RING CLASS FIELDS OF REAL QUADRATIC FIELDS AND CUSPIDAL VALUES OF RIGID MEROMORPHIC COCYCLES

HENRI DARMON AND JAN VONK

Abstract. *Rigid meromorphic cocycles* were introduced in [DV] as a possible foundation for extending the theory of complex multiplication to real quadratic fields. The present work defines the *cuspidal values* of rigid meromorphic cocycles and shows that they are p-units in a compositum of ring class fields of real quadratic fields, by exploiting the progress towards the p-adic Gross–Stark conjectures achieved in [DDP] and [DK].

Contents

Introduction

Let p be a rational prime and let $\mathcal{H}_p$ denote Drinfeld's p-adic upper half plane, a rigid analytic space whose $\mathbb{C}_p$-points are identified with $\mathbb{C}_p - \mathbb{Q}_p$. The discrete group $\Gamma := \mathrm{SL}_2(\mathbb{Z}[1/p])$ acts by Möbius transformations on $\mathcal{H}_p$ (with non-discrete orbits), and thus on the multiplicative group $\mathcal{M}^\times$ of non-zero rigid meromorphic functions on $\mathcal{H}_p$. This action preserves the subset $\mathcal{H}_p^{\mathrm{RM}} \subset \mathcal{H}_p$ of *real multiplication*, or RM points, namely, points of $\mathcal{H}_p$ that lie in a real quadratic field.

A *rigid meromorphic cocycle* is an element of the group $\mathrm{H}^1_f(\Gamma, \mathcal{M}^\times) \subset \mathrm{H}^1(\Gamma, \mathcal{M}^\times)$ consisting of classes whose restriction to the subgroup $\Gamma_\infty \subset \Gamma$ of upper-triangular matrices lies in $\mathrm{H}^1(\Gamma_\infty, \mathbb{C}_p^\times)$. Such a cohomology class admits a unique representative cocycle J (up to torsion) whose restriction to Γ_∞ is $\mathbb{C}_p^\times$-valued, and a rigid meromorphic cocycle is often conflated with this distinguished representative. Theorem 1 of [DV] asserts that the meromorphic function $J(S)$, where $S \in \Gamma$ is the standard matrix of order 4, has its zeroes and poles concentrated in a finite union of Γ-orbits in $\mathcal{H}_p^{\mathrm{RM}}$. This fact is used to associate to J a *field of definition*, denoted H_J, which lies in the compositum of the finite collection of ring class fields of real quadratic fields associated to the zeroes and poles of $J(S)$. If these zeroes and poles are concentrated at the roots of primitive binary quadratic forms of discriminant D with coefficients in $\mathbb{Z}[1/p]$, then J is said to be of *discriminant* D, and the field H_J is the narrow ring class field for the discriminant D, an abelian extension of $\mathbb{Q}(\sqrt{D})$ whose Galois group is identified with the group of classes of primitive binary quadratic forms of discriminant D.

1991 *Mathematics Subject Classification.* 11G18, 14G35.

Since the abelianisation of Γ_∞ is a group of rank one, the value

$$J[\infty] \quad := \quad J\begin{pmatrix} p & * \\ 0 & p^{-1} \end{pmatrix} \quad \in \mathbb{C}_p^\times \quad (\text{mod torsion})$$

is well-defined up to a root of unity. This invariant is called the *value* of J at ∞, and can be viewed as a somewhat degenerate instance of the RM values $J[\tau]$ explored in [DV]. The main result of this work is the counterpart of Conjecture 1 of [DV] in which RM points are replaced by cusps:

Theorem A. *If J is any rigid meromorphic cocycle, then the value $J[\infty]$ is algebraic. More precisely, a power of it belongs to $(\mathcal{O}_{H_J}[1/p])^\times$.*

The proof of Theorem A rests on the classification of rigid meromorphic cocycles described in [DV, Theorem 1.23], which applies to the larger and more easily described group $\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times)$ of rigid meromorphic cocycles modulo scalars. Indeed, no essential information is lost when replacing a rigid meromorphic cocycle by its image in this group, because the natural map

$$\mathrm{H}^1_f(\Gamma, \mathcal{M}^\times) \longrightarrow \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times)$$

has finite kernel (of exponent dividing 12).

Chapter 1 explains how the numerical invariants $J[\infty]$ defined above, as well as the RM values $J[\tau]$ attached in [DV] to any $\tau \in \mathcal{H}_p^{\mathrm{RM}}$, can be extended to arbitrary rigid meromorphic cocycles *modulo scalars*. Notably, a *parabolic lifting obstruction* $\bar{J}[r,s]$ of an element $\bar{J} \in \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times)$ at a pair (r,s) of cusps is defined, in such a way that

$$\bar{J}[0,\infty] = J[\infty]^2,$$

whenever $\bar{J}$ is the reduction modulo scalars of a genuine rigid meromorphic cocycle J.

An *RM divisor* is a finite formal linear combination with integer coefficients of elements of the quotient set $\Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}$. Chapter 2 associates to each such divisor a class $\bar{J} \in \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times)$, in such a way that, if $\bar{J}_\tau$ is the cocycle attached to the single RM point τ, the rigid meromorphic functions $\bar{J}_\tau(\gamma)$ have divisor supported on $\Gamma\tau$, for all $\gamma \in \Gamma$. The classes $\bar{J}_\tau$ are called *theta-cocycles*, and they are shown to generate $\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times)$ up to elements of $\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$, where $\mathcal{A}^\times \subset \mathcal{M}^\times$ is the group of rigid analytic functions on $\mathcal{H}_p$.

Chapter 3 completes the picture by describing $\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$, relating it to the cohomology of the open modular curve $Y_0(p)$, or equivalently by Poincaré duality, to the homology of $X_0(p)$ relative to the cusps. In particular, a distinguished rigid analytic cocycle $\bar{J}_{r,s}$ is attached to the image in this relative homology of the path from r to s on the complex upper half plane, for any pair (r,s) of distinct elements of $\mathbb{P}_1(\mathbb{Q})$.

Chapter 4 establishes a *reciprocity law* for rigid meromorphic cocycles modulo scalars, which asserts that

$$\bar{J}_\Delta[0,\infty] = \bar{J}_{0,\infty}[\Delta] := \prod_\tau \bar{J}_{0,\infty}[\tau]^{m_\tau},$$

where $\bar{J}_\Delta$ is the rigid meromorphic cocycle modulo scalars having $\Delta := \sum_\tau m_\tau \cdot (\tau)$ as a divisor. When $\bar{J}_\Delta$ lifts to a genuine rigid meromorphic cocycle $J_\Delta \in \mathrm{H}^1_f(\Gamma, \mathcal{M}^\times)$, the divisor Δ is said to be *principal*; in that case, $\bar{J}_{0,\infty}[\Delta]$ is shown to be a non-trivial power of $\bar{J}_{\mathrm{DR}}[\Delta]$, where $\bar{J}_{\mathrm{DR}} \in \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$ is the class constructed in [DV, §5], which lifts to a rigid analytic cocycle $J_{\mathrm{DR}} \in \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{A}^\times/p^{\mathbb{Z}})$ and is a simple variant of the *Dedekind–Rademacher cocycle* studied in [DD].

An immediate generalisation of the conjectures of [DD] described in [DV, §5.2] predicts that the values $J_{\mathrm{DR}}[\tau]$ are algebraic – more precisely, that a suitable power of $J_{\mathrm{DR}}[\tau]$ belongs to $(\mathcal{O}_{H_\tau}[1/p])^\times$. The key ingredient in the proof of Theorem A is the important recent work of Dasgupta and Kakde [DK] which proves this assertion by significantly strengthening the earlier results of [DDP] on the p-adic Gross–Stark conjecture for odd characters of totally real fields.

It is appropriate to insist on the indispensable role played by the theory of p-adic deformations of Galois representations in the proof of Theorem A — largely behind the scenes, since this ingredient is only exploited in [DDP] and [DK]. The availability of this technique accounts for why the p-adic Gross–Stark conjecture has seen so much progress while its archimedean counterpart remains seemingly intractable. The fact that logarithms of algebraic numbers in ring class fields of real quadratic fields are to be found in the Fourier coefficients of infinitesimal p-adic deformations of modular forms of weight one (cf. [DLR1], [DLR2]) likewise raises the hope that the conjectures of [DV] on real quadratic singular moduli may be open to attack in the framework of a still nascent theory of "p-adic mock modular forms".

To illustrate Theorem A, the smallest discriminant of a rigid meromorphic cocycle with non-trivial cuspidal values is $D = 12$, which has class number one but narrow class number two. For both $p = 5$ and 7, there is a unique p-adic cocycle $J_{(p)}$ of discriminant 12 with $J_{(p)}(S)$ having zeroes and poles only at the Γ-translates of $\tau := (1+\sqrt{3})/2$, defined by

$$J_{(p)}(S)(z) = \prod_{\substack{w\in\Gamma\tau \\ ww'<0}} \left(\frac{(z-1-w)(z-1+zw)}{(z-1+w)(z-1-zw)}\right)^{\mathrm{sign}(w)}.$$

The field of definition of $J_{(p)}$ is the narrow Hilbert class field of $\mathbb{Q}(\sqrt{3})$, which is the biquadratic field $\mathbb{Q}(\sqrt{3}, i)$. Numerical calculations, carried out to 100 digits of p-adic precision, indicate that

$$(1) \qquad J_{(5)}[\infty] \overset{?}{=} -1+2i \in \mathbb{Q}_5, \qquad\qquad J_{(7)}[\infty]^3 \overset{?}{=} (-13+3\sqrt{-3})/2 \in \mathbb{Q}_7.$$

When $p = 7$ and $D = 321$, the smallest positive discriminant of narrrow class number 6 in which p is inert, there are three distinct rigid meromorphic cocycles J_i (for $i = 1, 2, 3$) of discriminant 321, whose zeroes and poles are concentrated on the Γ-orbits of $\pm\tau_i$ where

$$\tau_1 = \frac{-17+\sqrt{321}}{2}, \qquad \tau_2 = \frac{-17+\sqrt{321}}{4}, \qquad \tau_3 = \frac{-17+\sqrt{321}}{8}.$$

Their cuspidal values, calculated to 20 digits of 7-adic accuracy, are

$$(2) \qquad \begin{aligned} J_1[\infty] &= 11055762063642167 && \pmod{7^{20}}, \\ J_2[\infty] &= 27863515261720344 - 24701001956851703\sqrt{321} && \pmod{7^{20}}, \\ J_3[\infty] &= 35228448313023684 - 11567417813120589\sqrt{321} && \pmod{7^{20}}. \end{aligned}$$

The quantity $J_1[\infty]$ belongs to $\mathbb{Q}_7$ while $J_2[\infty]$ and $J_3[\infty]^{-1}$ are conjugate to each other over the unramified quadratic extension of $\mathbb{Q}_7$. By their construction, these three quantities are 7-adic units, but it appears that $7^{-4}J_1[\infty]$, $J_2[\infty]$, and $J_3[\infty]^{-1}$ (calculated, this time, to 200 digits of 7-adic accuracy) satisfy the sextic polynomial with rational coefficients

$$(3) \qquad 7^4x^6 - 20976x^5 - 270624x^4 + 526859689x^3 - 649768224x^2 - 120922465776x + 7^{16},$$

whose splitting field is the narrow Hilbert class field of $\mathbb{Q}(\sqrt{321})$.

1. Cuspidal values and parabolic lifting obstructions

Let $J \in \mathrm{H}^1_f(\Gamma, \mathcal{M}^\times)$ be a rigid meromorphic cocycle, as defined in the introduction, and recall that $\Gamma_\infty \subset \Gamma$ is the group of upper triangular matrices in Γ.

