernel injects into  $\mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), \mathrm{MS}(\mathbb{Z}))^2$ , which is finite.

The fact that  $\Pi$  is a lattice in  $T(C_p)$  follows from the fact that the composition

$$\mathrm{MS}^{\Gamma_0(p)}(\mathbb{Z}) \xrightarrow{\mathrm{ST}^\times} \mathrm{MS}^\Gamma(\mathcal{A}^\times/C_p^\times) \xrightarrow{\mathrm{Obs}} \mathrm{H}^1(\Gamma, \mathrm{MS}(C_p^\times)) \xrightarrow{\vartheta^{-1}} \mathrm{MS}^{\Gamma_0(p)}(C_p^\times) \xrightarrow{\mathrm{ord}_p} \mathrm{MS}^{\Gamma_0(p)}(\mathbb{Z})$$

is the identity. □

The lattice  $\Pi$  and the torus  $T$  are closely related to the invariants defined and studied in [Das05] in a more general setting.

Recall that Proposition 4.10 shows that the natural divisor map on theta-symbols, which sends a theta-symbol to the  $\Gamma$ -orbit of the divisor of its values, provides a short exact sequence

$$1 \longrightarrow \mathrm{MS}^\Gamma(\mathcal{A}^\times/C_p^\times) \longrightarrow \mathrm{MS}^\Gamma(\mathcal{M}^\times/C_p^\times) \xrightarrow{\mathrm{Div}} \mathrm{Div}(\Gamma \backslash \mathcal{H}_p^{\mathrm{RM}}) \longrightarrow 1.$$

The following lemma provides a canonical splitting of this sequence.

**Lemma 5.3.** *If  $R \in \mathrm{Div}(\Gamma \backslash \mathcal{H}_p^{\mathrm{RM}})$  is an RM divisor, there is a unique theta-symbol  $J_R$  satisfying*

- (i)  $\mathrm{Divisor}(J_R) = R$ ;
- (ii)  $\mathrm{Obs}(J_R) \in \mathrm{H}^1(\Gamma, \mathrm{MS}(\mathcal{O}_{C_p}^\times))$ .

*Proof.* For  $\tau \in \mathcal{H}_p^{\mathrm{RM}}$ , the theta symbol  $J_\tau$  has divisor  $(\tau)$ , and hence, it is possible to construct a  $J_R^0$  with RM divisor equal to  $R$ . The theta-symbol  $J_R$  can then be defined by

$$J_R := J_R^0 \div K_R,$$

where we define

$$K_R := \mathrm{ST}^\times(\mathrm{ord}_p(\vartheta^{-1}(\mathrm{Obs}(J_R^0)))) \in \mathrm{MS}^\Gamma(\mathcal{A}^\times/C_p^\times).$$

Since  $K_R$  is analytic, the divisor of  $J_R$  is also  $R$ . On the other hand, since  $\text{Obs}(J_R^0)$  and  $\text{Obs}(K_R)$  have the same image under the map  $\text{ord}_p \circ \vartheta^{-1}$ , we see that  $\text{Obs}(J_R)$  is an element of  $H^1(\Gamma, \text{MS}(\mathcal{O}_{C_p}^\times))$ . The uniqueness of  $J_R$  satisfying these properties (i) and (ii) follows from the fact that  $\Pi$  is a lattice and hence that there are no non-trivial analytic theta cocycles  $J$  with  $\text{Obs}(J) \in H^1(\Gamma, \text{MS}(\mathcal{O}_{C_p}^\times))$ .  $\square$

The parabolic lifting obstruction attached to  $J$  is calculated by lifting  $J$  to a (not necessarily  $\Gamma$ -invariant)  $\mathcal{M}^\times$ -valued modular symbol  $\tilde{J}$ , and setting

$$(46) \quad \text{Obs}(J)(\gamma)\{r, s\} = \frac{\tilde{J}\{\gamma r, \gamma s\}(\gamma z)}{\tilde{J}\{r, s\}(z)}.$$

Note that the ratio on the right belongs to  $C_p^\times$  since the modular symbol  $\tilde{J}$  is  $\Gamma$ -invariant modulo multiplicative scalars.

### 5.2. Toric values

The quantities on the right of (46) depend on the choice of lift  $\tilde{J}$ , and indeed classes in  $H^1(\Gamma, \text{MS}(C_p^\times))$  need not admit a canonical representative 1-cocycle. It is useful to associate to such a class some well-defined numerical invariants arising from  $\mathbb{Q}$ -split tori in  $\Gamma$ . More precisely, let  $(r, s)$  be an ordered pair of elements of  $\mathbb{P}_1(\mathbb{Q})^2$ . The stabiliser  $\Gamma_{r,s}$  of  $(r, s)$  in  $\Gamma$  is the set of  $\mathbb{Z}[1/p]$ -valued points of a global split torus in  $\text{GL}_2(\mathbb{Q})$ , and is generated by a – unique, up to torsion – loxodromic element  $\gamma_{rs}$  having  $r$  as an attractive and  $s$  as a repulsive fixed point. Given a 1-cocycle  $\kappa \in Z^1(\Gamma, \text{MS}(C_p^\times))$ , the quantity

$$\kappa[r, s] := \kappa(\gamma_{rs})\{r, s\} \in C_p^\times$$

is trivial on coboundaries, and therefore leads to a well-defined numerical invariant.

**Definition 5.4.** The quantity  $\kappa[r, s]$  is called the *toric value* of  $\kappa \in H^1(\Gamma, \text{MS}(C_p^\times))$  at  $(r, s)$ .

The following lemma gives a simple formula for  $\text{Obs}(J)[r, s]$ .

**Lemma 5.5.** For all  $(r, s) \in \mathbb{P}_1(\mathbb{Q})^2$ ,

$$(47) \quad \text{Obs}(J)[r, s] = \tilde{J}\{r, s\}(\gamma_{rs}z) \div \tilde{J}\{r, s\}(z),$$

where  $\tilde{J}\{r, s\}$  is any rigid meromorphic function lifting  $J\{r, s\}(z)$ .

*Proof.* This is an immediate consequence of the definitions. It is worth reiterating the useful fact that the right hand side in (47) does not depend on the choice of lift  $\tilde{J}\{r, s\}$ , and hence that any choice would do.  $\square$

The following proposition asserts that the non-triviality of a class  $\kappa \in H^1(\Gamma, MS(C_p^\times))$  can be detected from its toric values.

