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Overconvergent generalised eigenforms of weight one and class fields of real quadratic fields

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A B S T R A C T

This article examines the Fourier expansions of certain non-classical p-adic modular forms of weight one: the eponymous *generalised eigenforms* of the title, so called because they lie in a generalised eigenspace for the Hecke operators. When this generalised eigenspace contains the theta series attached to a character of a real quadratic field K in which the prime p splits, the coefficients of the attendant generalised eigenform are expressed as p-adic logarithms of algebraic numbers belonging to an idoneous ring class field of K. This suggests an approach to "explicit class field theory" for real quadratic fields which is simpler than the one based on Stark's conjecture or its p-adic variants, and is perhaps closer in spirit to the classical theory of singular moduli.

1. Introduction and statement of the main result

Fourier coefficients of modular forms often describe interesting arithmetic functions. Classical examples are the partition function, the divisor function, and representation

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numbers of quadratic forms, which are related to Fourier coefficients of the Dedekind eta function, Eisenstein series, and theta series, respectively. A more modern instance is the appearance of Frobenius traces of two-dimensional Galois representations as Fourier coefficients of normalised eigenforms. Just as germane to an understanding of the present work are the topological and arithmetic intersection numbers of special cycles arising in the formulae of Hirzebruch–Zagier, Gross–Kohnen–Zagier, and their vast generalisations growing out of the "Kudla program".

The main theorem of this paper expresses the Fourier coefficients of certain p-adic modular forms of weight one as p-adic logarithms of algebraic numbers in ring class fields of real quadratic fields. It suggests an approach to "explicit class field theory" for real quadratic fields which is simpler than the one based on Stark's (still unproved) conjecture [17] or Gross's (more tractable) p-adic variant [13]. An analogy with the growing body of work on Fourier coefficients of incoherent Eisenstein series and weak harmonic Maass forms suggests that this approach is perhaps closer in spirit to the classical theory of singular moduli.

To set the stage for the main result, let K be a real quadratic field of discriminant $D > 0$ and let χ_K denote the even quadratic Dirichlet character associated to it. Let

$$\psi : G_K := \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathbb{C}^\times$$

be a ray class character (of order m, conductor $\mathfrak{f}_\psi$ and central character χ_ψ) which is of *mixed signature*, i.e., which is even at precisely one of the infinite places of K and odd at the other. Hecke's theta series $g := \theta_\psi$ attached to ψ is a holomorphic newform of weight one, level N and nebentype character χ with Fourier coefficients in $L := \mathbb{Q}(\mu_m)$, where

$$N = D \cdot \mathrm{Norm}_{K/\mathbb{Q}} \mathfrak{f}_\psi, \qquad \chi = \chi_K \chi_\psi.$$

Fix a prime $p = \mathfrak{p}\mathfrak{p}'$ which does not divide N and is split in $K/\mathbb{Q}$. The eigenform g is said to be *regular* at p if the Hecke polynomial

$$x^2 - a_p(g)x + \chi(p) = (x - \psi(\mathfrak{p}'))(x - \psi(\mathfrak{p})) =: (x - \alpha)(x - \beta)$$

has distinct roots. Assume henceforth that this regularity hypothesis holds, and let

$$g_\alpha(z) := g(z) - \beta g(pz), \qquad g_\beta(z) := g(z) - \alpha g(pz)$$

be the two distinct p-stabilisations of g, which are eigenvectors for the U_p operator with eigenvalues α and β respectively. Note that these stabilisations are both ordinary, since α and β are roots of unity.

Let $S_k(N, \chi)$ (resp. $S_k^{(p)}(N, \chi)$) denote the space of classical (resp. p-adic overconvergent) modular forms of weight k, level N and character χ, with coefficients in $\mathbb{C}_p$. The Hecke algebra $\mathbb{T}$ of level Np generated over $\mathbb{Q}$ by the operators T_ℓ with $\ell \nmid Np$ and U_q with

$q|Np$ acts naturally on the spaces $S_k(Np,\chi)$ and $S_k^{(p)}(N,\chi)$. The normalised eigenform $g_\alpha \in S_1(Np,\chi)$ gives rise to an algebra homomorphism $\varphi_{g_\alpha}: \mathbb{T} \longrightarrow L$ satisfying

$$\varphi_{g_\alpha}(T_\ell) = a_\ell(g) \quad \text{if } \ell \nmid Np, \qquad \varphi_{g_\alpha}(U_\ell) = \begin{cases} a_\ell(g) & \text{if } \ell|N; \\ \alpha & \text{if } \ell = p, \end{cases}$$

and g_α generates the one-dimensional eigenspace $S_1(Np,\chi)[g_\alpha]$ attached to this system of Hecke eigenvalues.