Lemma 1.1. *A rigid meromorphic cocycle admits a unique representative, up to torsion, whose restriction to Γ_∞ is scalar-valued.*

Proof. Any two such representatives differ by a coboundary $\varrho = d\eta$ whose restriction to Γ_∞ is scalar valued. Since the parabolic element

$$\begin{pmatrix} 1 & p^2-1 \\ 0 & 1 \end{pmatrix} = \left[\begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right]$$

is a commutator, it belongs to the kernel of ϱ and hence the function η satisfies $\eta(z+p^2-1) = \eta(z)$. It follows that η must be a constant, and therefore that $\varrho = 1$. □

The restriction of the distinguished representative J to Γ_∞ is a homomorphism to $\mathbb{C}_p^\times$. The abelianisation of Γ_∞ is isomorphic to $\mathbb{Z}/(p^2-1)\mathbb{Z} \times \mathbb{Z}$, and any matrix of the form $P_\infty = \left(\begin{smallmatrix} p & * \\ 0 & 1/p \end{smallmatrix}\right)$ generates an index $p-1$ subgroup of this abelianisation. In particular, the element $J(P_\infty) \in \mathbb{C}_p^\times$ does not depend on the choice of P_∞, up to torsion.

Definition 1.2. The *value* of J at ∞ is the quantity

$$J[\infty] := J(P_\infty) \in \mathbb{C}_p^\times \quad (\mathrm{mod\ torsion}).$$

This chapter will develop some preliminary results on cuspidal values, and in particular establish its close connection with certain cohomological lifting obstructions attached to rigid meromorphic cocycles modulo scalars.

1.1. Lifting obstructions.

As already mentioned in the introduction, it is fruitful to view the cuspidal values of Theorem A as instances of more general numerical invariants attached to classes in the group $\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times)$ of rigid meromorphic cocycles *modulo scalars.* The following lemma shows that one loses very little information in passing from a rigid meromorphic cocycle J to its image $\bar{J}$ in $\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times)$.

Lemma 1.3. *The natural homomorphism*

$$\mathrm{H}^1_f(\Gamma, \mathcal{M}^\times) \longrightarrow \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times) \tag{4}$$

has finite kernel of exponent dividing 12.

Proof. The group $\Gamma = \mathrm{SL}_2(\mathbb{Z}) *_{\Gamma_0(p)} \mathrm{SL}_2(\mathbb{Z})$ has finite abelianisation, of exponent dividing 12, since it is an amalgamated product of two copies of $\mathrm{SL}_2(\mathbb{Z})$ (cf. [Se, II. §1.4]). The result follows because the kernel of (4) is a quotient of $\mathrm{H}^1(\Gamma, \mathbb{C}_p^\times)$. □

Just as important is the cokernel of (4), which represents the *obstruction* to lifting a rigid meromorphic cocycle modulo scalars to a genuine rigid meromorphic cocycle. Recall that if Ω is a Γ-module, then $\mathrm{MS}(\Omega)$ denotes the Γ-module of *modular symbols* with values in Ω, i.e., the set of functions $m : \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q}) \longrightarrow \Omega$ satisfying

$$\begin{aligned} m\{r,t\} &= m\{r,s\} + m\{s,t\}, \\ m\{r,s\} &= -m\{s,r\}, \end{aligned}$$

for all $r, s, t \in \mathbb{P}_1(\mathbb{Q})$. The importance of modular symbols stems from the well-known map

$$\mathrm{MS}^\Gamma(\Omega) := \mathrm{H}^0(\Gamma, \mathrm{MS}(\Omega)) \longrightarrow \mathrm{H}^1_{\mathrm{par}}(\Gamma, \Omega)$$

sending the Γ-invariant Ω-valued modular symbol m to the cocycle $c(\gamma) := m\{\infty, \gamma\infty\}$ (cf. [DV, Lemma 1.3]). The assignment $\Omega \mapsto \mathrm{MS}(\Omega)$ is an exact functor on the category of Γ-modules. Taking the Γ-cohomology of the short exact sequence

$$1 \longrightarrow \mathbb{C}_p^\times \longrightarrow \mathcal{M}^\times \longrightarrow \mathcal{M}^\times/\mathbb{C}_p^\times \longrightarrow 1$$

and of the sequence obtained from it by applying the functor MS leads to the commutative diagram in which the first full row involves the parabolic cohomology of Γ:

$$\begin{array}{ccccccc} & & & & & & \mathrm{H}^1(\Gamma_\infty, \mathbb{C}_p^\times) \\ & & & & & & \downarrow \\ \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathbb{C}_p^\times) & \longrightarrow & \mathrm{MS}^\Gamma(\mathcal{M}^\times) & \longrightarrow & \mathrm{MS}^\Gamma(\mathcal{M}^\times/\mathbb{C}_p^\times) & \xrightarrow{\delta} & \mathrm{H}^1(\Gamma, \mathrm{MS}(\mathbb{C}_p^\times)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}^1(\Gamma, \mathbb{C}_p^\times) & \longrightarrow & \mathrm{H}^1(\Gamma, \mathcal{M}^\times) & \longrightarrow & \mathrm{H}^1(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times) & \xrightarrow{\delta} & \mathrm{H}^2(\Gamma, \mathbb{C}_p^\times). \end{array} \tag{5}$$

The vertical sequence is obtained as in [DV, Lemma 1.3]. The diagram is commutative, since the map $\mathrm{H}^1(\Gamma, \mathrm{MS}(\mathbb{C}_p^\times)) \longrightarrow \mathrm{H}^2(\Gamma, \mathbb{C}_p^\times)$ associates to $[\kappa]$ the class of the 2-cocycle defined by

$$(\gamma_1, \gamma_2) \mapsto \kappa(\gamma_1)\{\infty, \gamma_2\infty\}.$$

Definition 1.4. If $\bar{J} \in \mathrm{MS}^\Gamma(\mathcal{M}^\times/\mathbb{C}_p^\times)$, its image under δ is called the *parabolic lifting obstruction*, and its image in $\mathrm{H}^2(\Gamma, \mathbb{C}_p^\times)$ the *lifting obstruction*, attached to $\bar{J}$.

The terminology is justified by the fact that $\bar{J}$ lifts to a (parabolic) rigid meromorphic cocycle if and only if its (parabolic) lifting obstruction vanishes.

The next lemma analyses the rightmost column of diagram (5). The sequence

$$0 \longrightarrow \mathrm{H}^1(\Gamma_\infty, \mathbb{C}_p) \longrightarrow \mathrm{H}^1(\Gamma, \mathrm{MS}(\mathbb{C}_p)) \longrightarrow \mathrm{H}^2(\Gamma, \mathbb{C}_p) \longrightarrow 0$$

which is its additive counterpart is exact on the left and on the right because $\mathrm{H}^1(\Gamma, \mathbb{C}_p) = \mathrm{H}^2(\Gamma_\infty, \mathbb{C}_p) = 0$. The vector spaces that arise in it are equipped with a natural action of the Hecke operators T_n for $p \nmid n$, defined in the usual way via double cosets. They are also equipped with an action of involutions w_p and w_∞ obtained via conjugation by the matrices

$$W_p := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \qquad W_\infty := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Lemma 1.5. *(1) The group $\mathrm{H}^1(\Gamma_\infty, \mathbb{C}_p)$ is a one-dimensional $\mathbb{C}_p$-vector space which is Eisenstein as a Hecke module, i.e., the Hecke operator T_ℓ acts on it as multiplication by $\ell + 1$ for all primes $\ell \neq p$.*

(2) There are Hecke-equivariant isomorphisms

$$\mathrm{H}^2(\Gamma, \mathbb{C}_p) \;=\; \mathrm{H}^1(\Gamma_0(p), \mathbb{C}_p) \;\simeq\; M_2(\Gamma_0(p), \mathbb{C}_p) \oplus S_2(\Gamma_0(p), \mathbb{C}_p).$$

In particular, the rank of $\mathrm{H}^2(\Gamma, \mathbb{C}_p)$ is $2g + 1$, where g is the genus of $X_0(p)$.

(3) Under the above identifications, the involution w_p acting on $\mathrm{H}^2(\Gamma, \mathbb{C}_p)$ corresponds to the negative of the Atkin-Lehner involution at p on $\mathrm{H}^1(\Gamma_0(p), \mathbb{C}_p)$, and it fixes the Eisenstein subspace of $\mathrm{H}^1(\Gamma, \mathrm{MS}(\mathbb{C}_p))$.

(4) The involution w_∞ decomposes $\mathrm{H}^1(\Gamma, \mathrm{MS}(\mathbb{C}_p))$ into a direct sum of two eigenspaces of equal dimension $g + 1$:

$$\begin{aligned} \mathrm{H}^1(\Gamma, \mathrm{MS}(\mathbb{C}_p)) &= \mathrm{H}^1(\Gamma, \mathrm{MS}(\mathbb{C}_p))^+ \oplus \mathrm{H}^1(\Gamma, \mathrm{MS}(\mathbb{C}_p))^- \\ &\simeq M_2(\Gamma_0(p)) \oplus M_2(\Gamma_0(p)). \end{aligned}$$

It acts as -1 on $\mathrm{H}^1(\Gamma_\infty, \mathbb{C}_p)$ and as 1 on the Eisenstein subspace of $\mathrm{H}^2(\Gamma, \mathbb{C}_p)$.

Proof. Assertion (1) follows from the fact, already noted before Definition 1.2, that the abelianisation of Γ_∞ modulo torsion is free of rank one over $\mathbb{Z}$. The second follows from the exact sequence

$$\mathrm{H}^1(\Gamma, \mathbb{C}_p) \longrightarrow \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}_p)^2 \longrightarrow \mathrm{H}^1(\Gamma_0(p), \mathbb{C}_p) \longrightarrow \mathrm{H}^2(\Gamma, \mathbb{C}_p) \longrightarrow \mathrm{H}^2(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}_p)^2$$

which is the second exact sequence of [Se, II. §2.8, Prop. 13] applied to the action of Γ on the Bruhat–Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$, whose edge and vertex stabilisers are isomorphic to $\Gamma_0(p)$ and $\mathrm{SL}_2(\mathbb{Z})$ respectively, and whose fundamental region consists of a single closed edge. Assertion (3) is proved in [Dar, Lemma 1.4], while assertion (4) can be treated by the same arguments as in [Dar, Lemma 1.8]. □

Elements $\bar{J} \in \mathrm{MS}^\Gamma(\mathcal{M}^\times/\mathbb{C}_p^\times)$ can be "evaluated" at pairs of distinct elements of $\mathbb{P}_1(\mathbb{Q})$. Namely, if (r, s) is such an ordered pair, the stabiliser of (r, s) in Γ is the product of a finite torsion group with an infinite cyclic group of rank one. Let $\gamma_{r,s}$ be a generator (modulo torsion) of this stabiliser, having r as an attractive and s as a repulsive fixed point.

Definition 1.6. The *parabolic lifting obstruction* of $\bar{J}$ at (r, s) is the quantity

$$\bar{J}[r, s] := \tilde{J}\{r, s\}(\gamma_{r,s}z) \div \tilde{J}\{r, s\}(z) \in \mathbb{C}_p^\times \quad (\text{mod torsion}),$$

where $\tilde{J}\{r, s\}$ is an arbitrary lift of $\bar{J}\{r, s\} \in \mathcal{M}^\times/\mathbb{C}_p^\times$ to $\mathcal{M}^\times$.