**Proposition 5.6.** *Let  $\kappa$  be a class in  $H^1(\Gamma, MS(C_p^\times))$ . If  $\text{ord}_p \kappa[r, s] = \log_p \kappa[r, s] = 0$  for all  $(r, s) \in \mathbb{P}_1(\mathbb{Q})^2$ , then  $\kappa$  is a torsion class.*

*Proof.* If  $T$  is a Hecke operator and  $J \in H^1(\Gamma, MS(C_p^\times))$ , then the toric values of  $T(J)$  can be expressed as linear combinations of other toric values, and hence the kernel of the map toric evaluation map composed with  $\text{ord}_p$  and  $\log_p$ ,

$$(48) \quad H^1(\Gamma, MS(C_p^\times)) \longrightarrow \prod_{(r,s)} \mathbb{Z} \oplus C_p$$

is Hecke stable. Since the Hecke algebra acts semi-simply, the kernel is non-trivial if and only if it contains a Hecke eigenclass. This eigenclass cannot be a power of the lifting obstruction of the boundary symbol of (34), since

$$\text{Obs}(J_{\sharp})[0, \infty] = p^2.$$

Hence, it must be associated to a weight two cusp form  $f$  on  $\Gamma_0(p)$ . Denote by  $\kappa_f$  this eigenclass with trivial toric values. Let  $c$  be an integer prime to  $p$ , and let  $\nu \in \mathbb{Z}/c\mathbb{Z}$ . Proposition 2.16 of [Dar01] shows that the toric value  $\kappa_f[\infty, \nu/c]$ , which in *loc.cit.* is denoted  $W_{\Psi_\nu}$ , depends only on the image of  $\nu$  in the group  $(\mathbb{Z}/c\mathbb{Z})^\times / \langle p^{2\mathbb{Z}} \rangle$ , and that, for any Dirichlet character  $\chi$  of conductor  $c$  satisfying  $\chi(p) = \pm 1$ , we have

$$\begin{aligned} \sum_{\nu \in (\mathbb{Z}/c\mathbb{Z})^\times} \chi(\nu) \text{ord}_p \kappa_f[\infty, \nu/c] &\sim L(f, \chi, 1), \\ \sum_{\nu \in (\mathbb{Z}/c\mathbb{Z})^\times} \chi(\nu) \log_p \kappa_f[\infty, \nu/c] &\sim L(f, \chi, 1), \end{aligned}$$

where  $\sim$  denotes equality up to multiplication by a simple non-zero scalar depending on  $\kappa_f$ . Since there are infinitely many such Dirichlet characters (in fact, infinitely many quadratic ones) for which  $L(f, \chi, 1) \neq 0$ , it follows that there is no such  $\kappa_f$ , and hence that the map (48) is injective.  $\square$

### 5.3. The toric values of the lifting obstruction

An explicit formula for  $\text{Obs}(J_\tau)[r, s]$  will now be obtained, for unramified  $\tau$  in Proposition 5.7, and for ramified  $\tau$  in Proposition 5.9. It will play a central role in the analysis to follow, as well as in the numerical computation of toric values of the lifting obstruction, which give a computational pathway to the Gross–Stark units and Stark–Heegner points discussed in §3.7, as we will see in §5.4.

Recall the degree zero divisors  $\Delta_\tau^{\leq n}\{r, s\}$  supported on  $\mathcal{H}_p^{\leq n}$  that were used to define  $J_\tau\{r, s\}(z)$  as a limit of rational functions. For the calculation that follows, it will be useful to replace the affinoid cover  $\{\mathcal{H}_p^{\leq n}\}_{n \geq 0}$  by another cover involving affinoid domains that are better adapted to studying the action of the matrix  $\gamma_{rs}$ . To this end, consider the infinite path  $\mathcal{Y}(r, s)$  on the Bruhat–Tits tree going from  $r$  to  $s$ , viewed as ends of the tree. When  $p \nmid \det(r, s)$ , this path contains the standard vertex  $v_\circ$ , and consecutive vertices of  $\mathcal{Y}(r, s)$  can be numbered as  $\dots, v_{-1}, v_0, v_1, \dots$ , in such a way that  $v_0 = v_\circ$ . The connected subtree  $\mathcal{T}_j$  of  $\mathcal{T}$  having  $v_j$  as its root vertex, and containing no edge of  $\mathcal{Y}(r, s)$ , is characterised by the property

$$\text{ord}_p([z; (r) - (s)]) = j \text{ if and only if } \text{red}(z) \in \mathcal{T}_j.$$

Let  $\mathcal{Y}[v_i, v_j]$  be the subgraph of  $\mathcal{Y}(r, s)$  consisting of the path from  $v_i$  to  $v_j$ , with all the vertices between  $v_i$  and  $v_j$  included, and let  $\mathcal{Y}[v_i, v_j) := \mathcal{Y}[v_i, v_j] - \{v_j\}$ . The Bruhat–Tits tree can be expressed as the disjoint union

$$\mathcal{T} = \bigcup_{j=-\infty}^{\infty} (\mathcal{T}_j \cup \mathcal{Y}[v_j, v_{j+1})).$$

The matrix  $\gamma_{rs}$  preserves the path  $\mathcal{Y}(r, s)$ , sending the vertex  $v_j$  to  $v_{j+2e}$  where  $p^{\pm e}$  are the eigenvalues of  $\gamma_{rs}$ . It also maps the subtree  $\mathcal{T}_j$  to  $\mathcal{T}_{j+2e}$ . Hence the connected subgraph

$$\mathcal{T}_{[v_0, v_{2e})} := \mathcal{Y}[v_0, v_{2e}) \cup \mathcal{V}(\mathcal{T}_0) \cup \mathcal{E}(\mathcal{T}_1) \cup \dots \cup \mathcal{T}_{2e-1}$$

is a fundamental domain for the  $\Gamma_{r,s}$ -action on  $\mathcal{T}$ , and

$$\Omega(r, s) := \text{red}^{-1}(\mathcal{T}_{[v_0, v_{2e})}) = \{z \in \mathcal{H}_p \text{ with } 0 \leq \text{ord}_p([z; (r) - (s)]) < 2e\}$$

is a fundamental region in  $\mathcal{H}_p$  for this action. Let  $\mathcal{T}_j^{\leq n}$  denote the finite subtree of  $\mathcal{T}_j$  of depth  $n$  with  $v_j$  as root vertex, and let

$$\mathcal{T}_{[v_0, v_{2e})}^{\leq n} := \mathcal{Y}[v_0, v_{2e}) \cup \mathcal{T}_0^{\leq n} \cup \mathcal{T}_1^{\leq n-1} \cup \mathcal{T}_2^{\leq n} \cup \dots \cup \mathcal{T}_{2e-1}^{\leq n-1},$$

$$\Omega^{[n]}(r, s) := \text{red}^{-1} \left( \mathcal{T}_{[v_0, v_{2e}]}^{\leq n} \right) \subset \mathcal{H}_p.$$

The sets  $\Omega^{[n]}(r, s)$  form a nested sequence whose union is  $\Omega(r, s)$ , and likewise

$$(49) \quad \mathcal{H}_p^{[n]} := \text{red}^{-1} \left( \bigcup_{j=-n}^n \gamma_{rs}^j \mathcal{T}_{[v_0, v_{2e}]}^{\leq n} \right) = \bigcup_{j=-n}^n \gamma_{rs}^j \Omega^{[n]}(r, s),$$

satisfies

$$(50) \quad \mathcal{H}_p = \bigcup_{n \geq 1} \mathcal{H}_p^{[n]}.$$

We associate divisors  $\Delta_\tau^{[n]} \{r, s\}$  and  $\Pi_\tau^{[n]} \{r, s\}$  to the sets  $\mathcal{H}_p^{[n]}$  and  $\Omega^{[n]}(r, s)$  by setting

$$(51) \quad \Delta_\tau^{[n]} \{r, s\} := \sum_{w \in (\Gamma\tau) \cap \mathcal{H}_p^{[n]}} \delta_{r,s}(w) \cdot (w), \quad \Pi_\tau^{[n]} \{r, s\} := \sum_{w \in (\Gamma\tau) \cap \Omega^{[n]}(r, s)} \delta_{r,s}(w) \cdot (w).$$