In [4], Cho and Vatsal make the important observation that the Coleman–Mazur eigencurve is smooth but *not étale* over weight space at the points corresponding to g_α and g_β. (It is likely that the ramification degree of the weight map is always equal to two at these points under the regularity assumption: see [4,1], and Adel Bettina's forthcoming PhD thesis [2] for various results in this direction.) In particular, letting I_{g_α} be the kernel of φ_{g_α}, the submodule of $S_1^{(p)}(N,\chi)$ which is annihilated by $I_{g_\alpha}^2$, denoted $S_1^{(p)}(N,\chi)[[g_\alpha]]$, is two-dimensional and contains non-classical forms which do not lie in the image of the natural inclusion

$$S_1(Np,\chi)[g_\alpha] \hookrightarrow S_1^{(p)}(N,\chi)[[g_\alpha]].$$

An overconvergent form in $S_1^{(p)}(N,\chi)[[g_\alpha]]$ which is not a multiple of g_α is called a *generalised eigenform* attached to g_α, and is said to be *normalised* if its first Fourier coefficient is equal to zero. Such a normalised generalised eigenform, denoted

$$g_\alpha^\flat := \sum_{n=2}^{\infty} a_n(g_\alpha^\flat) q^n,$$

is uniquely determined by g_α up to scaling, and the Hecke operators act on it by the rule

$$T_\ell g_\alpha^\flat = a_\ell(g_\alpha) g_\alpha^\flat + a_\ell(g_\alpha^\flat) g_\alpha, \qquad U_q g_\alpha^\flat = a_q(g_\alpha) g_\alpha^\flat + a_q(g_\alpha^\flat) g_\alpha, \tag{1}$$

for all primes $\ell \nmid Np$ and all $q|Np$. Theorem 1 below shows that the Fourier coefficients $a_n(g_\alpha^\flat)$ are interesting arithmetic quantities with a bearing on explicit class field theory for K.

Let ψ' denote the character deduced from ψ by composing it with the involution in $\mathrm{Gal}(K/\mathbb{Q})$. The ratio $\psi_\heartsuit := \psi/\psi'$ is a totally odd ring class character of K. Let H denote the ring class field of K which is fixed by the kernel of $\psi_\heartsuit$, and set $G := \mathrm{Gal}(H/K)$.

If $\ell \nmid N$ is any rational prime which is inert in $K/\mathbb{Q}$, the corresponding prime ℓ of K splits completely in H/K, and the set Σ_ℓ of primes of H above ℓ is endowed with the structure of a principal G-set. Given $\lambda \in \Sigma_\ell$, let $u(\lambda) \in \mathcal{O}_H[1/\lambda]^\times \otimes \mathbb{Q}$ be any λ-unit of H satisfying $\mathrm{ord}_\lambda(u(\lambda)) = 1$. While $u(\lambda)$ is only defined up to units in $\mathcal{O}_H^\times$, the element

$$u(\psi_\heartsuit, \lambda) = \sum_{\sigma \in G} \psi_\heartsuit^{-1}(\sigma) \otimes u(\lambda)^\sigma \in L \otimes \mathcal{O}_H[1/\ell]^\times$$

is independent of the choice of generator $u(\lambda)$, since there are no genuine units in $L \otimes \mathcal{O}_H^\times$ in the eigencomponents for the totally odd character $\psi_\heartsuit$. The ℓ-unit $u(\psi_\heartsuit, \lambda)$ does depend on the choice of $\lambda \in \Sigma_\ell$. Section 2 below uses ψ to define a function $\eta : \Sigma_\ell \longrightarrow \mu_m$ for which the element

$$u(\psi_\heartsuit, \ell) := \eta(\lambda) \otimes u(\psi_\heartsuit, \lambda) \in L \otimes \mathcal{O}_H[1/\ell]^\times \tag{2}$$

depends only on the inert prime ℓ and not on the choice of prime $\lambda \in \Sigma_\ell$ above it.

Fix embeddings of L and of H into $\bar{\mathbb{Q}}_p$, and let

$$\log_p : L \otimes H^\times \longrightarrow \bar{\mathbb{Q}}_p$$

be the resulting p-adic logarithm on $H^\times$, extended to $L \otimes H^\times$ by L-linearity. The main result of this paper is

Theorem 1. *The normalised generalised eigenform $g_\alpha^\flat$ attached to g_α can be scaled in such a way that, for all primes $\ell \nmid N$,*

$$a_\ell(g_\alpha^\flat) = \begin{cases} 0 & \text{if } \chi_K(\ell) = +1; \\ \log_p u(\psi_\heartsuit, \ell) & \text{if } \chi_K(\ell) = -1. \end{cases}$$

More generally, for all $n \geq 2$ with $\gcd(n, N) = 1$,

$$a_n(g_\alpha^\flat) = \frac{1}{2} \sum_{\ell | n} \log_p u(\psi_\heartsuit, \ell) \cdot (\mathrm{ord}_\ell(n) + 1) \cdot a_{n/\ell}(g_\alpha), \tag{3}$$

where the sum runs over primes ℓ that are inert in K.