The terminology is justified by the fact that

$$\bar{J}[r, s] = \kappa(\gamma_{r,s})\{r, s\}, \tag{6}$$

where κ is the parabolic lifting obstruction of $\bar{J} \in \mathrm{MS}^\Gamma(\mathcal{M}^\times/\mathbb{C}_p^\times)$. The following lemma spells out the relation between cuspidal values and parabolic lifting obstructions:

Lemma 1.7. *Let J be a rigid meromorphic cocycle and let $\bar{J}$ be its natural image in* $\mathrm{MS}^\Gamma(\mathcal{M}^\times/\mathbb{C}_p^\times)$. *Then*

$$J[\infty]^2 = \bar{J}[0, \infty].$$

Proof. Let

$$S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma_{0\infty} := \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}.$$

The relation $S\gamma_{0\infty} = \gamma_{0\infty}^{-1}S$ shows, in light of the cocycle relation, that

$$J(S) \times SJ(\gamma_{0\infty}) = J(\gamma_{0\infty}^{-1}) \times \gamma_{0\infty}^{-1}J(S),$$

and hence, since $SJ(\gamma_{0\infty}) = J(\gamma_{0\infty}) = J(\gamma_{0\infty}^{-1})^{-1}$,

$$J(\gamma_{0\infty})^2 = J(S)(p^2z)/J(S)(z) = \tilde{J}\{0, \infty\}(\gamma_{0\infty}z)/\tilde{J}\{0, \infty\}(z).$$

The result then follows from Definition 1.6. □

1.2. **RM values of rigid cocycles modulo scalars.** Recall that the *value* of a rigid meromorphic cocycle J at an RM point τ is defined by setting

$$J[\tau] := J(\gamma_\tau)(\tau),$$

where γ_τ is a (suitably normalised) generator of the stabiliser of τ in Γ. These RM values are the main subject of [DV]. We conclude this chapter by extending the notion of RM values to cocycles *modulo scalars*.

Because the group $\mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}_p^\times)$ is of order ≤ 12 and $\mathrm{H}^2(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}_p^\times)$ is trivial, the restriction of $\bar{J}$ to $\mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), \mathcal{M}^\times/\mathbb{C}_p^\times)$ lifts to an element

$$\tilde{J} \in \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), \mathcal{M}^\times)$$

which is unique up to a 12-torsion element. Let $\mathcal{H}_p^{\mathrm{RM},\circ}$ denote the set of $\tau \in \mathcal{H}_p^{\mathrm{RM}}$ which are the root of a primitive *integral* binary quadratic form of discriminant prime to p. The matrix γ_τ then belongs to $\mathrm{SL}_2(\mathbb{Z})$, and this makes the following definition possible:

Definition 1.8. For all $\tau \in \mathcal{H}_p^{\mathrm{RM},\circ}$, the value of $\bar{J}$ at τ is

$$\bar{J}[\tau] := \tilde{J}(\gamma_\tau)(\tau).$$

When $\bar{J}$ lifts to a rigid meromorphic cocycle J, one has $\bar{J}[\tau] = J[\tau]$, but even when such a lift does not exist, the quantities $\bar{J}[\tau]$ are of great arithmetic interest, insofar as they are related to Gross–Stark units and Stark–Heegner points, as will be seen shortly.

2. Theta cocycles

We now recall the classification of elements of $\mathrm{H}^1(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times)$ that was largely carried out in [DV]. More precisely, this chapter describes certain elements

$$\bar{J}_\tau \in \mathrm{MS}^\Gamma(\mathcal{M}^\times/\mathbb{C}_p^\times),$$

called *theta cocycles*, which are indexed by elements $\tau \in \Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}$ and generate, together with $\mathrm{MS}^\Gamma(\mathcal{A}^\times/\mathbb{C}_p^\times)$, the group $\mathrm{MS}^\Gamma(\mathcal{M}^\times/\mathbb{C}_p^\times)$ up to torsion.

2.1. **The Bruhat–Tits tree.** Let $v_\circ$ be the standard vertex of the Bruhat–Tits tree $\mathcal{T}$ of $\mathrm{PGL}_2(\mathbb{Q}_p)$, whose stabiliser in Γ is $\mathrm{SL}_2(\mathbb{Z})$. For each integer $n \geq 0$, let $\mathcal{T}^{\leq n}$ denote the subgraph of $\mathcal{T}$ consisting of vertices and edges that are at distance $\leq n$ from $v_\circ$, and let $\mathcal{H}_p^{\leq n} \subset \mathcal{H}_p$ denote the inverse image of $\mathcal{T}^{\leq n}$ under the reduction map. The collection of $\mathcal{H}_p^{\leq n}$ gives an admissible covering of $\mathcal{H}_p$ by affinoid subsets which are stable under the action of $\mathrm{SL}_2(\mathbb{Z})$. If Π is any finite subgraph of $\mathcal{T}$, the stabiliser of Π in Γ is denoted Γ_Π. The groups Γ_Π are conjugate to finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and act discretely on $\mathcal{H}$.

If v is a vertex of $\mathcal{T}$, let $\mathcal{W}_v$ denotes the wide open subset corresponding to the vertex v, corresponding to the set of $z \in \mathcal{H}_p$ whose image under the reduction map is either v or one of the edges having v as an endpoint. When $v = v_\circ$, we denote this wide open simply by $\mathcal{W}_\circ$. The set of all $\mathcal{W}_v$ gives an admissible covering of $\mathcal{H}_p$ by wide open subsets, and the subgroup of Γ that preserves $\mathcal{W}_\circ$ is equal to $\mathrm{SL}_2(\mathbb{Z})$. More generally, the stabiliser of v (or $\mathcal{W}_v$) in Γ, denoted Γ_v, is conjugate to $\mathrm{SL}_2(\mathbb{Z})$.

2.2. **The functions $t_\Delta^\eta(z)$.** Fix an auxiliary base point $\eta \in \mathcal{H}_p$. Given $w \in \mathcal{H}_p$, let $t_w^\eta(z) := \frac{z-w}{\eta-w}$ be the rational function satisfying

$$\mathrm{Div}(t_w^\eta) = (w) - (\infty), \qquad t_w^\eta(\eta) = 1.$$

More generally, if $W := \sum m_i \cdot (w_i)$ is a divisor on $\mathcal{H}_p$, then the function

$$t_W^\eta(z) := \prod t_{w_i}^\eta(z)^{m_i}$$

is the unique rational function having $W - \deg(W) \cdot \infty$ as divisor which satisfies $t_W^\eta(\eta) = 1$. Note that when $W = (w_1) - (w_2)$, the rational function $t_W^\eta(z)$ can be expressed as the cross-ratio:

$$t_{(w_1)-(w_2)}^\eta(z) = (z, \eta; w_1, w_2) := \frac{(z-w_1)(\eta-w_2)}{(z-w_2)(\eta-w_1)},$$

and hence satisfies the Γ-equivariance property

$$t_{\gamma W}^{\gamma\eta}(\gamma z) = t_W^\eta(z), \qquad \text{for all } \gamma \in \Gamma,$$

which remains valid when W is replaced with any degree zero divisor.

2.3. **Theta cocycles.** Let $\tau \in \mathcal{H}_p$ be any RM point, let F be the real quadratic field that it generates, and let D be the discriminant of F. The element τ is the root of a unique (up to sign) primitive integral binary quadratic form, whose discriminant is of the form Dn^2p^{2t} for some positive integer n which is prime to p. The integer Dn^2 is called the discriminant of τ, and is denoted $\mathrm{disc}(\tau)$. In particular, the RM point τ is fundamental if its discriminant is equal to D. The field F shall be viewed both as a subfield of $\mathbb{R}$ and as a subfield of $\mathbb{C}_p$ which is not contained in $\mathbb{Q}_p$.

For $w \in F$, let (w, w') denote the geodesic on $\mathcal{H}$ joining w to its Galois conjugate w', which maps to a countable union of copies of the same basic closed geodesic on $\Gamma_\Pi\backslash\mathcal{H}$ for any subgroup $\Gamma_\Pi \subset \Gamma$. Likewise, if r, s are elements of $\mathbb{P}_1(\mathbb{Q})$, write $[r, s]$ for the hyperbolic geodesic on $\mathcal{H}$ joining these two elements, which maps to a compact (but not necessarily closed) geodesic on any quotient $\Gamma_\Pi\backslash\mathcal{H}$. The geodesics (w, w') and $[r, s]$ always intersect properly, and we set

$$\delta_{r,s}(w) := (w, w') \cdot [r, s] = \begin{cases} 1 & \text{if the two geodesics intersect positively;} \\ -1 & \text{if they intersect negatively;} \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

The infinite formal sum

$$\Delta_\tau\{r, s\} := \sum_{w \in \Gamma\tau} \delta_{r,s}(w) \cdot (w) \tag{8}$$

defines a Γ-invariant modular symbol with values in the Γ-module $\mathbb{Z}\langle\Gamma\tau\rangle$ of formal (possibly infinite) $\mathbb{Z}$-linear combinations of points of $\Gamma\tau$. Set

$$\Delta_\tau^v\{r, s\} := \sum_{w \in \Gamma\tau \cap \mathcal{W}_v} \delta_{r,s}(w) \cdot (w)$$

which defines a Γ_v-invariant modular symbol. The divisor $\Delta_\tau^v\{r, s\}$ has finite support, since for a fixed discriminant there are at most finitely many w such that $\delta_{r,s}(w) \neq 0$. We may therefore consider the function

$$(r, s) \mapsto \deg \Delta_\tau^v\{r, s\}$$

which defines an element in $\mathrm{MS}^{\Gamma_v}(\mathbb{Z})$. Since Γ_v has finite abelianisation, this space of modular symbols is trivial, and therefore $\deg \Delta_\tau^v\{r, s\}$ is identically zero for all v, τ and (r, s). This implies that we may write (non-canonically)

$$\Delta_\tau\{r, s\} = \sum_{j=1}^{\infty} (w_j^+) - (w_j^-), \tag{9}$$

where w_j^+ and w_j^- are contained in the same wide open $\mathcal{W}_v$.

Lemma 2.1. *For all $r, s \in \mathbb{P}_1(\mathbb{Q})$ and $\tau \in \mathcal{H}_p^{\mathrm{RM}}$, the infinite product*

$$J_\tau^\eta\{r, s\}(z) = \prod_{w \in \Gamma\tau} t_w^\eta(z)^{\delta_{r,s}(w)} := \prod_{j=1}^{\infty} t^\eta_{w_j^+ - w_j^-}(z)$$

converges uniformly to a rigid meromorphic function on any affinoid subset of $\mathcal{H}_p$. The rigid meromorphic functions $J_\tau^\eta\{r, s\}(z)$ satisfy the following properties:

(a) $J_\tau^\eta\{r, s\}$ is a modular symbol with values in $\mathcal{M}^\times$, i.e.,

$$J_\tau^\eta\{r, s\} \times J_\tau^\eta\{s, t\} = J_\tau^\eta\{r, t\} \qquad \text{for all } r, s, t \in \mathbb{P}_1(\mathbb{Q}).$$

(b) The rigid meromorphic function $J_\tau^\eta\{r,s\}(z)$ *is independent, up to multiplication by a non-zero scalar, of the choice of base point* η*, i.e., for all* $\eta, \eta' \in \mathcal{H}_p$,

$$J_\tau^\eta\{r,s\}(z) = \lambda \times J_\tau^{\eta'}\{r,s\}(z), \qquad \text{where } \lambda = J_\tau^\eta\{r,s\}(\eta').$$

In particular, the image of $J_\tau^\eta\{r,s\}$ *in* $\mathcal{M}^\times/\mathbb{C}_p^\times$ *does not depend on* η*, and shall therefore be denoted* $\bar{J}_\tau\{r,s\}$.