Note that, by (49),

$$(52) \quad \Delta_\tau^{[n]} \{r, s\} = \sum_{j=-n}^n \gamma_{rs}^j \Pi_\tau^{[n]} \{r, s\}.$$

**Proposition 5.7.** *For all unramified  $\tau \in \Gamma \backslash \mathcal{H}_p^{\text{RM}}$ ,*

$$\text{Obs}(J_\tau)[r, s] = \lim_{n \rightarrow \infty} [\Pi_\tau^{[n]}(r, s); (r) - (s)].$$

*Proof.* Choose a base point  $\eta$ , and assume for simplicity that its reduction is equal to the vertex  $v_\circ$ . This base point determines a lift of  $J_\tau \{r, s\}(z)$  to  $\mathcal{M}^\times$  by setting

$$\tilde{J}_\tau \{r, s\}(z) := \lim_{n \rightarrow \infty} [(z) - (\eta); \Delta_\tau^{\leq n} \{r, s\}],$$

following the definition given in Lemma 4.6. In light of (50) and Corollary 4.3, the definition of  $\tilde{J}_\tau \{r, s\}(z)$  remains unchanged after replacing  $\Delta_\tau^{\leq n} \{r, s\}$  by the divisors  $\Delta_\tau^{[n]} \{r, s\}$  of (51), i.e.,

$$\tilde{J}_\tau \{r, s\}(z) = \lim_{n \rightarrow \infty} [(z) - (\eta); \Delta_\tau^{[n]} \{r, s\}].$$

By definition of the lifting obstruction,

$$\text{Obs}(J_\tau)[r, s] := \tilde{J}_\tau \{r, s\}(\gamma_{rs}z) / \tilde{J}_\tau \{r, s\}(z)$$

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} [(\gamma_{rs}z) - (\eta); \Delta_\tau^{[n]} \{r, s\}] \div [(z) - (\eta); \Delta_\tau^{[n]} \{r, s\}] \\
 &= \lim_{n \rightarrow \infty} [(\gamma_{rs}z) - (z); \Delta_\tau^{[n]} \{r, s\}].
 \end{aligned}$$

By (52), we may rewrite this as

$$\begin{aligned}
 \text{Obs}(J_\tau)[r, s] &= \lim_{n \rightarrow \infty} \prod_{j=-n}^n [(\gamma_{rs}z) - (z); \gamma_{rs}^j \Pi_\tau^{[n]} \{r, s\}] \\
 &= \lim_{n \rightarrow \infty} \prod_{j=-n}^n [(\gamma_{rs}^{-j+1}z) - (\gamma_{rs}^{-j}z); \Pi_\tau^{[n]} \{r, s\}] \\
 &= \lim_{n \rightarrow \infty} [(\gamma_{rs}^{n+1}z) - (\gamma_{rs}^{-n}z); \Pi_\tau^{[n]} \{r, s\}] \\
 &= \lim_{n \rightarrow \infty} [(r) - (s); \Pi_\tau^{[n]} \{r, s\}].
 \end{aligned}$$

The proposition now follows from Weil reciprocity. □

**Corollary 5.8.** *If  $\tau$  is unramified and  $p \nmid \det(r, s)$ , then  $\text{Obs}(J_\tau)[r, s]$  belongs to  $\mathcal{O}_{C_p}^\times$ .*

*Proof.* The expression  $[\Pi_\tau^{[n]} \{r, s\}; (r) - (s)]$  occurring in Proposition 5.7 can be written as a finite product of  $[\Delta_\tau^v \{r, s\}; (r) - (s)]$  for various vertices  $v$ , which are  $p$ -adic units since  $\Delta_\tau^v \{r, s\}$  is a degree zero divisor supported on a single basic affinoid. □

To deal with the ramified case, the following ramified version of Proposition 5.7 will be needed. Since its proof is essentially identical to that of the unramified version, we leave it to the reader.

**Proposition 5.9.** *For all ramified  $\tau \in \Gamma \setminus \mathcal{H}_p^{\text{RM}}$ ,*

$$\begin{aligned}
 \text{Obs}(J_\tau^+)[r, s] &= \lim_{n \rightarrow \infty} [\Pi_\tau^{[n]}(r, s); (r) - (s)], \\
 \text{Obs}(J_\tau^-)[r, s] &= \lim_{n \rightarrow \infty} [\Pi_\tau^{[n]}(r, s); (r) - (s)],
 \end{aligned}$$

where the limits are taken over the odd and even values of  $n$ , respectively.

Unlike the lifting obstructions attached to  $J_\tau$  when  $\tau$  is unramified, the lifting obstructions attached to  $J_\tau^+$  and  $J_\tau^-$  need not be  $p$ -adic units. The following result computes their  $p$ -adic valuations. To state it, one associates to  $\tau$  and  $(r, s)$  the unique harmonic cocycle  $c_\tau \{r, s\}$  on the Bruhat–Tits tree  $\mathcal{T}$  satisfying, for all even oriented edges  $e$

$$c_\tau \{r, s\}(e) = \text{sgn}(e) \cdot \deg(\Delta_\tau^{U_e} \{r, s\}),$$



where  $U_e$  is the oriented annulus mapping to  $e$  under the reduction map.

**Proposition 5.10.** *For all ramified  $\tau \in \Gamma \setminus \mathcal{H}_p^{\text{RM}}$ ,*

$$(53) \quad \text{ord}_p(\text{Obs}(J_\tau^+)[r, s]) = -\text{ord}_p(\text{Obs}(J_\tau^-)[r, s]) = \frac{1}{2} \sum_{e \in \Upsilon[v_0, v_{2e}]} c_\tau\{r, s\}(e).$$

*In particular,  $\text{ord}_p(\text{Obs}(J_\tau)[r, s]) = 0$ .*

*Proof.* By Proposition 5.9,  $\text{Obs}(J_\tau^+)[r, s]$  can be written as the limit of  $[\Pi_\tau^{[n]}(r, s); (r) - (s)]$  for  $n$  odd, each of which is a finite product of factors of the form

$$(54) \quad [\Delta_\tau^v\{r, s\} \cap \Omega^{[n]}(r, s); (r) - (s)]$$

for various vertices  $v$  of *even* distance to  $v_0$ . There are two cases to consider: When  $v$  is a vertex that does not lie on the path  $\Upsilon(r, s)$ , we have

$$\Delta_\tau^v\{r, s\} \cap \Omega^{[n]}(r, s) = \Delta_\tau^v\{r, s\},$$

and all points  $w$  in the support of  $\Delta_\tau^v\{r, s\}$  satisfy  $\text{ord}_p[w; (r) - (s)] = j$  for the same  $j \in \mathbb{Z}$ . Since the divisor  $\Delta_\tau^v\{r, s\}$  is of degree zero, we find that

$$\text{ord}_p[\Delta_\tau^v\{r, s\}; (r) - (s)] = j \cdot \text{deg}(\Delta_\tau^v\{r, s\}) = 0.$$