The following two examples illustrate Theorem 1.

Example 1.1. Let ψ be the quadratic character of $K = \mathbb{Q}(\sqrt{21})$ of conductor $\mathfrak{f}_\psi := (3, \sqrt{21})$ attached to the quadratic extension $K(\sqrt{3 + \sqrt{21}})$ of K, so that $m = 2$ and $L = \mathbb{Q}$. The weight one modular form g attached to ψ is of level $N = 63$ and has for nebentype character the odd quadratic Dirichlet character χ_7 of conductor 7. The ring class character $\psi_\heartsuit$ is a genus character associated to K and the associated genus field is just the (narrow) Hilbert class field $H = \mathbb{Q}(\sqrt{-3}, \sqrt{-7})$ of K.

The prime $p = 5$ is split in K, and the roots of the associated Hecke polynomial are $\alpha = 1$ and $\beta = -1$. Hence the generalised eigenspace of g_α in $S_1^{(5)}(63, \chi_7)$ contains a normalised generalised eigenform $g_\alpha^\flat$ which is unique up to scaling. The fast algorithms of [15] for calculating with overconvergent modular forms were used to efficiently compute this generalised eigenform numerically with an accuracy of 50 significant 5-adic digits, producing a modular form whose first non-vanishing Fourier coefficient $a_2(g_\alpha^\flat)$ is equal to 1.

For primes $\ell < 300$ that split in K (including $\ell = p$) it was observed that $a_\ell(g_\alpha^\flat) = 0$. When $\chi_K(\ell) = -1$, it was observed that

$$a_\ell(g_\alpha^\flat) = \frac{\log_p(u(\psi_\heartsuit, \ell))}{\log_p(u(\psi_\heartsuit, 2))}, \tag{4}$$

where $u(\psi_\heartsuit, \ell)$ denotes a suitable fundamental ℓ-unit of norm 1 in H/K, for all inert $\ell \geq 2$. (The logarithm of such a unit is unique up to sign.) At the ramified primes we observed

$$a_3(g_\alpha^\flat) = 0, \qquad a_7(g_\alpha^\flat) = \frac{1}{2} \cdot \frac{\log_p(u(\psi_\heartsuit, 7))}{\log_p(u(\psi_\heartsuit, 2))}. \tag{5}$$

Here the 2-unit is $u(\psi_\heartsuit, 2) := (-3 + \sqrt{-7})/4$. The numerical values of the first few non-zero coefficients $a_\ell(g_\alpha^\flat)$ for $\ell > 2$ prime, and the values of $u(\psi_\heartsuit, \ell)$ verifying (4) and (5), are listed in the table below. The p-adic logarithms were calculated relative to the 5-adic embedding of H in which $\sqrt{21} \equiv -1 \bmod 5$.

ℓ	$a_\ell(g_\alpha^\flat) \bmod 5^{50}$	$u(\psi_\heartsuit, \ell)$
7	20012844832326722144621655295530693	$(1 - 4\sqrt{-3})/7$
11	9753260368539762436495550803834302	$(-3 - 4\sqrt{-7})/11$
13	80089851328257507529397205421800237	$(-1 + 15\sqrt{-3})/26$
19	5387308298676565974776485314728008	$(11 + 21\sqrt{-3})/38$
23	7951947833969991753485495228957006	$(9 + 8\sqrt{-7})/23$
29	59833461154145179184050173388767665	$(-27 - 4\sqrt{-7})/29$
31	8202575240226165174943058721099781	$(-13 + 35\sqrt{-3})/62$

Example 1.2. Let χ be a Dirichlet character of conductor 145 with order 4 at the prime 5 and order 2 at the prime 29. The space $S_1(145, \chi)$ is one-dimensional and spanned by the modular form

$$g = q + iq^4 + iq^5 + (-i - 1)q^7 - iq^9 + (-i + 1)q^{13} - q^{16} - q^{20} + \cdots.$$

It is the theta series attached to a quartic character of $K = \mathbb{Q}(\sqrt{29})$ ramified at one of the primes above (5). Level 145 is the smallest where one encounters weight one theta series attached to a character of a real quadratic field, but not to a character of any imaginary quadratic field. (There are two non-conjugate such forms, the other of which appears in [6, Example 4.1].)

The prime $p = 13$ is split in K and the roots of the Hecke polynomial for this prime are $\alpha = 1$ and $\beta = -i$. We view g_α as a 13-adic modular form using the embedding of $L = \mathbb{Q}(i)$ into $\mathbb{Q}_{13}$ for which $i \equiv 5 \bmod 13$. The coefficients of the normalised eigenform $g_\alpha^\flat$, scaled so that $a_2(g_\alpha)^\flat = 1$, are given in the second column of the table below for the inert primes $\ell = 3, 11, 17$ and 19 of K.