(c) The modular symbol $J_\tau^\eta\{r,s\}(z)$ *satisfies the* Γ*-invariance properties*

$$J_\tau^{\gamma\eta}\{\gamma r, \gamma s\}(\gamma z) = J_\tau^\eta\{r,s\}(z) \qquad \text{for all } \gamma \in \Gamma,$$

and hence

$$\bar{J}_\tau\{\gamma r, \gamma s\}(\gamma z) = \bar{J}_\tau\{r,s\}(z), \qquad \text{for all } \gamma \in \Gamma.$$

Proof. For all $k > 1$, we have that

$$f_k - f_{k-1} = \left(t^\eta_{w_k^+ - w_k^-}(z) - 1\right) \cdot f_{k-1}, \qquad \text{where } f_k(z) := \prod_{j=N}^{k} t^\eta_{w_j^+ - w_j^-}(z). \tag{10}$$

Now fix $n \geq 1$, and suppose that $z \in \mathcal{H}_p^{\leq n}$. Choose an arbitrary $N > 0$. Since $t^\eta_{w_k^+ - w_k^-}(z)$ is the cross-ratio of w_k^+, w_k^-, η, and z, which is invariant under $\mathrm{GL}_2(\mathbb{Q}_p)$, we may assume without loss of generality, after acting by an appropriate element of this group, that for all k large enough, we have

(i) η belongs to $\mathcal{W}_\circ$,
(ii) w_k^+ and w_k^- are both congruent to 0 modulo p^N,
(iii) $z \in \mathbb{C}_p$ satisfies $\mathrm{ord}_p(z) \geq -n'$, with $n' = n + O_\eta(1)$.

In this case, we readily obtain the estimate

$$\left| t^\eta_{w_k^+ - w_k^-}(z) - 1 \right| \leq C_\eta p^{n-N}, \qquad \text{for all } z \in \mathcal{H}_p^{\leq n}.$$

for some constant C_η that only depends on η. It follows from (10) that

$$\lim_{k\to\infty} \|f_k - f_{k-1}\|_{\mathcal{H}_p^{\leq n}} = 0,$$

so that the sequence of partial products f_k converges uniformly on $\mathcal{H}_p^{\leq n}$ to a nowhere vanishing rigid analytic function. It follows that the infinite product defining $J_\tau^\eta\{r,s\}(z)$ converges to a meromorphic function on $\mathcal{H}_p^{\leq n}$. Since this is true for all $n \geq 1$ and since the collection $\{\mathcal{H}_p^{\leq n}\}_{n\geq 1}$ is an admissible affinoid covering of $\mathcal{H}_p$, the first statement follows.

The remaining properties are formal consequences of the definitions, following the arguments in [GVdP, II. §2]. See also [DV, §3.3]. □

Lemma 2.1 implies that $\bar{J}_\tau$ is a Γ-invariant modular symbol with values in $\mathcal{M}^\times/\mathbb{C}_p^\times$. As before, we use the same notation $\bar{J}_\tau$ to designate the associated parabolic cohomology class, satisfying

$$\bar{J}_\tau(\gamma)(z) := \bar{J}_\tau\{\infty, \gamma\infty\}(z), \qquad \text{for all } \gamma \in \Gamma.$$

Definition 2.2. The class $\bar{J}_\tau$ is called the *theta cocycle* associated to $\tau \in \Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}$.

One of the most important results of [DV] is that the cocycles $\bar{J}_\tau$ generate $\mathrm{MS}^\Gamma(\mathcal{M}^\times/\mathbb{C}_p^\times)$ up to analytic cocycles, see [DV, Lemma 3.12, Theorem 3.13].

Proposition 2.3. *Up to elements of* $\mathrm{MS}^\Gamma(\mathcal{A}^\times/\mathbb{C}_p^\times)$*, every class in* $\mathrm{MS}^\Gamma(\mathcal{M}^\times/\mathbb{C}_p^\times)$ *can be expressed as a finite product of cocycles of the form* $\bar{J}_\tau$.

3. Analytic cocycles

The description of $\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times)$ implicit in Proposition 2.3 will now be made more precise by examining the group

$$\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times) = \mathrm{MS}^\Gamma(\mathcal{A}^\times/\mathbb{C}_p^\times)$$

of rigid analytic cocycles modulo scalars.

3.1. **The Schneider–Teitelbaum lift.** The logarithmic derivative map embeds the multiplicative group $\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$ into the $\mathbb{C}_p$-vector space $\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{A}_2)$, where $\mathcal{A}_2$ denotes the rigid analytic differentials on $\mathcal{H}_p$ equipped with the "weight two action" of Γ. Let

$$U := \{z \in \mathbb{C}_p \text{ with } 1/p < |z| < 1\} \subset \mathcal{H}_p \tag{11}$$

denote the standard annulus, whose stabiliser in Γ is equal to $\Gamma_0(p)$. The p-adic annular residue $\omega \mapsto \mathrm{res}_U(\omega)$, as described for instance in [Sch, §II] or [Te], determines a map

$$\mathrm{res}_U : \mathcal{A}_2 \longrightarrow \mathbb{C}_p$$

Theorem 3.1. *The maps*

$$\mathrm{res}_U : \mathrm{H}^1(\Gamma, \mathcal{A}_2) \longrightarrow \mathrm{H}^1(\Gamma_0(p), \mathbb{C}_p), \qquad \mathrm{res}_U : \mathrm{MS}^\Gamma(\mathcal{A}_2) \longrightarrow \mathrm{MS}^{\Gamma_0(p)}(\mathbb{C}_p)$$

induced by the p-adic annular residue are isomorphisms of $\mathbb{C}_p$-vector spaces.

The proof of the first isomorphism is described in [DV, §2], and the second follows from exactly the same principles. Both rest on the construction of explicit inverses to the above maps, referred to as *Schneider–Teitelbaum lifts*. Theorem 3.1 leads to the construction of various explicit analytic cocycles, as described in [DV, §5]:

(i) Let

$$E_2^{(p)}(z) = \frac{p-1}{12} + 2\sum_{n\geq 1} \sigma_1^{(p)}(n) e^{2\pi i n z}, \qquad \text{where } \sigma_1^{(p)}(n) = \sum_{p\nmid d | n} d$$

be the weight two Eisenstein series on $\Gamma_0(p)$, and let $\omega_{\mathrm{Eis}} := 2\pi i E_2^{(p)}(z)dz$ be the associated differential of the third kind on the modular curve $X_0(p)$. The periods of ω_{Eis} are encoded in the *Dedekind–Rademacher homomorphism* $\varphi_{\mathrm{DR}} : \Gamma_0(p) \longrightarrow \mathbb{Z}$ defined by

$$\varphi_{\mathrm{DR}} := (2\pi i)^{-1} \int_{z_0}^{\gamma z_0} \omega_{\mathrm{Eis}}. \tag{12}$$

The *Dedekind–Rademacher cocycle* $\bar{J}_{\mathrm{DR}} \in \mathrm{H}^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$ of [DV, §5.3] is the multiplicative Schneider–Teitelbaum lift of φ_{DR}.

(ii) Let $f \in S_2(\Gamma_0(p))$ be a normalised cuspidal newform with fourier coefficients in a field $K_f \subset \mathbb{R}$, and let $\omega_f := 2\pi i f(z)dz$ be the associated regular differential on $X_0(p)$. The real analytic differentials

$$\omega_f^+ := \frac{1}{2}(\omega_f + \bar{\omega}_f), \qquad \omega_f^- := \frac{1}{2}(\omega_f - \bar{\omega}_f)$$

give rise to modular symbols φ_f^+ and $\varphi_f^- \in \mathrm{MS}^{\Gamma_0(p)}(K_f)$, defined by

$$\varphi_f^+\{r,s\} := (\Omega_f^+)^{-1} \int_r^s \omega_f^+, \qquad \varphi_f^-\{r,s\} = (\Omega_f^-)^{-1} \int_r^s \omega_f^-, \tag{13}$$

where Ω_f^+ and Ω_f^- are the so-called *real and imaginary periods* attached to f. The requirement that $\varphi_f^\pm$ be K_f-valued only determines these modular symbols up to multiplication by $K_f^\times$, but we can further insist that

(a) φ_f^- takes values in $\mathcal{O}_{K_f}$, and maps surjectively to $\mathbb{Z}$ when $K_f = \mathbb{Q}$.
(b) $\Omega_f^+\Omega_f^- = \Omega_f := -\langle\omega_f^+, \omega_f^-\rangle$.

When f has integer Fourier coefficients, these conditions determine Ω_f^- and Ω_f^+ up to a sign. The resulting Schneider–Teitelbaum lifts are denoted $\bar{J}_f^+$ and $\bar{J}_f^- \in \mathrm{MS}^\Gamma(\mathcal{A}^\times/\mathbb{C}_p^\times)$. They are called the *elliptic modular cocycles* attached to f, and are described in [DV, §5.4]. Note that we can exploit the natural injection $\mathrm{MS}^\Gamma(\mathcal{A}^\times/\mathbb{C}_p^\times) \longrightarrow H^1_{\mathrm{par}}(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$ to view $\bar{J}_f^\pm$ as elements of the latter group when this is convenient.

3.2. **The winding cocycle.** The primary goal of this section is to further enrich the theory of rigid analytic cocycles by describing the construction of certain explicit (not necessarily parabolic) elements

$$\bar{J}_\rho \in \mathrm{H}^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times),$$

which are indexed by ordered pairs $\rho = (r, s)$ of distinct cusps in $\mathbb{P}_1(\mathbb{Q})$.

The *determinant* of a pair (r, s) of distinct elements of $\mathbb{P}_1(\mathbb{Q})$ is $ad - bc$, where $r = a/b$ and $s = c/d$ are expressions for r and s as fractions in lowest terms, adopting the usual convention that $\infty = 1/0$. It is an integer that is well-defined up to sign, hence shall always be normalised to be positive. If (r, s) and (r', s') are Γ-equivalent, then their determinants differ by a power of p. Let Σ_ρ denote the Γ-orbit of the pair ρ, and let $\Sigma_\rho^{(m)} \subset \Sigma_\rho$ be the subset of pairs (r, s) with $\mathrm{ord}_p(\det(r, s)) = m$. It is not hard to see that $\Sigma_\rho^{(m)}$ is non-empty for all $m \geq 0$ and that

$$\Sigma_\rho = \bigcup_{m=0}^{\infty} \Sigma_\rho^{(m)}.$$

For each pair (r, s), and a base point $\eta \in \mathcal{H}_p$, let $t^\eta_{r,s}(z)$ be the unique rational function having $(r) - (s)$ as a divisor and satisfying $t^\eta_{r,s}(\eta) = 1$. One also defines functions $t_{r,s}(z)$ by expressing $r = \frac{a}{b}$ and $s = \frac{c}{d}$ as fractions in lowest terms, in such a way that $ad - bc > 0$, and setting

$$t_{r,s}(z) = \frac{bz - a}{dz - c}.$$

The function $t_{r,s}(z)$ depends only on the pair (r, s) and its divisor is equal to $(r) - (s)$. Hence it differs from $t^\eta_{r,s}(z)$ by a constant. More precisely

$$t_{r,s}(z) = t_{r,s}(\eta) \cdot t^\eta_{r,s}(z) = \left(\frac{b\eta - a}{d\eta - c}\right) \cdot t^\eta_{r,s}(z).$$

It shall be useful to examine the restrictions of the functions $t^\eta_{r,s}(z)$ and $t_{r,s}(z)$ on certain affinoid subsets of $\mathcal{H}_p$. We shall follow the notations from §2.1.

Lemma 3.2. *Let $m > n \geq 0$ be integers. For all $(r, s) \in \Sigma_\rho^{(m)}$,*

(1) the restriction of $t_{r,s}(z)$ to $\mathcal{H}_p^{\leq n}$ takes values in $w + p^{m-n}\mathcal{O}_{\mathbb{C}_p}$ for some $w \in \mathbb{Z}_p^\times$;
(2) the restriction of $t^\eta_{r,s}(z)$ to $\mathcal{H}_p^{\leq n}$ takes values in $1 + p^{m-n}\mathcal{O}_{\mathbb{C}_p}$.