When  $v = v_j$  is on the path  $\Upsilon(r, s)$  the quantity (54) is itself a product of factors indexed by the points  $w$  in the support of  $\Delta_\tau^v\{r, s\}$ . The valuation of  $[w; (r) - (s)]$  is

$$\begin{aligned} j & \quad \text{if } e \text{ does not lie on } \Upsilon(r, s), \\ j - 1/2 & \quad \text{if } e = (v_{j-1}, v_j), \\ j + 1/2 & \quad \text{if } e = (v_j, v_{j+1}). \end{aligned}$$

When  $j \neq 0$  we again have  $\Delta_\tau^v\{r, s\} \cap \Omega^{[n]}(r, s) = \Delta_\tau^v\{r, s\}$ , which has degree zero. Taking the weighted sum of all these valuations, it follows that

$$\begin{aligned} \text{ord}_p[\Delta_\tau^v\{r, s\}; (r) - (s)] &= -\frac{1}{2}c_\tau\{r, s\}(v_j, v_{j-1}) + \frac{1}{2}c_\tau\{r, s\}(v_j, v_{j+1}) \\ &= \frac{1}{2}c_\tau\{r, s\}(v_{j-1}, v_j) + \frac{1}{2}c_\tau\{r, s\}(v_j, v_{j+1}). \end{aligned}$$

When  $j = 0$  on the other hand, the valuation simply becomes  $\frac{1}{2}c_\tau\{r, s\}(v_0, v_1)$ . This proves the desired result for  $\text{Obs}(J_\tau^+)[r, s]$ . The result for  $\text{Obs}(J_\tau^-)[r, s]$  follows in exactly the same way, interchanging even an odd vertices.  $\square$

### 5.4. A reciprocity law

The main result of this section is the following reciprocity law.

**Theorem 5.11.** *Let  $\tau \in \mathcal{H}_p$  be an RM point, and  $(r, s)$  a pair of elements in  $\mathbb{P}_1(\mathbb{Q})$  with  $p \nmid \det(r, s)$ , and let*

$$\begin{aligned} J_\tau &\in \mathrm{H}^1(\Gamma, \mathcal{M}^\times/C_p^\times), \\ J_{r,s} &\in \mathrm{H}^1(\Gamma, \mathcal{A}^\times/C_p^\times), \end{aligned}$$

be the associated rigid meromorphic, resp. analytic, theta-cocycles. Then

$$\mathrm{Obs}(J_\tau)[r, s] = J_{r,s}[\tau] \pmod{\text{torsion}}.$$

*Proof.* Suppose first that  $\tau$  is unramified. By Proposition 5.7, it is enough to show that

$$J_{r,s}[\tau] = \lim_{n \rightarrow \infty} [\Pi_\tau^{[n]} \{r, s\}; (r) - (s)].$$

We begin by treating the case where  $\tau$  is even. We can then assume without loss of generality, after translating  $\tau$  by a suitable element of  $\Gamma$ , that  $\tau$  reduces to the vertex  $v_\circ$ . The automorph  $\gamma_\tau$  then belongs to  $\mathrm{SL}_2(\mathbb{Z})$ , and one has

$$J_{r,s}[\tau] = J_{r,s}^\circ(\gamma_\tau)(\tau),$$

where  $J_{r,s}^\circ$  is the lift of  $J_{r,s}$  to an element of  $\mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathcal{A}^\times)$ . By the formula for  $J_{r,s}^\circ$  given in Proposition 3.23,

$$(55) \quad J_{r,s}[\tau] = \prod_{n=0}^{\infty} J_{r,s}^{(n)}(\gamma_\tau)(\tau), \quad \text{where } J_{r,s}^{(n)}(\gamma_\tau)(z) := \prod_{(u,v) \in \Sigma_{r,s}^{(n)}} [z; (u) - (v)]^{[u,v] \cdot [\xi, \gamma_\tau \xi]}.$$

Let  $(\Gamma/\Gamma_{r,s})_n$  denote the set of  $\alpha \in \Gamma/\Gamma_{r,s}$  satisfying  $\mathrm{ord}_p(\det(\alpha r, \alpha s)) \leq n$ . We may then rewrite (55) as asserting that  $J_{r,s}[\tau] = \lim_{n \rightarrow \infty} J_{r,s}^{\leq n}(\gamma_\tau)(\tau)$ , where

$$(56) \quad J_{r,s}^{\leq n}(\gamma_\tau)(\tau) = \prod_{\alpha \in (\Gamma/\Gamma_{r,s})_n} [\tau; (\alpha r) - (\alpha s)]^{[\alpha r, \alpha s] \cdot [\xi, \gamma_\tau \xi]}.$$

Since  $\gamma_\tau$  belongs to the group  $\mathrm{SL}_2(\mathbb{Z})$  which preserves the determinant of  $(r, s)$ , it acts on the set  $(\Gamma/\Gamma_{r,s})_n$  by left multiplication, and therefore

$$(57) \quad J_{r,s}^{\leq n}(\gamma_\tau)(\tau) = \prod_{\alpha \in \gamma_\tau^{\mathbb{Z}} \setminus (\Gamma/\Gamma_{r,s})_n} \prod_{j=-\infty}^{\infty} [\tau; (\gamma_\tau^j \alpha r) - (\gamma_\tau^j \alpha s)]^{[\gamma_\tau^j \alpha r, \gamma_\tau^j \alpha s] \cdot [\xi, \gamma_\tau \xi]}$$

$$(58) \quad = \prod_{\alpha \in \gamma_\tau^{\mathbb{Z}} \backslash (\Gamma / \Gamma_{r,s})_n} \prod_{j=-\infty}^{\infty} [\tau; (\alpha r) - (\alpha s)]^{[\alpha r, \alpha s] \cdot [\gamma_\tau^{-j} \xi, \gamma_\tau^{-j+1} \xi]}$$

$$(59) \quad = \prod_{\alpha \in \gamma_\tau^{\mathbb{Z}} \backslash (\Gamma / \Gamma_{r,s})_m} [\tau; (\alpha r) - (\alpha s)]^{[\alpha r, \alpha s] \cdot [\tau', \tau]}$$

$$(60) \quad = \prod_{\alpha \in \gamma_\tau^{\mathbb{Z}} \backslash (\Gamma / \Gamma_{r,s})_m} [(\alpha^{-1} \tau); (r) - (s)]^{[r, s] \cdot [\alpha^{-1} \tau', \alpha^{-1} \tau]} \pmod{p^{\mathbb{Z}}}.$$

In this series of equalities, (58) and (60) follow from the  $SL_2(\mathbb{Z})$  and  $\Gamma$ -equivariance properties of the expression  $[z; (r) - (s)]$ , respectively, and (59) follows from the identity

$$\begin{aligned} \sum_{j=-\infty}^{\infty} [\alpha r, \alpha s] \cdot [\gamma_\tau^{-j} \xi, \gamma_\tau^{-j+1} \xi] &= \lim_{M \rightarrow \infty} [\alpha r, \alpha s] \cdot [\gamma_\tau^{-M} \xi, \gamma_\tau^{M+1} \xi] \\ &= [\alpha r, \alpha s] \cdot [\tau', \tau], \end{aligned}$$

since  $\tau$  is the attractive fixed point in  $\mathcal{H}$  for  $\gamma_\tau$  and  $\tau'$  is its repulsive fixed point.