The ring class field H of conductor 5 is a cyclic quartic extension of K given by

$$H = K(\sqrt{5}, \delta) \qquad \text{where } \delta^2 = \frac{\sqrt{145} - 15}{32}.$$

Let σ be the generator of $\mathrm{Gal}(H/K)$ defined by

$$\sigma(\sqrt{5}) = -\sqrt{5}, \qquad \sigma(\delta) = -\frac{1}{4}(3\sqrt{5} + \sqrt{29})\delta.$$

We embed H in the quartic unramified extension

$$\mathbb{Q}_{13^4} = \mathbb{Q}(\alpha) \qquad \text{where } \alpha^4 + 3\alpha^2 + 12\alpha + 2 = 0$$

of $\mathbb{Q}_{13}$, in such a way that

$$\sqrt{29} \equiv 9, \qquad \sqrt{5} \equiv 8\alpha^3 + 2\alpha^2 + 7\alpha + 10, \qquad \delta \equiv \alpha^3 + 5\alpha^2 + 6\alpha + 10 \pmod{13}.$$

For $\ell = 3, 11, 17$ and 19, it was verified that

$$a_\ell(g_\alpha^\flat) = \frac{\log_{13}(u(\psi_\heartsuit, \ell))}{\log_{13}(u(\psi_\heartsuit, 2))}$$

to 20-digits of 13-adic precision, where (denoting the group operation in $L \otimes H^\times$ additively)

$$u(\psi_\heartsuit, \ell) := u_\ell + i \otimes \sigma(u_\ell) - \sigma^2(u_\ell) - i \otimes \sigma^3(u_\ell),$$

for a suitable ℓ-unit u_ℓ of H. The 2-unit u_2 is given by

$$u_2 := \frac{1}{2}(-\sqrt{5} - \sqrt{29} + 6)\delta + \frac{1}{8}(\sqrt{29} - 7)\sqrt{5} + \frac{1}{8}(\sqrt{29} + 1),$$

and the others are listed in the last column of the table below.

ℓ	$a_\ell(g_\alpha^\flat) \bmod 13^{20}$	u_ℓ
3	12915196799386050150007	$(\sqrt{5} + \sqrt{29} - 4)\delta + \frac{1}{4}(\sqrt{29} - 4)\sqrt{5} + \frac{1}{4}(2\sqrt{29} - 13)$
11	35241433186275777328 42	$\left(\frac{1}{4}\left((\sqrt{29} + 1)\sqrt{5} + (-\sqrt{29} + 11)\right)\delta + \frac{1}{4}\left(\sqrt{5} - 1\right)\right)^4$
17	2294079923934379645 10	$\left((16\sqrt{29} + 84)\sqrt{5} + (36\sqrt{29} + 200)\right)\delta + \frac{1}{4}(11\sqrt{29} + 63)\sqrt{5} + \frac{1}{4}(15\sqrt{29} + 83)$
19	15142834827825079965585	$\left(\frac{1}{4}\left((3\sqrt{29} - 13)\sqrt{5} + (-15\sqrt{29} + 85)\right)\delta + \frac{1}{8}(3\sqrt{29} - 15)\sqrt{5} + \frac{1}{8}(7\sqrt{29} - 35)\right)^2$

Remark 1.3. Theorem 1 was inspired by the work of Bellaïche and Dimitrov [1] on the geometry of the Coleman–Mazur eigencurve at classical weight one points. Theorem 1.1. of [1] asserts that this eigencurve is étale over weight space at any classical weight one

point for which p is regular, *unless* g is a theta series of a real quadratic field in which p splits. This explains why Theorem 1 above focuses on this setting, the only "regular at p" scenario where a non-trivial generalised eigenform co-exists with its classical weight one counterpart.

Remark 1.4. Every (totally odd) ring class character of K can be written as ψ'/ψ for a suitable ray class character ψ (of mixed signature). (Cf. for example Lemma 6.7 of [7].) Hence all ring class fields of K can be generated by exponentials of the Fourier coefficients of $g_\alpha^\flat$ for suitable real dihedral newforms g of weight one attached to K, in much the same way that the ring class fields of a quadratic imaginary K can be generated by values of the modular function $j(z)$ at arguments $z \in K$. The existence of the generalised eigenforms $g_\alpha^\flat$ of Theorem 1 can therefore be envisioned as an eventual pathway to "explicit class field theory" for real quadratic fields.