Proof. If $r = \frac{a}{b}$ and $s = \frac{c}{d}$, the fact that $ad - bc = p^m$ implies that the primitive vectors (a, b) and (c, d) in $\mathbb{Z}^2$ are proportional to each other modulo p^m. Hence there exists $v \in \mathbb{Z}_p^\times$ for which $(a, b) = v \cdot (c, d) + p^m(e, f)$ for some $(e, f) \in \mathbb{Z}^2$. It follows that

$$t_{r,s}(z) = v + p^m \frac{fz - e}{dz - c}.$$

But as z ranges over $\mathcal{H}_p^{\leq n}$, the rational function $\frac{fz-e}{dz-c}$ takes values in $p^{-n}\mathcal{O}_{\mathbb{C}_p}$, and the first statement follows. For the second it suffices to note that $t^\eta_{r,s}(z)$ is proportional to $t_{r,s}(z)$ by a factor in $\mathcal{O}_{\mathbb{C}_p}^\times$ and therefore takes constant values on $\mathcal{H}_p^{\leq n}$ modulo p^{m-n}, combined with the fact that $t^\eta_{r,s}(\eta) = 1$. □

If ξ_1 and ξ_2 are two points of the extended upper-half plane $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}_1(\mathbb{Q})$, the symbol $[\xi_1, \xi_2]$ is used to denote the hyperbolic geodesic segment on $\mathcal{H}$ going from ξ_1 to ξ_2. The intersection of two (open or closed) geodesic segments on $\mathcal{H}$ is defined in the natural way, as in (7). A point $\xi \in \mathcal{H}^*$ is said to be *ρ-admissible* if it does not lie on any geodesic in $\Gamma\rho$. Clearly, all p-admissible ξ belong to $\mathcal{H}$, and the set of p-admissible base points is preserved by the action of Γ. Since the non-admissible points are contained in a countable union of sets of measure zero, the existence of admissible base points is clear. For computational purposes it can be desirable to dispose of concrete constructions, such as are provided by the following lemma when $\rho = (0, \infty)$:

Lemma 3.3. *Suppose that $\xi \in \mathcal{H}$ is the root of a primitive integral binary quadratic form $[A, B, C]$ of discriminant $\Delta := B^2 - 4AC < 0$, and that*

(1) the prime p is inert in the imaginary quadratic field $\mathbb{Q}(\sqrt{\Delta})$;

(2) the class of $[A, B, C]$ is of order > 2 in the class group attached to Δ.

Then ξ is ρ-admissible, where $\rho = (0, \infty)$.

Proof. If ξ lies on a Γ-translate of $[0, \infty]$, then there is a Γ-translate ξ' of ξ which lies on the geodesic $[0, \infty]$, and hence ξ' is the root of a primitive binary quadratic form of the type $Ax^2 + Cy^2$, which is of discriminant Dp^{2t} for some $t \geq 0$ and of order 2 in the associated class group. It follows from the second assumption that ξ cannot lie on such a geodesic. □

Fix a ρ-admissible base point $\xi \in \mathcal{H}$, and set

$$J_\rho^\eta(\gamma)(z) = \prod_{(r,s) \in \Sigma_\rho} \left(t_{r,s}^\eta(z)\right)^{[r,s] \cdot [\xi, \gamma\xi]}.$$

Lemma 3.4. *For each $\gamma \in \Gamma$, the infinite product defining $J_\rho^\eta(\gamma)$ converges to a rigid analytic function on $\mathcal{H}_p$ and its image $\bar{J}_\rho(\gamma)$ in $\mathcal{A}^\times/\mathbb{C}_p^\times$ satisfies a cocycle condition modulo scalars, namely*

$$\bar{J}_\rho(\gamma_1\gamma_2) = \bar{J}_\rho(\gamma_1) \times \gamma_1 \cdot \bar{J}_\rho(\gamma_2).$$

Proof. Observe first that $\Gamma_\circ := \mathrm{SL}_2(\mathbb{Z})$ acts on the set $\Sigma_\rho^{(m)}$ by Möbius transformations, and that there are finitely many orbits for this action:

$$\Sigma_\rho^{(m)} \quad = \quad \Gamma_\circ \cdot (r_1, s_1) \quad \sqcup \quad \Gamma_\circ \cdot (r_2, s_2) \quad \sqcup \quad \cdots \quad \sqcup \quad \Gamma_\circ \cdot (r_\ell, m_\ell).$$

But the cardinality of the set

$$\{\alpha \in \Gamma_\circ \text{ such that } [\alpha r, \alpha s] \cdot [\xi, \gamma\xi] = \pm 1\}$$

represents the number of intersection points between the images of the geodesics $[r, s]$ and $[\xi, \gamma\xi]$ in the quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$. Since this number is finite, it follows that the product

$$J_{\rho,m}^\eta(\gamma)(z) := \prod_{(r,s) \in \Sigma_\rho^{(m)}} \left(t_{r,s}^\eta(z)\right)^{[r,s] \cdot [\xi, \gamma\xi]}$$

has finitely many factors that are $\neq 1$, and hence is a rational function of z. To prove convergence of

$$J_\rho^\eta(\gamma)(z) := \prod_{m=0}^{\infty} J_{\rho,m}^\eta(\gamma)(z)$$

as a rigid meromorphic function of $z \in \mathcal{H}_p^{\leq n}$ it suffices to show that the restriction of $J_{\rho,m}^\eta(\gamma)$ to $\mathcal{H}_p^{\leq n}$ converges uniformly to 1 as $m \longrightarrow \infty$. But this follows directly from Lemma 3.2. We have thus showed that the infinite product defining $J_\rho^\eta(z)$ converges absolutely and uniformly on affinoid subsets of $\mathcal{H}_p$. The cocycle condition for $\bar{J}_\rho$ then follows by exactly the same reasoning as was used in the previous chapter to show that $\bar{J}_\tau$ belongs to $\mathrm{H}^1(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times)$. $\square$

Lemma 3.5. *The class of $\bar{J}_\rho$ in $\mathrm{H}^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$ does not depend on the choice of base point $\eta \in \mathcal{H}_p$ and of admissible base point $\xi \in \mathcal{H}$ that were made to define it.*

Proof. Changing the base point η to η' merely multiplies the functions $J_\rho^\eta(\gamma)$ by a non-zero scalar, and hence does not affect the cocycle $\bar{J}_\rho \in Z^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$. As for replacing ξ by ξ' in the definition of J_ρ^η, a direct calculation reveals that the associated cocycles differ by the coboundary dF, where

$$F(z) = \prod_{(r,s)\in\Sigma} \left(t_{r,s}^\eta(z)\right)^{[r,s]\cdot[\xi,\xi']} \in \mathcal{A}^\times.$$

$\square$

We now turn to the question of lifting $\bar{J}_\rho \in \mathrm{H}^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$ to a genuine analytic cocycle

$$J_\rho \overset{?}{\in} \mathrm{H}^1(\Gamma, \mathcal{A}^\times).$$

In general, the cocycle $\bar{J}_\rho$ need not admit such a lift, but its restriction to $\Gamma_\circ := \mathrm{SL}_2(\mathbb{Z})$ does, by the triviality of $\mathrm{H}^2(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}_p^\times)$. To describe this lift, we first restrict $\bar{J}_\rho$ to $\mathrm{H}^1(\Gamma_\circ, \mathcal{A}^\times/\mathbb{C}_p^\times)$ and construct an explicit lift of it to a class

$$J_\rho^\circ \in \mathrm{H}^1(\Gamma_\circ, \mathcal{A}^\times).$$

The first part of Lemma 3.2 suggests that replacing $t_{r,s}^\eta$ by $t_{r,s}$ in the definition of J_ρ^η leads to an infinite product which need not converge in general. However, we have

Proposition 3.6. *For all $\gamma \in \Gamma_\circ$, the infinite product*

$$J_\rho^\circ(\gamma)(z) := \prod_{m=0}^{\infty} J_{\rho,m}^\circ(\gamma)(z), \qquad \text{where } J_{\rho,m}^\circ(\gamma)(z) := \prod_{(r,s)\in\Sigma_\rho^{(m)}} (t_{r,s}(z))^{[r,s]\cdot[\xi,\gamma\xi]}$$

converges to a rigid analytic function on $\mathcal{H}_p$, up to 12-th roots of unity, and gives rise to an element of $\mathrm{H}^1(\Gamma_\circ, \mathcal{A}^\times/\mu_{12})$.

Proof. For integers $m > n \geq 0$, consider the restriction of $J_{\rho,m}^\circ(\gamma)(z)$ to the affinoid $\mathcal{H}_p^{\leq n}$. By Lemma 3.2, this restriction is constant modulo p^{m-n} and hence its mod p^{m-n} reduction defines a cocycle in $\mathrm{H}^1(\Gamma_\circ, (\mathbb{Z}/p^{m-n}\mathbb{Z})^\times)$. Since the abelianisation of $\Gamma_\circ$ is of order 12, it follows that

$$J_{\rho,m}^\circ(\gamma)(z)|_{\mathcal{H}_p^{\leq n}} \in \mu_{12} \pmod{p^{m-n}}.$$

The convergence of the infinite product (up to 12 th roots of unity) follows. $\square$

3.3. **Decomposition in cohomology.** Recall the *multiplicative Schneider–Teitelbaum lift*

$$L_{\mathrm{ST}}^\times : \mathrm{H}^1(\Gamma_0(p), \mathbb{Z}) \longrightarrow \mathrm{H}^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$$

that is described in [DV, §5.3]. For any pair $\rho = (r, s)$ of distinct elements $r, s \in \mathbb{P}_1(\mathbb{Q})$, let $[r, s]$ be the image of the geodesic path from r to s in the relative homology of $X_0(p)$ relative to the cusps, and let $\varphi_\rho : \Gamma_0(p) \longrightarrow \mathbb{Z}$ be the homomorphism defined by

$$\varphi_\rho(\gamma) = [r, s] \cdot \gamma,$$

where $\cdot$ denotes the intersection pairing

$$\mathrm{H}_1(X_0(p); \{0, \infty\}, \mathbb{Z}) \times \mathrm{H}_1(Y_0(p), \mathbb{Z}) \longrightarrow \mathbb{Z}.$$

Proposition 3.7. *The cocycle* $\bar{J}_\rho$ *is the image of* φ_ρ *under* $L_{\mathrm{ST}}^\times$*:*

$$\bar{J}_\rho = L_{\mathrm{ST}}^\times(\varphi_\rho).$$

Proof. Recall the standard annulus U of (11) having $\Gamma_0(p)$ as its stabiliser in Γ. The inverse of the Schneider–Teitelbaum lift takes a cocycle $\bar{J} \in \mathrm{H}^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$ to the homomorphism

$$\phi_J : \Gamma_0(p) \longrightarrow \mathbb{Z}, \qquad \phi_J(\gamma) := \mathrm{res}_U(\mathrm{dlog}\, \bar{J}(\gamma)),$$

where res_U is the p-adic annular residue attached to U. Consider the infinite product expression of Proposition 3.6 for J_ρ° and observe that the terms $\mathrm{dlog}\, J_{\rho,m}^\circ(\gamma)$ for $m \geq 1$ contribute nothing to the annular residue at U: indeed, two cusps r, s for which $\det(r,s) = p^m$ either both belong to U or to its complement, and hence $\mathrm{res}_U(\mathrm{dlog}\, t_{r,s}(z)) = 0$ for such pairs. On the other hand,

$$\mathrm{res}_U(\mathrm{dlog}\, t_{r,s}(z)) = \begin{cases} 1 & \text{if } r \notin \mathbb{Z}_p, s \in \mathbb{Z}_p, \\ -1 & \text{if } r \in \mathbb{Z}_p, s \notin \mathbb{Z}_p. \end{cases}$$

Hence, any pair (r,s) for which the residue of $\mathrm{dlog}\, t_{r,s}(z)$ is equal to 1 is of the form $(\alpha r_0, \alpha s_0)$, for some $\alpha \in \Gamma_0(p)$, where $(r_0, s_0) \in \Sigma_{\rho,0}$ is any pair for which $r_0 \notin \mathbb{Z}_p$ and $s_0 \in \mathbb{Z}_p$.