Since the subgraph  $\mathcal{T}_{[v_0, v_{2e}]}$  is a fundamental region in  $\mathcal{T}$  for the action of the matrix  $\gamma_{rs}$ , the set of matrices  $\alpha \in \Gamma$  for which  $\alpha^{-1} v_0 \in \mathcal{T}_{[v_0, v_{2e}]}$  gives system of representatives for the quotient  $\Gamma / \Gamma_{r,s}$ . After identifying the quotient with this particular system of representatives, the natural map  $\alpha \mapsto \alpha^{-1} \tau$  yields a bijection

$$\Gamma_\tau \backslash (\Gamma / \Gamma_{r,s})_m \longrightarrow (\Gamma \tau) \cap \Omega^{(m)}(r, s),$$

where we recall that  $\Omega^{(m)}$  is the truncated fundamental region for the action of  $\Gamma_{r,s}$ . We may then rewrite (60) as

$$\begin{aligned} J_{r,s}^{\leq m}(\gamma_\tau)(\tau) &= \prod_{w \in (\Gamma \tau) \cap \Omega^{(m)}(r, s)} [w; (r) - (s)]^{\delta_{r,s}(w)} \\ &= [\prod_\tau^{\leq m} \{r, s\}; (r) - (s)] \pmod{p^{\mathbb{Z}}}. \end{aligned}$$

We have therefore proved that

$$J_{r,s}[\tau] = \text{Obs}(J_\tau)[r, s],$$

up to powers of  $p$ . On the other hand,

1. The left hand side belongs to  $\mathcal{O}_{C_p}^\times$ , by Cor. 3.24.

2. The right hand side belongs to  $\mathcal{O}_{C_p}^\times$ , by Cor. 5.8.

The theorem follows. The arguments for  $\tau$  ramified are similar. □

*Remark 5.12.* It is in general not expected that the toric values of the lifting obstruction  $\text{Obs}(J_\tau)[r, s]$  are algebraic numbers. However, the reciprocity law of Theorem 5.11 implies that they are very closely related to the *Gross–Stark units* and *Stark–Heegner points* discussed in §3. To see why, note that the reciprocity law equates the toric value to the RM value of the toric cocycle  $J_{r,s}$ , which is an analytic theta cocycle, and therefore a combination of the cocycles defined in §3.7. Through the Schneider–Teitelbaum lift, the precise linear combination may be computed easily in homology, which was done in [DPV1]. For instance, for  $(r, s) = (0, \infty)$  one finds that

$$(61) \quad J_{0,\infty} = \left(\frac{12}{p-1}\right) \cdot J_{\text{DR}} + \sum_f L_{\text{alg}}(f, 1) \cdot J_f^- ,$$

where the sum runs over a basis of normalised eigenforms for  $S_2(\Gamma_0(p))$ , and additive notation is used to denote the group operation in  $H^1(\Gamma, \mathcal{A}^\times/C_p^\times) \otimes \overline{\mathbb{Q}}_p$ . The quantity  $L_{\text{alg}}(f, 1)$  is the algebraic part of the L-value attached to  $f$ , and may easily be computed in practice.

The explicit description provided by Proposition 5.7 makes the toric values of the lifting obstructions eminently suited to explicit computation, using the modular symbol algorithms that were described in [DV21, § 2.5]. We illustrate this here with a few examples.

**Example 5.13.** The smallest discriminant of a rigid meromorphic theta-symbol whose parabolic lifting obstruction has non-trivial toric values is  $D = 12$ , which has class number one but narrow class number two. The primes  $p = 5, 7$  are both inert for this discriminant, so that  $\tau := (1 + \sqrt{3})/2$  is contained in the corresponding  $p$ -adic upper half plane  $\mathcal{H}_p$ . For these primes all terms of (61) corresponding to cusp forms must vanish, since  $S_2(\Gamma_0(p)) = 0$ . This means that the  $p$ -adic cocycle  $J_{0,\infty}$  is a multiple of the Dedekind–Rademacher cocycle  $J_{\text{DR}}$  for both of these primes, and as such it should have algebraic RM values. Numerical calculations, carried out to 100 digits of  $p$ -adic precision, suggest that

$$(62) \quad \text{Obs}(J_\tau)[0, \infty] \stackrel{?}{=} -1+2i \in \mathbb{Q}_5, \quad \text{Obs}(J_\tau)[0, \infty]^3 \stackrel{?}{=} (-13+3\sqrt{-3})/2 \in \mathbb{Q}_7.$$

(Of course, numerical evaluations of sort, no matter what precision they are carried out to, never yield a proof of such identities. The identities above

should in principle be provable using the methods of [DPV2], but we have not attempted to work out the details.)

When  $p = 7$  and  $D = 321$ , the smallest positive discriminant of narrow class number 6 in which  $p$  is inert, there are three distinct rigid meromorphic cocycles  $J_i$  (for  $i = 1, 2, 3$ ) of discriminant 321, whose zeroes and poles are concentrated on the  $\Gamma$ -orbits of  $\pm\tau_i$  where

$$\tau_1 = \frac{-17 + \sqrt{321}}{2}, \quad \tau_2 = \frac{-17 + \sqrt{321}}{4}, \quad \tau_3 = \frac{-17 + \sqrt{321}}{8}.$$

Their cuspidal values, calculated to 20 digits of 7-adic accuracy, are

(63)

$$\begin{aligned} \text{Obs}(J_1)[0, \infty] &= 11055762063642167 \pmod{7^{20}}, \\ \text{Obs}(J_2)[0, \infty] &= 27863515261720344 - 24701001956851703\sqrt{321} \pmod{7^{20}}, \\ \text{Obs}(J_3)[0, \infty] &= 35228448313023684 - 11567417813120589\sqrt{321} \pmod{7^{20}}. \end{aligned}$$

The first quantity belongs to  $\mathbb{Q}_7$  while the second and third are conjugate to each other (up to inversion) over the unramified quadratic extension of  $\mathbb{Q}_7$ . All three quantities are 7-adic units, but to 200 digits of 7-adic accuracy it appears that, up suitable powers of 7, all three of them satisfy the same sextic polynomial with rational coefficients

(64)

$$7^4x^6 - 20976x^5 - 270624x^4 + 526859689x^3 - 649768224x^2 - 120922465776x + 7^{16},$$

whose splitting field is the narrow Hilbert class field of  $\mathbb{Q}(\sqrt{321})$ .

**Example 5.14.** Let  $\tau = 2\sqrt{2}$ , which has discriminant 32, and is contained in the 11-adic upper half plane. Let  $\bar{J}_\tau$  be the 11-adic theta cocycle attached to  $\tau$ . Then the reciprocity law proved in Theorem 5.11 implies that

$$\text{Obs}(J_\tau)[0, \infty] = \frac{6}{5}J_{\text{DR}}[\tau] + \frac{1}{5}J_E[\tau]$$

where  $E$  is the modular curve of level 11 given explicitly in Weierstraß form by

$$E : y^2 + y = x^3 - x^2 - 10x - 20$$

This means that whereas the quantity  $\text{Obs}(J_\tau)[0, \infty]$  itself need not be algebraic, we may apply suitable Hecke operators to obtain algebraic quantities,

according to the results discussed in §3.7. We observed to a high 11-adic precision that

$$\begin{cases} \text{Obs}((T_2 + 2)J_\tau)[0, \infty] &= J_{\text{DR}}[\tau]^6 &= \left(\frac{\sqrt{-2}-3}{11}\right)^6 &\in \mathbb{C}_{11}^\times \\ \text{Obs}((T_2 - 3)J_\tau)[0, \infty] &= J_E^-[ \tau ]^{-1} &= (2\sqrt{-2}, 4\sqrt{-2} - 5) &\in \mathbb{C}_{11}^\times/q_E^{\mathbb{Z}} \end{cases}$$

### 6. Generating series

#### 6.1. A Gross–Kohnen–Zagier formula for Stark–Heegner points

Let  $-d_1$  be a fixed negative fundamental discriminant prime to  $p$ . If  $-d_2$  is a second negative discriminant such that  $(-d_2/p) \neq (-d_1/p)$ , we can associate to the pair  $(-d_1, -d_2)$  the RM divisors  $\mathbb{D}_{-d_1, -d_2}^\pm$  as in (4). If  $J$  is any analytic theta-cocycle, the values  $J[\mathbb{D}_{-d_1, -d_2}^\pm]$  are defined in the obvious way, by extending the definition of the RM values  $J[\tau_Q]$  by multiplicativity.