Remark 1.5. Comparing (3) with the formula for the coefficient denoted $a_n(\phi)$ in Theorem 1 of [14] reveals a strong analogy between Theorem 1 and a fundamental result of Kudla, Rapoport, and Yang on the Fourier coefficients of central derivatives of incoherent Eisenstein series of weight one. When Eisenstein series are replaced by a weight one cuspidal newform g, the notion of *mock modular forms* provides a fruitful substitute for Kudla's incoherent Eisenstein series. The mock modular form attached to g, denoted $g^\sharp$, is the holomorphic part of a *harmonic weak Maass form* mapping to g under an appropriate differential operator. Recent work of Bill Duke, Stephan Ehlen, Yingkun Li, and Maryna Viazovska relates the Fourier coefficients of $g^\sharp$ to the complex logarithms of algebraic numbers belonging to the field cut out by the adjoint Ad_g of the two-dimensional Artin representation attached to g. The articles [10,12,11,19] focus largely on the case where g is a theta series of an imaginary quadratic field. The algebraic numbers whose logarithms arise in the Fourier expansion of $g^\sharp$ then belong to abelian extensions of imaginary rather than real quadratic fields, and the proofs in [10,12,11,19] rely crucially on the theory of complex multiplication. Such a theory is unavailable for real quadratic fields, and the techniques exploited in the present work, based on deformations of Galois representations, are thus quite different, substantially simpler, and fundamentally p-adic in nature.

Remark 1.6. The concluding section of [10] makes an experimental study of $g^\sharp$ when g is an octahedral newform of level 283, in which the Stark conjecture for the Artin L-function attached to Ad_g (as described, for instance, in Section 6 of [17]) plays an essential role. The case where g is the theta series of a character of a real quadratic field is treated extensively in [16], where it is explained that the Fourier coefficients of $g^\sharp$ are expected to be logarithms of algebraic numbers belonging to K rather than to some non-trivial ring class field of K. This (disappointing, at least for explicit class field theory) feature of the archimedean context can be traced to the fact that the L-function of the representation induced from $\psi_\heartsuit$ does not vanish at $s = 0$, and that there are no non-trivial Stark units in the $\psi_{\heartsuit}$-isotypic part of the unit group of H. As pointed out in [4] and [1], it is precisely this phenomenon which leads to the existence of the overconvergent generalised eigenform $g_\alpha^\flat$ of Theorem 1.

Remark 1.7. There are other instances where the properties of weak harmonic Maass forms resonate with those of overconvergent modular forms like $g_\alpha^\flat$. For example, Bruinier and Ono study the Fourier coefficients of the holomorphic part of the weak harmonic Maass form of weight 1/2 attached to a classical modular form g of weight 3/2 whose Shimura lift has rational coefficients and hence corresponds to an elliptic curve over $\mathbb{Q}$. The main result of [3] relates these Fourier coefficients to Heegner points on the elliptic curve defined over a varying collection of quadratic fields. The p-adic logarithms of the same Heegner points are realised in [8] as the Fourier coefficients of a "modular form of weight $3/2+\varepsilon$" arising as an infinitesimal p-adic deformation of g over weight space. It would be interesting to flesh out the rather tantalising analogy between weak harmonic Maass forms and p-adic deformations of classical eigenforms. To what extent can the latter be envisaged as non-archimedean counterparts of the former?

Remark 1.8. The ℓ-units $u(\psi_{\heartsuit},\ell)$ are precisely the Gross–Stark units studied in [5]. The latter reference proposes a *conjectural* analytic formula for their ℓ-adic logarithms refining Gross's ℓ-adic analogue of the Stark conjectures. Theorem 1 above holds unconditionally and concerns the p-adic logarithms of the same ℓ-units, for primes $p \neq \ell$. It therefore bears no direct connection with the Gross–Stark conjecture, even though its proof, like that of the Gross–Stark conjecture given in [9] and [18], relies crucially on the deformation theory of p-adic Galois representations.

We close the introduction with the following corollary of Theorem 1, which shows that a naive version of the q-expansion principle fails for the generalised eigenspace $S_1^{(p)}(N,\chi)[[g_\alpha]]$.

Corollary 2. *Let $S \subset S_1^{(p)}(N,\chi)[[g_\alpha]]$ be a $\bar{\mathbb{Q}}$-vector space which is stable under all the Hecke operators. Then either S is contained in $S_1(Np,\chi)[g_\alpha]$, or it is infinite-dimensional over $\bar{\mathbb{Q}}$.*

Proof. If S is not contained in $S_1(Np,\chi)[g_\alpha]$, it contains a non-zero (not necessarily normalised) generalised eigenform h of the form $g_\alpha^\flat + \lambda g_\alpha$, where $g_\alpha^\flat$ is a normalised generalised eigenform and $\lambda \in \mathbb{C}_p$. By (1) combined with the stability of S under the Hecke operators,

$$(T_\ell - a_\ell(g_\alpha))h = a_\ell(g_\alpha^\flat)g_\alpha$$

also belongs to S, for all primes $\ell \nmid Np$. Theorem 1 implies that the forms $\Omega \log_p(u(\psi_{\heartsuit},\ell))g_\alpha$ (for a suitable $\Omega \neq 0$) belong to S, for all $\ell \nmid Np$ which are inert in K. The corollary follows from the linear independence over $\bar{\mathbb{Q}}$ of the p-adic logarithms of algebraic numbers.