It follows that

$$\begin{aligned} \mathrm{res}_U(\mathrm{dlog}\, J_{r,s}(\gamma)) &= \sum_{\alpha \in \Gamma_0(p)} (+1)[\alpha r_0, \alpha s_0] \cdot [\xi, \gamma\xi] + \sum_{\alpha \in \Gamma_0(p)} (-1)[\alpha s_0, \alpha r_0] \cdot [\xi, \gamma\xi] \\ &= 2 \sum_{\alpha \in \Gamma_0(p)} [\alpha r_0, \alpha s_0] \cdot [\xi, \gamma\xi]. \end{aligned}$$

In this last expression one can recognise the intersection product of the relative homology class $[r,s]$ with the class of γ in $\mathrm{H}_1(Y_0(p), \mathbb{Z})$. The proposition follows. ☐

We now specialise to the case where $(r,s) = (0, \infty)$, and describes the decomposition of the cohomology class $\varphi_{[0,\infty]}$ relative to the $\bar{\mathbb{Q}}$- basis $(\varphi_{\mathrm{DR}}, \varphi_f^\pm)$ for $\mathrm{MS}^{\Gamma_0(p)}(\bar{\mathbb{Q}})$ described in (12) and (13).

Lemma 3.8. *The homomorphism* $\varphi_{[0,\infty]}$ *is equal to*

$$\varphi_{[0,\infty]} = \left(\frac{p-1}{12}\right)^{-1} \varphi_{\mathrm{DR}} + \sum_f L_{\mathrm{alg}}(f,1) \cdot \varphi_f^-,$$

where the sum runs over a basis of normalised eigenforms for $S_2(\Gamma_0(p))$*.*

Proof. Recall the canonical identifications

$$\mathrm{H}_1(Y_0(p); \{0,\infty\}, \mathbb{C}) \longrightarrow \mathrm{H}_c^1(Y_0(p))^\vee \longrightarrow \mathrm{H}_{\mathrm{dR}}^1(Y_0(p)),$$

where H_c^1 denotes the deRham cohomology with compact support and the superscript $\vee$ denotes the $\mathbb{C}$-linear dual. The first identification arises from the integration pairing and the second from Poincaré duality. Let $G_{[0,\infty]}$ be the class in $\mathrm{H}_{\mathrm{dR}}^1(Y_0(p))$ corresponding to $\gamma_{[0,\infty]}$ under this identification, which is characterised by the equivalent conditions

$$\int_\gamma G_{[0,\infty]} = \gamma \cdot [0,\infty], \qquad \text{for all } \gamma \in \mathrm{H}_1(Y_0(p), \mathbb{Z}), \tag{14}$$

$$\langle G_{[0,\infty]}, \omega \rangle = \int_0^\infty \omega, \qquad \text{for all } \omega \in \mathrm{H}_c^1(Y_0(p)). \tag{15}$$

Let λ_{Eis} and $\lambda_f^\pm \in \mathbb{C}$ be the coordinates of $G_{[0,\infty]}$ relative to the basis of $\mathrm{H}_{\mathrm{dR}}^1(Y_0(p))$consisting of ω_{Eis} and of the classes ω_f^+ and ω_f^- as f ranges over the normalised weight two eigenforms on $\Gamma_0(p)$:

$$G_{[0,\infty]} = \lambda_{\mathrm{Eis}}\omega_{\mathrm{Eis}} + \sum_f (\lambda_f^+\omega_f^+ + \lambda_f^-\omega_f^-). \tag{16}$$

Let $\gamma \in \mathrm{H}_1(Y_0(p), \mathbb{Z})$ be the class attached to the standard (upper-triangular) parabolic element of $\Gamma_0(p)$, which is orthogonal to the cuspidal classes ω_f^+ and ω_f^-. Applying (14) to this class and substituting for the expansion (16) of $G_{[0,\infty]}$, one obtains

$$\left(\frac{2\pi i(p-1)}{12}\right)\cdot \lambda_{\mathrm{Eis}} = 1 \quad \text{and hence} \quad \lambda_{\mathrm{Eis}} = \left(\frac{2\pi i(p-1)}{12}\right)^{-1}. \tag{17}$$

The class $G_{[0,\infty]} - \lambda_{\mathrm{Eis}}\omega_{\mathrm{Eis}}$ belongs to $\mathrm{H}^1_{\mathrm{dR}}(X_0(p))$ and can therefore be paired against any element of the de Rham cohomology of $X_0(p)$. Applying (15) with $\omega = \omega_f^-$ and substituting for (16) once again, yields

$$-\Omega_f\lambda_f^+ = \int_0^\infty \omega_f^- = 0, \quad \text{and hence} \quad \lambda_f^+ = 0. \tag{18}$$

The same calculation with $\omega = \omega_f^+$ reveals that

$$\Omega_f\lambda_f^- = \int_0^\infty \omega_f^+, \quad \text{and hence} \quad \lambda_f^- = (\Omega_f)^{-1}\int_0^\infty \omega_f^+ = L_{\mathrm{alg}}(f,1)(\Omega_f^-)^{-1}. \tag{19}$$

We have thus obtained

$$G_{[0,\infty]} = \left(\frac{2\pi i(p-1)}{12}\right)^{-1}\omega_{\mathrm{Eis}} + \sum_f L_{\mathrm{alg}}(f,1)\cdot(\Omega_f^-)^{-1}\omega_f^-, \tag{20}$$

where the sum is taken over a basis of eigenforms for f. The lemma now follows from (14) and the definitions in (12) and (13). □

Corollary 3.9. *The rigid analytic cocycle* $\bar{J}_{0,\infty}$ *is equal to*

$$\bar{J}_{0,\infty} = \left(\frac{p-1}{12}\right)^{-1}\cdot \bar{J}_{\mathrm{DR}} + \sum_f L_{\mathrm{alg}}(f,1)\cdot \bar{J}_f^-,$$

where the sum runs over a basis of normalised eigenforms for $S_2(\Gamma_0(p))$*, and additive notation is used to denote the group operation in* $\mathrm{H}^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)\otimes K_f$*.*

Proof. This follows by applying the Schneider–Teitelbaum lift to the identity in Lemma 3.8. □

The restrictions of the cocycles $\bar{J}_{\mathrm{DR}}$ and $\bar{J}_f^-$ to $\Gamma_\circ := \mathrm{SL}_2(\mathbb{Z})$ lift uniquely to $\mathcal{A}^\times$-valued cocycles. In the special case where f has integer Fourier coefficients, and hence corresponds to an elliptic curve E_f via the Eichler–Shimura construction, the value

$$P_f^-(\tau) := J_f^-[\tau] \in \mathcal{O}_{\mathbb{C}_p}^\times$$

is called the *Stark–Heegner point* associated to J_f^- and $\tau \in \mathcal{H}_p^{\mathrm{RM}}$. Its image in $E(\mathbb{C}_p)$ under the Tate uniformisation is conjectured to be a global point in $E(H_\tau)\otimes\mathbb{Q}$, which belongs to the minus eigenspace for the action of complex conjugation on the narrow ring class field H_τ.

Corollary 3.10. *For all RM points* $\tau \in \mathcal{H}_p^\circ$*,*

$$\log(T_n\bar{J}_{0,\infty}[\tau]) = \sigma_1^{(p)}(n)\left(\frac{p-1}{12}\right)^{-1}\cdot\log(J_{\mathrm{DR}}[\tau]) + \sum_f a_n(f)L_{\mathrm{alg}}(f,1)\cdot\log(P_f^-(\tau)).$$

Proof. Since the cocycles J_{DR} and J_f^- are Hecke eigenclasses with T_n-eigenvalues $\sigma_1^{(p)}(n)$ and $a_n(f)$ respectively, Corollary (3.9) shows that, modulo torsion,

$$T_n(\bar{J}_{0,\infty}) = \sigma_1^{(p)}(n)\left(\frac{p-1}{12}\right)^{-1} \cdot \bar{J}_{\mathrm{DR}} + \sum_f a_n(f) L_{\mathrm{alg}}(f,1) \cdot \bar{J}_f^-.$$

The result now follows by evaluating these cocycles at τ and taking the p-adic logarithms of these values on both sides. □

Corollary 3.11. *Let Δ be a principal divisor on $\Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}$. Then*

$$\log(T_n\bar{J}_{0,\infty}[\Delta]) = \sigma_1^{(p)}(n)\left(\frac{p-1}{12}\right)^{-1} \cdot \log(J_{\mathrm{DR}}[\Delta]).$$

Proof. This follows from the fact that $P_f^-[\Delta] = J_f^-[\Delta]$ is torsion for all principal divisors Δ. □

4. A Weil reciprocity law for rigid meromorphic cocycles

The main result of this chapter is

Theorem 4.1. *Let $\bar{J}_\tau$ be the theta-cocycle associated to an unramified RM point $\tau \in \mathcal{H}_p^\circ$, and let $\bar{J}_\rho$ be the analytic cocycle associated to the pair $\rho = (r,s)$ of elements of $\mathbb{P}_1(\mathbb{Q})$. Then*

$$\bar{J}_\tau[\rho] = \bar{J}_\rho[\tau] \pmod{\text{torsion}}.$$

Proof. The proof follows directly by comparing Propositions 4.2 and 4.5 below, in which the left and right hand sides in Theorem 4.1 are calculated independently. □

It will be assumed for simplicity that p is inert in F, so that the elements in $\mathcal{H}_p \cap F$ map to vertices under the reduction map to the Bruhat–Tits tree $\mathcal{T}$. The case where p is ramified in F is also interesting and is only omitted for the sake of brevity. For simplicity, propositions 4.2 and 4.5 are only proved in the case where $\rho = (0,\infty)$, the details of the proof for general ρ, which proceed along very similar lines, being left to the reader.

Proposition 4.2. *For all unramified $\tau \in \Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}$,*

$$\bar{J}_\tau[0,\infty] \quad = \prod_{w \in \Pi_\tau(0,\infty)} w^{\delta(w)},$$

where $\Pi_\tau(0,\infty) = \{w \in \Sigma_\tau(0,\infty)$ with $\mathrm{ord}_p(w) = 0$ or $1.\}$.