Let  $\mathbb{M}(p) \subset M_{3/2}(4p) \otimes \mathbb{Q}_p$  denote Kohnen’s subspace tensored with  $\mathbb{Q}_p$ , consisting of modular forms whose Fourier coefficients are supported on integers  $d_2 \equiv 0, 3 \pmod{4}$ . It can be decomposed as

$$\mathbb{M}(p) = \mathbb{M}(p)^+ \oplus \mathbb{M}(p)^-,$$

where  $\mathbb{M}(p)^+$  (resp.  $\mathbb{M}(p)^-$ ) consists of modular forms whose Fourier coefficients are supported on integers  $d_2$  for which  $(-d_2/p) \neq 1$  (resp.  $(-d_2/p) \neq -1$ ).<sup>1</sup> Then  $\mathbb{M}(p)^+$  is the +1-eigenspace for the  $U_{p^2}$  operator (Cf. [Gr87, §12]), i.e., a form  $g = \sum c(D)q^D$  in this space satisfies

$$(65) \quad c(Dp^{2m}) = c(D).$$

The following result can be viewed as an analogue of the Gross–Kohnen–Zagier theorem for Stark–Heegner points:

**Theorem 6.1.** *Let  $J$  be any analytic theta-cocycle.*

1. *If  $\left(\frac{-d_1}{p}\right) = 1$ , then the generating series*

$$\Theta_{-d_1, J}(q) := \sum_{d_2 \geq 1} \log_p J[\mathbb{D}_{-d_1, -d_2}^+] q^{d_2}$$

*belongs to  $\mathbb{M}(p)^+$ .*

---

<sup>1</sup>Note that over  $\mathbb{C}$ , the space obtained by the conditions defining  $\mathbb{M}(p)^+$  is denoted instead  $M_{\mathbb{C}}^*$  in [Gr87] and its cuspidal subspace is denoted  $S_{3/2}(p)^-$  in [Ko82].

2. If  $\left(\frac{-d_1}{p}\right) = -1$ , then the generating series

$$\Theta_{-d_1, J}(q) := \sum_{d_2 \geq 1} \log_p J[\mathbb{D}_{-d_1, -d_2}^-] q^{d_2}$$

belongs to  $\mathbb{M}(p)^-$ .

*Sketch of proof.* Assume first that  $\left(\frac{-d_1}{p}\right) = 1$ . Since the Hecke algebra of weight two and level  $p$  acts semisimply on the space of rigid analytic theta cocycles, we may assume, after tensoring this space with  $\bar{\mathbb{Q}}$ , that  $J$  is a Hecke eigenclass. If  $J$  is the class  $J_f^+$  or  $J_f^-$  attached to a cusp form  $f$  of weight 2 on  $\Gamma_0(p)$ , then  $J_f^+[\mathbb{D}_{-d_1, -d_2}^+] = 1$ , while the modularity assertion for  $J_f^-$  follows from [DT08, Theorem 5.1 (a)]. More precisely, after replacing the fundamental discriminant  $-D_2$  of loc.cit. by  $-d_1$  and the varying discriminants  $-D_1$  by  $-d_2$ , this theorem asserts that there is a modular form  $\theta := \sum b(d_2)q^{d_2} \in \mathbb{M}(p)^+$  for which

$$\log_p J_f^-[\mathbb{D}_{-d_1, -d_2}^+] = b(d_2),$$

for all  $d_2$  satisfying

$$(66) \quad \text{ord}_p(d_2) \leq 1, \quad d_2 = 0, 3 \pmod{4}, \quad \left(\frac{-d_2}{p}\right) \neq 1.$$

In applying [DT08, Theorem 5.1 (a)], it must be noted that somewhat stronger assumptions are made in its statement than needed. Namely:

- It is assumed that  $-d_1$  and  $-d_2$  are both fundamental and relatively prime to each other. This stronger hypothesis on  $-d_2$  is only used in the proof of [DT08, Theorem 5.1 (b)], where it is necessary that the character  $\omega_{-d_1}$  be a *genus character* in order to invoke [BD09, Thm. 1]. The proof of [DT08, Theorem 5.1 (a)], where only Kohnen’s formula is used, applies to the more general  $-d_2$  of Theorem 6.1.
- It is assumed that  $f$  has rational Fourier coefficients and hence corresponds to an elliptic curve, but the proof applies just as well to arbitrary cuspidal eigenforms after extending scalars to the field generated by the Fourier coefficients of  $f$ .
- Finally, Theorem 5.1 of [DT08] applies only to the cuspidal eigenclasses, leaving out the boundary theta-symbol  $J_{\sharp}$  of §3.7.1 and the Dedekind–Rademacher cocycle  $J_{\text{DR}}$  of §3.7.3. The case where  $J = J_{\sharp}$  is trivial since  $J_{\sharp}[\mathbb{D}_{-d_1, -d_2}^+] = 1$ , while the case where  $J = J_{\text{DR}}$  is handled in [Pa10, Theorem 5.2.] and the discussion that follows it, which shows

that the associated generating series  $\Theta_{-d_1, J_{\text{DR}}}(q)$  is an Eisenstein series of weight  $3/2$  on  $\Gamma_0(4p)$ .

The modularity of the coefficients  $\log_p J_f^-[\mathbb{D}_{-d_1, -d_2}^+]$  is thus established, for all rigid analytic theta-cocycles  $J$  and for all  $d_2$  satisfying (66). The result then follows for all  $d_2$  from the rule (5) satisfied by the quantities  $\mathbb{D}_{-d_1, -d_2}^+$ , combined with the fact that the Fourier coefficients of forms in  $\mathbb{M}(p)^+$  satisfy the same relation, by (65).

This handles the first assertion. The argument for the case  $\left(\frac{-d_1}{p}\right) = -1$  proceeds along the same lines. □

*Remark 6.2.* While the argument above falls somewhat short of being a complete proof because of its reliance on the slightly weaker results in [BD09] and [DT08], a simpler and more direct route to Theorem 6.1 is available, which will be treated in forthcoming work. It builds on the approach of [BDG], and produces the generating series  $\Theta_{-d_1, J}(q)$  as the first derivative of a  $p$ -adic family of Kudla-Millson theta series in weight  $3/2$ .