2. Proof of Theorem 1

The theta series $g = \theta_\psi$ corresponds to an odd, irreducible, two-dimensional Artin representation

$$\varrho : G_{\mathbb{Q}} \longrightarrow \mathbf{GL}_2(L),$$

obtained by inducing ψ from G_K to $G_{\mathbb{Q}}$. The two-dimensional L-vector space underlying ϱ decomposes as a direct sum of one-dimensional representations ψ and ψ' when restricted to G_K. Fix an element τ_0 in the complement $G_{\mathbb{Q}} \setminus G_K$ of G_K in $G_{\mathbb{Q}}$, and let e_1 and e_2 be eigenvectors for the G_K-action attached to ψ and ψ' respectively, chosen so that $e_1 = \varrho(\tau_0)e_2$. Relative to this basis,

$$\varrho_{|G_K} = \begin{pmatrix} \psi & 0 \\ 0 & \psi' \end{pmatrix}, \qquad \varrho_{|G_{\mathbb{Q}} \setminus G_K} = \begin{pmatrix} 0 & \eta' \\ \eta & 0 \end{pmatrix} \tag{6}$$

where η and η' are L-valued functions on $G_{\mathbb{Q}} \setminus G_K$ given by the rule $\eta(\tau) := \psi(\tau_0\tau)$ and $\eta'(\tau) := \psi(\tau\tau_0^{-1})$.

Let L_p denote a p-adic completion of L and let $L_p[\varepsilon]$ denote the ring of dual numbers, for which $\varepsilon^2 = 0$. The theorems of Cho–Vatsal [4] and Bellaïche–Dimitrov [1] show that the tangent space $H^1_{\mathrm{ord}}(\mathbb{Q}, \mathrm{Ad}^0(\varrho))$ of the universal ordinary deformation space attached to ϱ with constant determinant, denoted as $t_{\mathcal{D}'}$ in [1, Definition 2.1], is one-dimensional over L_p. This means that there is a unique (up to conjugation, and replacing ε by a non-zero multiple) ordinary lift

$$\tilde{\varrho} : G_{\mathbb{Q}} \longrightarrow \mathbf{GL}_2(L_p[\varepsilon])$$

of ϱ satisfying

$$\det(\tilde{\varrho}) = \det(\varrho). \tag{7}$$

We begin by observing that $\tilde{\varrho}$ can be written in the form

$$\tilde{\varrho}_{|G_K} = \begin{pmatrix} \psi & \psi'\kappa' \cdot \varepsilon \\ \psi\kappa \cdot \varepsilon & \psi' \end{pmatrix}, \qquad \tilde{\varrho}_{|G_{\mathbb{Q}} \setminus G_K} = \begin{pmatrix} d_1 \cdot \varepsilon & \eta' \\ \eta & d_2 \cdot \varepsilon \end{pmatrix}. \tag{8}$$

The fact that the diagonal entries of $\tilde{\varrho}_{|G_K}$ remain "constant", i.e., belong to L and are equal to those of ϱ, follows from (7) and the fact that this is true of the lower right-hand matrix entry, which is unramified at p by the ordinarity of $\tilde{\varrho}$, since there are no non-trivial homomorphisms from G_K to L_p that are unramified at a prime above p. It likewise ensures that the anti-diagonal entries of $\tilde{\varrho}_{|G_{\mathbb{Q}} \setminus G_K}$ are equal to those of $\varrho_{|G_{\mathbb{Q}} \setminus G_K}$, i.e., are described by the functions η and η'.

Lemma 2.1. *The functions* κ *and* κ' *belong to* $H^1(K, L_p(\psi_\heartsuit^{-1}))$ *and to* $H^1(K, L_p(\psi_\heartsuit))$ *respectively. Their restrictions to* G_H *are related by the rule*

$$\kappa'(\sigma) = \frac{\eta'(\tau)}{\eta(\tau)}\kappa(\tau\sigma\tau^{-1}), \tag{9}$$

for any $\tau \in G_{\mathbb{Q}} \setminus G_K$*. The class* κ *is unramified at* $\mathfrak{p}$*, and the class* κ' *is unramified at* $\mathfrak{p}'$*.*