Remark 4.3. Although this infinite product does not converge absolutely, it can be defined to be

$$\prod_{w\in\Pi_\tau(0,\infty)} w^{\delta(w)} := \prod_{w\in\Pi_\tau^+(0,\infty)} w^+/w^-.$$

Where we define here, and in what follows, $\Pi_\tau^+(0,\infty)$ and $\Pi_\tau^-(0,\infty)$ to be the set of positive and negative elements of $\Pi_\tau(0,\infty)$, respectively. Here, the order of the infinite product on the right hand side is taken as follows: Note that for each w in the index set, $\mathrm{disc}(w) = Dp^{2m}$ for some $m \geq 0$. We can write $\Pi_\tau(0,\infty)$ as a disjoint union of finite sets

$$\Pi_\tau(0,\infty) := \bigcup_{m=0}^{\infty} \Pi_\tau^m(0,\infty), \qquad \text{where} \quad \Pi_\tau^m(0,\infty) := \{w \in \Pi_\tau(0,\infty) : \mathrm{disc}(w) = Dp^{2m}\}.$$

We then define

$$\prod_{w\in\Pi_\tau(0,\infty)} w^{\delta(w)} \quad := \quad \prod_{m=0}^{\infty} \prod_{w\in\Pi_\tau^m(0,\infty)} w^{\delta(w)}.$$

Proof. By definition,

$$\bar{J}_\tau[0,\infty] := J_\tau^\eta\{0,\infty\}(p^2z)/J_\tau^\eta\{0,\infty\}(z) = \prod_{w\in\Sigma_\tau(0,\infty)} (t_w(p^2z)/t_w(z))^{\delta(w)}, \tag{21}$$

where

$$\Sigma_\tau(0,\infty) := \left\{w \in \Gamma\tau \text{ with } ww' < 0\right\}, \qquad \text{and} \qquad \delta(w) := \mathrm{sign}(w),$$

and we have abbreviated t_w^η to t_w, suppressing the base point $\eta \in \mathcal{H}_p$ in order to lighten the notations. We begin by observing that $\Sigma_\tau(0,\infty)$ is stable under multiplication by p^2, which corresponds to the action by the diagonal matrix $P_\infty \in \Gamma$. Note that $\Pi_\tau(0,\infty)$ is a system of representatives for the orbits under this action. From (9) there is a (non-canonical) involution ι on the set $\Pi_\tau(0,\infty)$ which interchanges $\Pi_\tau^+(0,\infty)$ and $\Pi_\tau^-(0,\infty)$ and preserves the fibers of the reduction map from $\Pi_\tau(0,\infty)$ to the Bruhat–Tits tree of $\mathrm{GL}_2(\mathbb{Q}_p)$. Fix such an involution ι and, for all $w^+ \in \Pi_\tau^+(0,\infty)$, let $w^- := \iota(w^+)$. Note that in particular w^+/w^- belongs to $\mathcal{O}_{\mathbb{C}_p}^\times$. With these notations in place, (21) can be rewritten as:

$$\bar{J}_\tau[0,\infty] := \prod_{w^+\in\Pi_\tau^+(0,\infty)} \prod_{j=-\infty}^{\infty} \frac{t_{p^{2j}w^+}(p^2z)t_{p^{2j}w^-}(z)}{t_{p^{2j}w^+}(z)t_{p^{2j}w^-}(p^2z)}. \tag{22}$$

In order to simplify the inner product we invoke the following lemma:

Lemma 4.4. *For all w^+ and w^- in $\mathcal{H}_p$,*

$$\prod_{j=-\infty}^{\infty} \frac{t_{p^{2j}w^+}(p^2z)t_{p^{2j}w^-}(z)}{t_{p^{2j}w^+}(z)t_{p^{2j}w^-}(p^2z)} = w^+/w^-.$$

Proof. The general factor in the product is just the cross-ratio of $p^{2j}w^+$, $p^{2j}w^-$, z and p^2z, and can be rewritten as

$$X := \prod_{j=-\infty}^{\infty} \frac{(p^2z - p^{2j}w^+)(z - p^{2j}w^-)}{(z - p^{2j}w^+)(p^2z - p^{2j}w^-)} = \prod_{j=-\infty}^{\infty} \frac{(z - p^{2(j-1)}w^+)(z - p^{2j}w^-)}{(z - p^{2j}w^+)(z - p^{2(j-1)}w^-)}.$$

This expression is the limit as $N \to \infty$ of the telescoping products

$$X_N := \prod_{j=-N}^{N} \frac{(z - p^{2(j-1)}w^+)(z - p^{2j}w^-)}{(z - p^{2j}w^+)(z - p^{2(j-1)}w^-)} = \frac{(z - p^{2(-N-1)}w^+)(z - p^{2N}w^-)}{(z - p^{2N}w^+)(z - p^{2(-N-1)}w^-)}.$$

This latter expression for X_N makes it apparent that

$$X = \lim_{N\longrightarrow\infty} X_N = w^+/w^-,$$

as was to be shown. □

Proposition 4.2 now follows from rewriting (22) using Lemma 4.4. □

Proposition 4.5. *For all unramified $\tau \in \Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}$,*

$$\bar{J}_{0,\infty}[\tau] \quad = \prod_{w\in\Pi_\tau(0,\infty)} w^{\delta(w)}.$$

Proof. Set $\rho := (0,\infty)$, and let $\tilde{\Gamma}$ denote the subgroup of $\mathrm{GL}_2(\mathbb{Z}[1/p])$ consisting of matrices with positive determinant. By Proposition 3.6,

$$\bar{J}_\rho[\tau] = J_\rho^\circ(\gamma_\tau)(\tau) = \prod_{m=0}^{\infty} J_{\rho,m}^\circ(\gamma_\tau)(\tau),$$

where

$$J^{\circ}_{\rho,m}(\gamma_\tau)(\tau) := \prod_{(r,s)\in\Sigma_\rho^{(m)}} (t_{r,s}(z))^{[r,s]\cdot[\xi,\gamma\xi]}. \tag{23}$$

We begin by reinterpreting the index set of this product. For any $m \geq 0$, define

$$\tilde{\Gamma}_m := \left\{ \gamma = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \tilde{\Gamma} \quad \text{with } a,b,c,d \in \mathbb{Z}, \quad p \nmid \gcd(a,b), \quad p \nmid \gcd(c,d), \quad ad - bc = p^m \right\}.$$

The map $\alpha \mapsto (\alpha 0, \alpha\infty)$ defines a bijection between the sets $\tilde{\Gamma}_m$ and $\Sigma_\rho^{(m)}$. We may now rewrite (23) in a series of steps:

$$\begin{aligned}
J^{\circ}_{\rho,m}(\gamma_\tau)(\tau) &:= \prod_{\alpha\in\tilde{\Gamma}_m} t_{\alpha 0,\alpha\infty}(\tau)^{[\alpha 0,\alpha\infty]\cdot[\xi,\gamma_\tau\xi]} && (24)\\
&= \prod_{\alpha\in\gamma_\tau^{\mathbb{Z}}\backslash\tilde{\Gamma}_m} \prod_{j=-\infty}^{\infty} t_{\gamma_\tau^j\alpha 0,\gamma_\tau^j\alpha\infty}(\tau)^{[\gamma_\tau^j\alpha 0,\gamma_\tau^j\alpha\infty]\cdot[\xi,\gamma_\tau\xi]}. && (25)\\
&= \prod_{\alpha\in\gamma_\tau^{\mathbb{Z}}\backslash\tilde{\Gamma}_m} \prod_{j=-\infty}^{\infty} t_{\alpha 0,\alpha\infty}(\tau)^{[\alpha 0,\alpha\infty]\cdot[\gamma_\tau^{-j}\xi,\gamma_\tau^{-j+1}\xi]} && (26)\\
&= \prod_{\alpha\in\gamma_\tau^{\mathbb{Z}}\backslash\tilde{\Gamma}_m} t_{\alpha 0,\alpha\infty}(\tau)^{[\alpha 0,\alpha\infty]\cdot[\tau',\tau]} && (27)\\
&= \prod_{\alpha\in\gamma_\tau^{\mathbb{Z}}\backslash\tilde{\Gamma}_m} t_{0,\infty}(\alpha^{-1}\tau)^{[0,\infty]\cdot[\alpha^{-1}\tau',\alpha^{-1}\tau]} && (28)
\end{aligned}$$

Here, (25) is justified by the fact that the set $\tilde{\Gamma}_m$ is stable under left multiplication by $\Gamma_\circ$, and (26) and (28) follow from the equivariance property $t_{\gamma_\tau r,\gamma_\tau s}(\gamma_\tau z) = t_{r,s}(z)$. Finally, (27) follows from the identity

$$\begin{aligned}
\sum_{j=-\infty}^{\infty} [\alpha 0,\alpha\infty]\cdot[\gamma_\tau^{-j}\xi,\gamma_\tau^{-j+1}\xi] &= \lim_{M\longrightarrow\infty} [\alpha 0,\alpha\infty]\cdot[\gamma_\tau^{-M}\xi,\gamma_\tau^{M+1}\xi]\\
&= [\alpha 0,\alpha\infty]\cdot[\tau',\tau],
\end{aligned}$$

since τ is the attractive fixed point in $\mathcal{H}$ for γ_τ and τ' is its repulsive fixed point.

Now we observe that the map $\alpha \mapsto \alpha^{-1}\tau =: w$ identifies the set $\gamma_\tau^{\mathbb{Z}}\backslash\tilde{\Gamma}_m$ with the set

$$\left\{ w \in \tilde{\Gamma}\tau, \qquad \mathrm{ord}_p(w) = 0, \qquad \mathrm{disc}(\tau) = Dp^{2m} \right\}.$$

The function $\delta(w) = [0,\infty]\cdot[w',w]$ on this set is supported on those w for which $ww' < 0$, and therefore we may rewrite (28) as

$$\bar{J}_\rho[\tau] = \prod_{w\in\tilde{\Pi}_\tau(0,\infty)} t_{0,\infty}(w)^{[0,\infty]\cdot[w',w]},$$

where $\tilde{\Pi}_\tau(0,\infty) := \{w \in \tilde{\Gamma}\tau \quad \text{s.t.} \quad ww' < 0, \quad \mathrm{ord}_p(w) = 0\}$. Note however that there is a bijection

$$\rho : \Pi_\tau(0,\infty) \longrightarrow \tilde{\Pi}_\tau(0,\infty), \qquad \rho(w) := \begin{cases} w & \text{if } \mathrm{ord}_p(w) = 0, \\ w/p & \text{if } \mathrm{ord}_p(w) = 1 \end{cases}$$

Since $t_{0,\infty}(w) = w$ and $[0,\infty]\cdot[w',w] = \delta(w)$, the proposition follows. □

The reciprocity law can be used to prove Theorem A of the introduction:

Theorem 4.6. *If J is any rigid meromorphic cocycle, then the value $J[\infty]$ is algebraic. More precisely, a power of it belongs to $(\mathcal{O}_{H_J}[1/p])^\times$.*

Proof. Let Δ denote the divisor of J. Then we have the following chain of equalities in $\mathbb{C}_p^\times$ modulo torsion:

$$J[\infty]^2 = \bar{J}_\Delta[0,\infty] = \bar{J}_{0,\infty}[\Delta] = \bar{J}_{\mathrm{DR}}[\Delta],$$

where

(1) the first equality follows from 1.7;
(2) the second is a consequence of the reciprocity law of Theorem 4.1 in the case where $\rho = (0,\infty)$;
(3) the third follows from Corollary 3.11 in the case $n = 1$.

On the other hand, the main result of [DK], building on the Galois deformation techniques that were used in [DDP] to prove the p-adic Gross–Stark conjecture, shows that the p-adic logarithm of $\bar{J}_{\mathrm{DR}}[\Delta]$ agrees with the logarithm of a suitable p-unit in the ring class field H_Δ attached to Δ. The theorem follows. □

Example. Let $\tau = 2\sqrt{2}$, which has discriminant 32, and is contained in the 11-adic upper half plane. Let $\bar{J}_\tau$ be the 11-adic theta cocycle attached to τ. Then

$$\bar{J}_\tau[0,\infty] = \frac{6}{5} J_{\mathrm{DR}}[\tau] + \frac{1}{5} J_E^-[\tau]$$

where $E : y^2 + y = x^3 - x^2 - 10x - 20$ is the modular curve $X_0(11)$. Using the algorithms described in [DV, §3.5], we may compute the cocycles $\bar{J}_\tau$ explicitly, and we verified to a high 11-adic precision that

$$\begin{cases} (T_2+2)(\bar{J}_\tau)[0,\infty] &= J_{\mathrm{DR}}[\tau]^6 &= \left(\frac{\sqrt{-2}-3}{11}\right)^6 & \in \mathbb{C}_{11}^\times \\ (T_2-3)(\bar{J}_\tau)[0,\infty] &= J_E^-[\tau]^{-1} &= (2\sqrt{-2}, 4\sqrt{-2}-5) & \in \mathbb{C}_{11}^\times / q_E^{\mathbb{Z}} \end{cases}$$

5. The p-adic uniformisation of $X_0(p)$

We conclude by drawing a parallel between the classical theory of p-adic uniformisation of Mumford curves by p-adic Schottky groups $\Gamma \subset \mathrm{SL}_2(\mathbb{Q}_p)$ acting discretely on $\mathcal{H}_p$, and the theory of rigid meromorphic cocycles. Up to a few significant differences between the two settings, one passes from one to the other by "shifting the degree of cohomology by one".