*Remark 6.3.* When  $\left(\frac{-d_1}{p}\right) = 1$ , the generating series

$$\Theta_{-d_1, J}^b(q) := \sum_{d_2 \geq 0} \log_p J[\mathbb{D}_{-d_1, -d_2}^-] q^{d_2}$$

is not a classical modular form of weight  $3/2$ , but is best envisaged as a *p-adic mock modular form* whose “shadow” is a suitable kernel of the Shimura–Shintani correspondence between  $\mathbb{M}(p)^-$  and  $S_2(p)^-$ , where  $S_2(p)^-$  denotes the  $(-1)$ -eigenspace for the  $U_p$  operator on forms of weight two and level  $p$ . The full generating series

$$\sum_{d_2 \geq 0} \log_p J[\mathbb{D}_{-d_1, -d_2}] q^{d_2}$$

exhibits a somewhat subtle behaviour: it is a classical modular form of weight  $3/2$  along  $\mathbb{M}(p)^+$ , and a “ $p$ -adic mock modular form” along  $\mathbb{M}(p)^-$ . For details, see the discussion in [DT08].

**Corollary 6.4.** *Let  $\iota := \left(\frac{-d_1}{p}\right)$ . For all  $\phi = \sum_{d \gg 0} c(d)q^d \in M_{1/2}^{\parallel}(4p)$ , the divisor*

$$\mathbb{D}_{-d_1, \phi} := \sum_{d_2 \geq 0} c(-d_2) \mathbb{D}_{-d_1, -d_2}^{\iota}$$

*satisfies*

$$\log_p J[\mathbb{D}_{-d_1, \phi}] = 0, \quad \text{for all } J \in H^1(\Gamma, \mathcal{A}^\times / C_p^\times).$$



*Proof.* This is a direct consequence of Serre duality in light of Theorem 6.1. Namely, the value

$$\log_p J[\mathbb{D}_{-d_1, \phi}] = \sum_{d_2 \geq 0} c(-d_2) \cdot \log_p J[\mathbb{D}_{-d_1, -d_2}^\iota]$$

is the zero-th Fourier coefficient of  $\phi(q)\Theta_{-d_1, J}(q)$ , a weakly holomorphic modular form of weight 2 and level  $p$  with poles only at the cusp  $\infty$ . This coefficient is the residue of the associated regular differential

$$\phi(q)\Theta_{-d_1, J}(q) \frac{dq}{q} \in \Omega^1(X_0(p) - \{\infty\}),$$

and is therefore zero by the residue theorem. □

### 6.2. A principality criterion

Let  $\mathbb{D}$  be an arbitrary RM divisor. The following criterion for  $\mathbb{D}$  to be principal plays a crucial role in our argument.

**Proposition 6.5.** *If  $\log_p J[\mathbb{D}] = 0$  for all analytic theta-symbols  $J \in \text{MS}^\Gamma(\mathcal{A}^\times/C_p^\times)$ , then  $\mathbb{D}$  is a principal divisor, i.e., there is a meromorphic modular symbol  $J_{\mathbb{D}} \in H^1(\Gamma, \mathcal{M}^\times) \otimes \mathbb{Q}$  satisfying*

$$\text{Divisor}(J_{\mathbb{D}}) = \mathbb{D}.$$

*Proof.* The hypothesis implies in particular that

$$\log_p J_{r,s}[\mathbb{D}] = 0, \quad \text{for all } (r, s) \in \mathbb{P}_1(\mathbb{Q})^2.$$

By the reciprocity law of Theorem 5.11, it follows that

$$\log_p \text{Obs}(J_{\mathbb{D}})[r, s] = 0, \quad \text{for all } (r, s) \in \mathbb{P}_1(\mathbb{Q})^2,$$

where  $J_{\mathbb{D}}$  is the rigid meromorphic theta-symbol having  $\mathbb{D}$  as a divisor. Proposition 5.6 then implies that  $\log_p \text{Obs}(J_{\mathbb{D}}) = 0$ , and therefore that  $J_{\mathbb{D}}$  lifts to a meromorphic modular symbol in  $H^1(\Gamma, \mathcal{M}^\times) \otimes \mathbb{Q}$ . The proposition follows. □

### 6.3. A real quadratic Borchers lift

The results of the previous sections can now be combined to prove the following result. Recall that we have fixed a negative fundamental discriminant  $-d_1$  and set  $\iota := \left(\frac{-d_1}{p}\right)$ .

**Theorem 6.6.** *For all  $\phi = \sum_{d \gg -\infty} c_\phi(d)q^d \in M_{1/2}^{\ll}(4p)$  with integer Fourier coefficients, there is a rigid meromorphic cocycle  $J_{-d_1, \phi} \in H^1(\Gamma, \mathcal{M}^\times) \otimes \mathbb{Q}$  satisfying*

$$\text{Divisor}(J_{-d_1, \phi}) = \sum_{d_2 \geq 0} c_\phi(-d_2) \mathbb{D}_{-d_1, -d_2}^\iota,$$

where the sum is taken over the negative discriminants  $-d_2$  satisfying  $\left(\frac{-d_2}{p}\right) \neq \iota$ .

*Proof.* Corollary 6.4 implies that  $\log_p J[\mathbb{D}_{-d_1, \phi}] = 0$  for all rigid analytic theta-symbols  $J$ , and the principality criterion of Proposition 6.5 then shows that the divisor  $\mathbb{D}_{-d_1, \phi}$  is principal, i.e., arises as the divisor of some  $J_{-d_1, \phi}$  in the space  $H^1(\Gamma, \mathcal{M}^\times) \otimes \mathbb{Q}$ . □

### Appendix A. The parallel with Mumford’s $p$ -adic theta functions

The purpose of this motivational appendix is to draw the parallel between the  $p$ -adic theta functions that arise in the theory of  $p$ -adic uniformisation of Mumford curves by  $p$ -adic Schottky groups acting discretely on  $\mathcal{H}_p$ , and rigid meromorphic theta-cocycles. Up to a few significant differences between the two settings, one passes from one notion to the other by “shifting the degree of cohomology by one”.

Firstly, let  $\Gamma \subset \text{SL}_2(\mathbb{Q}_p)$  be a  $p$ -adic Schottky group acting freely and discretely on  $\mathcal{H}_p$  and on the Bruhat–Tits tree, and let  $X_\Gamma$  be the associated Mumford curve over  $\mathbb{Q}_p$ , whose  $\mathbb{C}_p$ -points are identified with the quotient  $\Gamma \backslash \mathcal{H}_p$ . As before, let  $\mathcal{A}^\times$  and  $\mathcal{M}^\times$  denote the multiplicative groups of rigid analytic and rigid meromorphic functions on  $\mathcal{H}_p$ , respectively. The theory of  $p$ -adic theta-functions associates to any degree zero divisor  $\Delta$  of  $\mathcal{H}_p$  a rigid meromorphic function

$$\theta_\Delta(z) := \prod_{\gamma \in \Gamma} [(z) - (\eta); \gamma\Delta].$$

The divisor of such a function is  $\Gamma$ -invariant. In fact,  $\theta_\Delta$  is invariant modulo scalars, i.e., it belongs to  $H^0(\Gamma, \mathcal{M}^\times / \mathbb{C}_p^\times)$ . It lifts to an element of  $H^0(\Gamma, \mathcal{M}^\times)$ , i.e., to a rational function on  $X_\Gamma$ , if and only if the image of  $\Delta$  in  $X_\Gamma(\mathbb{C}_p)$  is a principal divisor. The obstruction to  $\Delta$  being principal is measured by the automorphy factor  $\kappa_\Delta : \Gamma \rightarrow \mathbb{C}_p^\times$  of  $\theta_\Delta$ , defined by

$$\theta_\Delta(\gamma z) = \kappa_\Delta(\gamma) \theta_\Delta(z), \quad \text{for } \gamma \in \Gamma.$$