Proof. The first assertion is standard, and follows directly from the fact that ϱ is a homomorphism on G_K. The second follows by a similar argument, from a direct calculation of the anti-diagonal entries of $\tilde{\varrho}(\tau\sigma\tau^{-1}) = \tilde{\varrho}(\tau)\tilde{\varrho}(\sigma)\tilde{\varrho}(\tau)^{-1}$, using the fact that $\psi(\sigma) = \psi'(\sigma)$ when $\sigma \in G_H$ to simplify the calculation. Finally, the fact that ϱ is ordinary at the rational prime p (relative to a p-adic embedding of H which sends K to its completion at $\mathfrak{p}$) implies that κ is unramified at $\mathfrak{p}$. The relation (9) between κ and κ' implies that κ' is unramified at $\mathfrak{p}'$, since any $\tau \in G_{\mathbb{Q}} \setminus G_K$ interchanges the primes $\mathfrak{p}$ and $\mathfrak{p}'$ of K above p. □

The space $H^1_{\mathfrak{p}'}(K, L_p(\psi_\heartsuit))$ of global classes that are unramified at $\mathfrak{p}'$ is one-dimensional over L_p, and restriction to G_H gives an isomorphism

$$H^1_{\mathfrak{p}'}(K, L_p(\psi_\heartsuit)) \longrightarrow \hom_{\mathfrak{p}'}(G_H, L_p(\psi_\heartsuit))^{\mathrm{Gal}(H/K)}. \tag{10}$$

The target of this restriction map can be described explicitly in terms of global class field theory, which identifies $\hom_{\mathfrak{p}'}(G_H, L_p)$ with the continuous homomorphisms from the group $\mathbb{A}_H^\times$ of idèles of H which are trivial on principal elements and on $\mathcal{O}_{H_v}^\times$ for all primes $v \nmid \mathfrak{p}$ of H. The space $\hom_{\mathfrak{p}'}(G_H, L_p)$ is of dimension $t := \frac{[H:K]}{2}$. To describe it more concretely, let σ_∞ denote complex conjugation in $G = \mathrm{Gal}(H/K)$ (which is well-defined, independently of the choice of a complex embedding of H). Given a choice $\sigma_1, \dots, \sigma_t$ of coset representatives for $\langle\sigma_\infty\rangle$ in G, let κ_j denote the idèle class character whose restriction to $\mathcal{O}_{H_v}^\times$ is trivial for all $v \nmid \mathfrak{p}$ and whose restriction to $H_{\mathfrak{p}}^\times := (H \otimes_K K_{\mathfrak{p}})^\times$ is equal to

$$\kappa_j(a) = \log_p(a^{\sigma_j}/a^{\sigma_j\sigma_\infty}).$$

The elements $\kappa_1, \dots, \kappa_t$ form an L_p basis for $\hom_{\mathfrak{p}'}(G_H, L_p)$, and the one-dimensional target of (10) is spanned by the function

$$\kappa_{\psi_\heartsuit} := \sum_{j=1}^{t} \psi_\heartsuit^{-1}(\sigma_j)\kappa_j,$$

whose restriction to $H_{\mathfrak{p}}^\times$ is given by

$$\kappa_{\psi_\heartsuit}(a) = \sum_{\sigma\in G} \psi_\heartsuit^{-1}(\sigma) \cdot \log_p(a^\sigma). \tag{11}$$

Lemma 2.2. *For all primes ℓ of $\mathbb{Q}$ that are inert in K, and all $\lambda \in \Sigma_\ell$,*

$$\kappa_{\psi_\heartsuit}(\mathrm{Frob}_\lambda) = \log_p(u(\psi_\heartsuit, \lambda)).$$

Proof. By global class field theory, the value of $\kappa_{\psi_\heartsuit}(\mathrm{Frob}_\lambda)$ is equal to the image of $\kappa_{\psi_\heartsuit}$ on the idèle which is equal to the inverse of a local uniformiser of H_λ at λ, and to 1 everywhere else. This idèle class agrees, modulo the kernel of $\kappa_{\psi_\heartsuit}$, with the idèle which is trivial at all places of H except $\mathfrak{p}$ and equal to $u(\lambda)$ in $H_\mathfrak{p}^\times$. The result now follows from (11). □

Assume from now on that κ' has been scaled so that it is equal to the class $\kappa_{\psi_\heartsuit}$ of (11). The Galois representation $\tilde{\varrho}$ comes from an overconvergent weight one eigenform

$$\tilde{g}_\alpha = g_\alpha + g_\alpha^\flat \cdot \varepsilon$$

with coefficients in the ring $L_p[\varepsilon]$ of dual numbers. The modular form $g_\alpha^\flat$ is a generalised eigenform in the sense of the introduction, and its Fourier coefficients acquire a Galois-theoretic interpretation via the identity

$$\mathrm{trace}(\tilde{\varrho}(\sigma_\ell)) = a_\ell(g_\alpha) + a_\ell(g_\alpha^\flat) \cdot \varepsilon, \tag{12}$$

where $\ell \nmid N$ is any rational prime and $\sigma_\ell \in G_\mathbb{Q}$ is a Frobenius element at ℓ (attached to an arbitrary embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_\ell$). Note that although $\tilde{\varrho}$ is always ramified at p, (8) ensures that the trace of $\tilde{\varrho}(\sigma_p)$ does not depend on the choice of Frobenius element at p.