5.1. **Theta functions.** Firstly, let $\Gamma \subset \mathrm{SL}_2(\mathbb{Q}_p)$ be a p-adic Schottky group acting freely and discretely on $\mathcal{H}_p$ and on the Bruhat–Tits tree, and let X_Γ be the associated Mumford curve over $\mathbb{Q}_p$, whose $\mathbb{C}_p$-points are identified with the quotient $\Gamma\backslash\mathcal{H}_p$. As before, let $\mathcal{A}^\times$ and $\mathcal{M}^\times$ denote the multiplicative groups of rigid analytic and rigid meromorphic functions on $\mathcal{H}_p$, respectively. The theory of p-adic theta-functions associates to any degree zero divisor Δ of $\mathcal{H}_p$ a rigid meromorphic function

$$\theta_\Delta(z) := \prod_{\gamma\in\Gamma} t^\eta_{\gamma\Delta}(z)$$

on $\mathcal{H}_p$ which is invariant modulo scalars, i.e., belongs to $\mathrm{H}^0(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times)$. This class lifts to an element of $\mathrm{H}^0(\Gamma, \mathcal{M}^\times)$, i.e., to a rational function on X_Γ, if any only if the image of Δ in $X_\Gamma(\mathbb{C}_p)$ is a principal divisor. The obstruction to Δ being principal is measured by the automorphy factor $\kappa_\Delta : \Gamma \longrightarrow \mathbb{C}_p^\times$ of θ_Δ, satisfying

$$\theta_\Delta(\gamma z) = \kappa_\Delta(\gamma)\theta_\Delta(z), \qquad \text{for } \gamma \in \Gamma.$$

The group $\mathrm{H}^1(\Gamma, \mathbb{C}_p^\times)$ is isomorphic to $(\mathbb{C}_p^\times)^g$ where g is the genus of X_Γ, and the group Π_Γ of automorphy factors arising from elements of $\mathrm{H}^0(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$ forms a sublattice in $\mathrm{H}^1(\Gamma, \mathbb{C}_p^\times)$ which is commensurable with the Tate period lattice of the Jacobian of X_Γ. One thus obtains a p-adic uniformisation of this Jacobian as the quotient of $\mathrm{H}^1(\Gamma, \mathbb{C}_p^\times)$ by Π_Γ, and the lifting obstruction κ_Δ attached to $\Delta \in \mathrm{Div}^0(\mathcal{H}_p)$ encodes the image of Δ in $\mathrm{Jac}(X_\Gamma)$. This discussion is summarised in the commutative diagram

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C}_p^\times & \longrightarrow & \mathbb{C}_p^\times & \longrightarrow & \mathrm{H}^0(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times) & \xrightarrow{\delta} & \Pi_\Gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C}_p^\times & \longrightarrow & \mathrm{H}^0(\Gamma, \mathcal{M}^\times) & \longrightarrow & \mathrm{H}^0(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times) & \xrightarrow{\delta} & \mathrm{H}^1(\Gamma, \mathbb{C}_p^\times) & & \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{Div}} & & \downarrow{\scriptstyle \mathrm{Div}} & & \downarrow & & \\
 & & 0 & \longrightarrow & P(X_\Gamma) & \longrightarrow & \mathrm{Div}^0(X_\Gamma) & \longrightarrow & \mathrm{Jac}(X_\Gamma) & \longrightarrow & 0, \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}
\tag{29}
$$

where δ is the connecting homomorphism arising from in the long exact Γ-cohomology exact sequence, and

$$\Pi_\Gamma := \delta(\mathrm{H}^0(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)) \subset \mathrm{H}^1(\Gamma, \mathbb{C}_p^\times).$$

5.2. **Theta cocycles.** The analogue of (29) in the theory of rigid meromorphic cocycles is obtained by setting $\Gamma := \mathrm{SL}_2(\mathbb{Z}[1/p])$. This group is too large to act discretely on $\mathcal{H}_p$ or on the Bruhat–Tits tree without fixed points. Indeed, the vertex and edge stablisers in Γ are conjugate to $\mathrm{SL}_2(\mathbb{Z})$ and to the Hecke congruence group $\Gamma_0(p)$, respectively. Because of this, the groups $\mathrm{H}^0(\Gamma, \mathcal{A}^\times)$ and $\mathrm{H}^0(\Gamma, \mathcal{M}^\times)$ contain only the constant functions, and it becomes natural to replace the group $\mathrm{H}^0(\Gamma, \mathcal{M}^\times)$ of rigid meromorphic functions on X_Γ when Γ is a p-adic Schottky group, with the group $\mathrm{H}^1(\Gamma, \mathcal{M}^\times)$ of rigid meromorphic cocycles.

The theory of theta cocycles described in Chapter 2 associates to any divisor Δ of $\mathcal{H}_p$ *consisting of RM points* a rigid meromorphic cocycle

$$J_\Delta(z) \in \mathrm{H}^1(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times)$$

modulo multiplicative scalars, and shows that all such cocycles are obtained in this way. The divisor Δ is said to be *principal* if the class J_Δ lifts to genuine rigid meromorphic cocycle in $\mathrm{H}^1(\Gamma, \mathcal{M}^\times)$. The obstruction to Δ being principal is thus measured by the lifting obstruction $\kappa_\Delta \in \mathrm{H}^2(\Gamma, \mathbb{C}_p^\times)$ and its parabolic counterpart that were introduced and described in Chapter 1. Lemma 1.5 (2), shows that the group $\mathrm{H}^2(\Gamma, \mathbb{C}_p^\times)$ maps with finite kernel to $\mathrm{H}^1(\Gamma_0(p), \mathbb{C}_p^\times)$, suggesting that it could serve as the domain for a p-adic uniformisation of $J_0(p)$ (or even, of the generalised Jacobian of the open curve $Y_0(p)$). Indeed, it turns out that the group Π_Γ generated by the lifting obstructions of *analytic* cocycles in $\mathrm{H}^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$ forms a sublattice in $\mathrm{H}^2(\Gamma, \mathbb{C}_p^\times)$ which is commensurable with two copies of the Tate period lattice of $J_0(p)$, augmented by the discrete group generated by $p^{\mathbb{Z}}$. As explained in [Dar, §2] and in [Das], this is essentially a reformulation of the "exceptional zero conjecture" of Mazur, Tate and Teitelbaum [MTT] which was proved by Greenberg and Stevens [GS]. One thus obtains a kind of p-adic uniformisation of two copies of this Jacobian (along with a multiplicative factor of $\mathbb{C}_p^\times/p^{\mathbb{Z}}$) as a rigid analytic quotient of $\mathrm{H}^2(\Gamma, \mathbb{C}_p^\times)$ by the lattice Π_Γ. The lifting obstruction κ_τ attached to any $\tau \in \mathcal{H}_p^{\mathrm{RM}}$ encodes the images of the Gross–Stark units and the Stark–Heegner points attached to τ, in the generalised Jacobian of $Y_0(p)$. This discussion can be summarised in the following commutative diagram in the category of abelian groups up to isogeny, where morphisms are decreed to be isomorphisms if they have finite kernels and cokernels:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^1(\Gamma, \mathcal{A}^\times) & \longrightarrow & \mathrm{H}^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times) & \xrightarrow{\delta} & \Pi_\Gamma \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^1(\Gamma, \mathcal{M}^\times) & \longrightarrow & \mathrm{H}^1(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times) & \xrightarrow{\delta} & \mathrm{H}^2(\Gamma, \mathbb{C}_p^\times) \\
\downarrow{\scriptstyle \mathrm{Div}} & & \downarrow{\scriptstyle \mathrm{Div}} & & \downarrow \\
P(\Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}) & \longrightarrow & \mathrm{Div}^0(\Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}) & \longrightarrow & \mathrm{Jac}(X_0(p))(\mathbb{C}_p)^2 \oplus \mathbb{C}_p^\times/p^{\mathbb{Z}} \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array} \tag{30}$$

The method used in this paper to prove the algebraicity of $J[\infty]$ for $J \in \mathrm{H}^1_f(\Gamma, \mathcal{M}^\times)$ rests crucially on the theory of deformations of p-adic Galois representations, and appears to shed little light on the geometric structures that might underly the p-adic uniformisation of $J_0(p)$ by $\mathrm{H}^2(\Gamma, \mathbb{C}_p^\times)$ suggested by (30).

References

[Dar] H. Darmon. *Integration on $\mathcal{H}_p \times \mathcal{H}$ and arithmetic applications.* Annals of Math. (2) **154** (2001), no. 3, 589–639. ↑6, 20.

[Das] S. Dasgupta. *Stark–Heegner points on modular Jacobians.* Ann. Sci. Ecole Norm. Sup. (4) **38** (2005), no. 3, 427–469. ↑20.

[DD] H. Darmon and S. Dasgupta. *Elliptic units for real quadratic fields* . Annals of Mathematics (2) **163** (2006), no. 1, 301–346. ↑2.

[DDP] S. Dasgupta, H. Darmon, and R. Pollack. *Hilbert modular forms and the Gross–Stark conjecture.* Ann. of Math. (2) **174** (2011), no. 1, 439–484. ↑1, 3, 19.

[DK] S. Dasgupta and M. Kakde. Work in progress. ↑1, 3, 19.

[DLR1] H. Darmon, A. Lauder and V. Rotger. *Overconvergent generalised eigenforms of weight one and class fields of real quadratic fields.* Advances in Mathematics **283** (2015) 130–142. ↑3.

[DLR2] H. Darmon, A. Lauder and V. Rotger. *First order p-adic deformations of weight one newforms.* to appear in "Heidelberg conference on L-functions and automorphic forms", Bruinier and Kohnen (eds.) ↑3.

[DV] H. Darmon and J. Vonk. *Singular moduli for real quadratic fields: a rigid analytic approach.* Submitted. ↑1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 19.

[GS] R. Greenberg and G. Stevens. *p-adic L-functions and p-adic periods of modular forms.* Invent. Math. **111** (1993), no. 2, 407–447. ↑20.

[GVdP] L. Gerritzen and M. van der Put. Schottky groups and Mumford curves. Lecture Notes in Mathematics, **817**. Springer, Berlin, 1980. ↑9.

[MTT] B. Mazur, J. Tate, and J. Teitelbaum. *On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer.* Invent. Math. **84** (1986), no. 1, 1–48. ↑20.

[Sch] P. Schneider. *Rigid-analytic L-transforms.* Number theory, Noordwijkerhout 1983, 216–230, Lecture Notes in Math., 1068, Springer, Berlin, 1984. ↑10.

[Se] J-P. Serre. Trees. Translated from the French original by John Stillwell. Corrected 2nd printing of the 1980 English translation. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. x+142 pp. ↑4, 6.

[Te] J.T. Teitelbaum. *Values of p-adic L-functions and a p-adic Poisson kernel.* Invent. Math. **101** (1990), no. 2, 395–410. ↑10.

H. D.: Montreal, Canada
E-mail address: darmon@math.mcgill.ca

J.V.: Montreal, Canada
E-mail address: jan.vonk@math.mcgill.com