The group  $H^1(\Gamma, \mathbb{C}_p^\times)$  is isomorphic to  $(\mathbb{C}_p^\times)^g$  where  $g$  is the genus of  $X_\Gamma$ , and the group  $\Pi_\Gamma$  of automorphy factors arising from elements of  $H^0(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$  forms a sublattice in  $H^1(\Gamma, \mathbb{C}_p^\times)$  which is commensurable with the Tate period lattice of the Jacobian of  $X_\Gamma$ . One obtains a  $p$ -adic uniformisation of this Jacobian as the quotient of  $H^1(\Gamma, \mathbb{C}_p^\times)$  by  $\Pi_\Gamma$ , and the lifting obstruction  $\kappa_\Delta$  attached to  $\Delta \in \text{Div}^0(\mathcal{H}_p)$  encodes the image of  $\Delta$  in  $\text{Jac}(X_\Gamma)$ . This discussion is summarised in the following commutative diagram:

(67)

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathbb{C}_p^\times & \longrightarrow & \mathbb{C}_p^\times & \longrightarrow & H^0(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times) \xrightarrow{\delta} \Pi_\Gamma \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathbb{C}_p^\times & \longrightarrow & H^0(\Gamma, \mathcal{M}^\times) & \longrightarrow & H^0(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times) \xrightarrow{\delta} H^1(\Gamma, \mathbb{C}_p^\times) \\
 & & \downarrow & & \downarrow \text{Div} & & \downarrow \\
 & & 0 & \longrightarrow & \text{Prin}(X_\Gamma) & \longrightarrow & \text{Div}^0(X_\Gamma) \longrightarrow \text{Jac}(X_\Gamma) \longrightarrow 0, \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

where  $\delta$  is the connecting homomorphism arising from in the long exact  $\Gamma$ -cohomology exact sequence, and

$$\Pi_\Gamma := \delta(H^0(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)) \subset H^1(\Gamma, \mathbb{C}_p^\times).$$

The analogue of (67) in the setting of rigid meromorphic cocycles is obtained by replacing the  $p$ -adic Schottky group  $\Gamma$  by the multiplicative group of a  $\mathbb{Z}[1/p]$ -order in an *indefinite* quaternion algebra over  $\mathbb{Q}$ . The Ihara group  $\Gamma := \text{SL}_2(\mathbb{Z}[1/p])$  is the simplest, prototypical example of such a group. It is too large to act discretely on  $\mathcal{H}_p$ , or on the Bruhat–Tits tree without fixed points. Indeed, the vertex and edge stablisers in  $\Gamma$  are conjugate to  $\text{SL}_2(\mathbb{Z})$  and to the Hecke congruence group  $\Gamma_0(p)$ , respectively. Because of this, the groups  $H^0(\Gamma, \mathcal{A}^\times)$  and  $H^0(\Gamma, \mathcal{M}^\times)$  contain only the constant functions, and it becomes natural to replace the group  $H^0(\Gamma, \mathcal{M}^\times)$  of rigid meromorphic functions on  $X_\Gamma$  when  $\Gamma$  is a  $p$ -adic Schottky group, with the group  $H^1(\Gamma, \mathcal{M}^\times)$  of rigid meromorphic cocycles.

The theory of theta-cocycles described in §4 associates to any divisor  $\Delta$

of  $\mathcal{H}_p$  consisting of *RM points* a rigid meromorphic cocycle

$$J_\Delta(z) \in H^1(\Gamma, \mathcal{M}^\times / C_p^\times)$$

modulo multiplicative scalars, and shows that all such cocycles are obtained in this way. The divisor  $\Delta$  is said to be *principal* if the class  $J_\Delta$  lifts to genuine rigid meromorphic cocycle in  $H^1(\Gamma, \mathcal{M}^\times)$ . The obstruction to  $\Delta$  being principal is measured by the lifting obstruction  $\kappa_\Delta \in H^1(\Gamma, \text{MS}(C_p^\times))$  and its non-parabolic counterpart in  $H^2(\Gamma, C_p^\times)$ . The group  $H^2(\Gamma, C_p^\times)$  maps with finite kernel to  $H^1(\Gamma_0(p), C_p^\times)$ , suggesting that it could serve as the domain for a  $p$ -adic uniformisation of  $J_0(p)$  (or even, of the generalised Jacobian of the open curve  $Y_0(p)$ ). Indeed, it turns out that the group  $\Pi_\Gamma$  generated by the lifting obstructions of *analytic* cocycles in  $H^1(\Gamma, \mathcal{A}^\times / C_p^\times)$  forms a sublattice in  $H^2(\Gamma, C_p^\times)$  which is commensurable with two copies of the Tate period lattice of  $J_0(p)$ , augmented by the discrete group generated by  $p^\mathbb{Z}$ . As explained in [Dar01, §2] and in [Das05], this is essentially a reformulation of the “exceptional zero conjecture” of Mazur, Tate and Teitelbaum [MTT86] which was proved by Greenberg and Stevens [GS93]. One obtains a kind of  $p$ -adic uniformisation of two copies of this Jacobian (along with a multiplicative factor of  $C_p^\times / p^\mathbb{Z}$ ) as a rigid analytic quotient of  $H^2(\Gamma, C_p^\times)$  by the lattice  $\Pi_\Gamma$ . The lifting obstruction  $\kappa_\tau$  attached to any  $\tau \in \mathcal{H}_p^{\text{RM}}$  encodes the images of the Gross–Stark units and the Stark–Heegner points attached to  $\tau$ , in the generalised Jacobian of  $Y_0(p)$ .

This discussion can be summarised in the following commutative diagram in the category of abelian groups up to isogeny, where morphisms are decreed to be isomorphisms if they have finite kernels and cokernels:

(68)

$$\begin{array}{ccccc}
 & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow \\
 H^1(\Gamma, \mathcal{A}^\times) & \longrightarrow & H^1(\Gamma, \mathcal{A}^\times / C_p^\times) & \xrightarrow{\delta} & \Pi_\Gamma \\
 \downarrow & & \downarrow & & \downarrow \\
 H^1(\Gamma, \mathcal{M}^\times) & \longrightarrow & H^1(\Gamma, \mathcal{M}^\times / C_p^\times) & \xrightarrow{\delta} & H^2(\Gamma, C_p^\times) \\
 \downarrow \text{Div} & & \downarrow \text{Div} & & \downarrow \\
 \text{Prin}(\Gamma \backslash \mathcal{H}_p^{\text{RM}}) & \longrightarrow & \text{Div}^0(\Gamma \backslash \mathcal{H}_p^{\text{RM}}) & \longrightarrow & \text{Jac}(X_0(p))(C_p)^2 \oplus C_p^\times / p^\mathbb{Z} \\
 \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & 0
 \end{array}$$

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