If $\ell \nmid N$ is a prime of K for which $\chi_K(\ell) = 1$, and hence factors as a product of two primes λ and λ' of K, then the Frobenius element Frob_ℓ attached to ℓ belongs to G_K, and hence by (8),

$$\mathrm{trace}(\tilde{\varrho}(\mathrm{Frob}_\ell)) = \psi(\lambda) + \psi'(\lambda) = a_\ell(g_\alpha).$$

The first assertion in Theorem 1 when ℓ is a split prime follows in light of (12).

Let now $\ell \nmid N$ be a prime of K for which $\chi_K(\ell) = -1$. Let H_ϱ be the abelian extension of K which is fixed by the kernel of ϱ. The function η gives rise to a function on the primes of H_ϱ above ℓ by setting $\eta(\lambda) := \eta(\sigma_\lambda)$, where $\sigma_\lambda \in \mathrm{Gal}(H_\varrho/\mathbb{Q})$ is the ℓ-power Frobenius automorphism attached to the prime λ. Observe that, if $\lambda_1 = \sigma\lambda_2$ for some $\sigma \in \mathrm{Gal}(H_\varrho/K)$, then

$$\eta(\lambda_1) = \eta(\sigma_{\lambda_1}) = \eta(\sigma\sigma_{\lambda_2}\sigma^{-1}) = \psi_\heartsuit^{-1}(\sigma)\eta(\lambda_2). \tag{13}$$

In particular, the value $\eta(\lambda)$ depends only on the restriction of the ideal λ to the ring class field $H \subset H_\varrho$, and therefore η can also be viewed as a function on the sets Σ_ℓ of primes of H that were described in the introduction.

Fix now an embedding $\iota_\ell : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_\ell$. It determines a Frobenius element $\sigma_\ell \in G_{\mathbb{Q}}$, whose square is the Frobenius element in G_K attached to the prime ℓ of K. Let $\lambda \in \Sigma_\ell$ denote the prime of H above ℓ determined by ι_ℓ. By (12) and (8),

$$\operatorname{trace}(\tilde{\varrho}(\sigma_\ell)) = (d_1(\sigma_\ell) + d_2(\sigma_\ell)) \cdot \varepsilon = a_\ell(g_\alpha^\flat) \cdot \varepsilon, \tag{14}$$

while a direct calculation of $\tilde{\varrho}(\sigma_\ell^2)$ using (8) shows that

$$\begin{aligned} &\begin{pmatrix} \psi(\sigma_\ell^2) & \psi'(\sigma_\ell^2)\kappa'(\sigma_\ell^2) \cdot \varepsilon \\ \psi(\sigma_\ell^2)\kappa(\sigma_\ell^2) \cdot \varepsilon & \psi'(\sigma_\ell^2) \end{pmatrix} \\ &\quad = \begin{pmatrix} \eta(\sigma_\ell)\eta'(\sigma_\ell) & \eta'(\sigma_\ell)(d_1(\sigma_\ell) + d_2(\sigma_\ell)) \cdot \varepsilon \\ \eta'(\sigma_\ell)(d_1(\sigma_\ell) + d_2(\sigma_\ell)) \cdot \varepsilon & \eta(\sigma_\ell)\eta'(\sigma_\ell) \end{pmatrix}. \end{aligned}$$

Comparing the upper right hand corners in this equality of matrices yields

$$\psi'(\sigma_\ell^2)\kappa'(\sigma_\ell^2) = \eta'(\sigma_\ell)(d_1(\sigma_\ell) + d_2(\sigma_\ell)),$$

and hence by (14),

$$a_\ell(g_\alpha^\flat) = \eta(\sigma_\ell)\kappa'(\sigma_\ell^2) = \eta(\lambda)\kappa_{\psi_\heartsuit}(\mathrm{Frob}_\lambda) = \eta(\lambda)\log_p u(\psi_\heartsuit, \lambda) = \log_p u(\psi_\heartsuit, \ell),$$

where the penultimate equality follows from Lemma 2.2, and the last from the definition of $u(\psi_\heartsuit, \ell)$ given in (2). The first assertion in Theorem 1 follows. The second assertion is a direct consequence of the multiplicativity properties of the Fourier coefficients of $\tilde{g}_\alpha$, now that the values of $a_\ell(\tilde{g}_\alpha)$ for ℓ prime are known.

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