Mathematische Zeitschrift (2021) 298:385–418
https://doi.org/10.1007/s00209-020-02603-8

Mathematische Zeitschrift

Generalised Heegner cycles and the complex Abel–Jacobi map

Massimo Bertolini[1] · Henri Darmon[2] · David Lilienfeldt[2] · Kartik Prasanna[3]

Received: 25 March 2019 / Accepted: 26 July 2020 / Published online: 17 September 2020

Abstract
Generalised Heegner cycles were introduced in Bertolini et al. (Duke Math J 162(6), 1033–1148, 2013) as a variant of Heegner cycles on Kuga–Sato varieties. The first main result of this article is a formula for the image of these cycles under the complex Abel–Jacobi map in terms of explicit line integrals of modular forms on the complex upper half-plane. The second main theorem uses this formula to show that the Chow group and the Griffiths group of the product of a Kuga–Sato variety with an elliptic curve with complex multiplication are not finitely generated. More precisely, it is shown that the subgroup generated by the image of generalised Heegner cycles has infinite rank in the group of null-homologous cycles modulo both rational and algebraic equivalence.

Keywords Algebraic cycles · Abel–Jacobi map · Chow group · Griffiths group · Complex multiplication · Kuga–Sato varieties · Generalised Heegner cycles

During the preparation of this article, KP was supported partially by NSF grants DMS-1015173 and DMS-0854900 and DL was supported partially by the Institut des Sciences Mathématiques, Montréal, Canada.

✉ Henri Darmon
henri.darmon@mcgill.ca

Massimo Bertolini
massimo.bertolini@uni-due.de

David Lilienfeldt
david.lilienfeldt@mail.mcgill.ca

Kartik Prasanna
kartik.prasanna@gmail.com

1 Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann-Strasse 9, 45127 Essen, Germany

2 Department of Mathematics and Statistics, McGill University, Burnside Hall, 805 Sherbrooke Street West, Montreal, QC H3A 0G4, Canada

3 Department of Mathematics, University of Michigan, 2074 East Hall, 530 Church Street, Ann Arbor, MI 48109, USA

Contents

1 Introduction

In a previous work [1], the authors introduced a distinguished collection of null-homologous, codimension $r+1$ cycles on the $(2r+1)$-dimensional variety

$$X_r := W_r \times A^r,$$

where W_r is the Kuga–Sato variety obtained from the r-fold fiber power of the universal elliptic curve over a modular curve C, and A is a fixed elliptic curve with complex multiplication. Referred to as *generalised Heegner cycles* in [1] because of their close affinity with the Heegner cycles on Kuga–Sato varieties studied in [13,14] and [18], they are indexed by isogenies $\varphi : A \longrightarrow A'$. The cycle Δ_φ labeled by φ is supported on the fiber $(A')^r \times A^r$ above a point of C attached to A', and is equal, roughly speaking, to the r-fold self-product of the graph of φ.

One may consider the images of the Δ_φ under the *p-adic Abel–Jacobi map*

$$\mathrm{AJ}_p : \mathrm{CH}^{r+1}(X_r)_0(\mathbf{C}_p) \longrightarrow J^{r+1}(X_r/\mathbf{C}_p) := \mathrm{Fil}^{r+1} H^{2r+1}_{\mathrm{dR}}(X_r/\mathbf{C}_p)^\vee \tag{1}$$

whose domain is the Chow group of null-homologous codimension $r+1$ cycles on X_r over $\mathbf{C}_p := \widehat{\bar{\mathbf{Q}}}_p$ and whose target is the $\mathbf{C}_p$-linear dual of the middle step in the de Rham cohomology $H^{2r+1}_{\mathrm{dR}}(X_r/\mathbf{C}_p)$ relative to the Hodge filtration. The main result of [1] is a formula relating $\mathrm{AJ}_p(\Delta_\varphi)$ to special values of certain p-adic Rankin L-series. (An analogous formula for the p-adic heights of the same cycles was later obtained in [16].) A key ingredient in [1], made explicit in Section 3 of loc.cit., is a description of the relevant p-adic Abel–Jacobi images in terms of p-adic integration of higher weight modular forms, à la Coleman.

The goal of the present article is to give an analogous description of the image of the cycles Δ_φ under the complex Abel–Jacobi map

$$\mathrm{AJ}_{\mathbf{C}} : \mathrm{CH}^{r+1}(X_r)_0(\mathbf{C}) \longrightarrow J^{r+1}(X_r/\mathbf{C}) := \frac{\mathrm{Fil}^{r+1} H^{2r+1}_{\mathrm{dR}}(X_r/\mathbf{C})^\vee}{\mathrm{Im}\, H_{2r+1}(X_r(\mathbf{C}), \mathbf{Z})}, \tag{2}$$

where $J^{r+1}(X_r/\mathbf{C})$ is the $r+1$ Griffiths intermediate Jacobian. This map is defined in terms of complex integration of differential forms attached to classes in $H^{2r+1}_{\mathrm{dR}}(X_r/\mathbf{C})$. One of the main results of this work is Theorem 1 of Sect. 9, which gives a formula for $\mathrm{AJ}_{\mathbf{C}}(\Delta_\varphi)$ in terms of explicit line integrals of modular forms on the complex upper half-plane. An application of this formula is given in Theorem 2 of Sect. 10, where it is shown that the Chow group of homologically trivial cycles (resp. the Griffiths group when $r \geq 2$) of X_r over $\bar{\mathbf{Q}}$ has infinite rank. More precisely, it is proved that the subgroup generated by the image of generalised Heegner cycles in these groups has infinite rank. A second motivation for publishing a detailed proof of Theorem 1 is that this result forms the basis for the numerical calculations of Chow-Heegner points carried out in [2, §3]. It may also be useful in further numerical explorations of generalised Heegner cycles—for instance, in extending the calculations of [9] beyond the more "traditional" setting of Heegner cycles on Kuga–Sato varieties.

The proof of Theorem 2 follows closely that of Theorem 4.7 of [14] which treats the case of "usual" Heegner cycles on a Kuga–Sato threefold, and rests on an ingenious method of Bloch. The most significant difference lies in the setting that is treated: whereas Schoen's cycles are indexed by arbitrary quadratic orders of varying discriminant, ours are forced by necessity to be indexed by (not necessarily maximal) orders of a fixed imaginary quadratic field.

The present work can be compared with [3], which studies the position of generalised Heegner cycles relative to the coniveau filtration on the relevant Chow groups, constructing non-torsion elements in the Griffiths group by methods that are purely p-adic, relying crucially on p-adic Hodge theoretic invariants and their relation to p-adic L-functions. In contrast, the approach described herein rests on a blend of complex and p-adic techniques, and the results obtained are more general if somewhat more qualitative.

The preliminary Sects. 2 and 3 provide an overview of the theory of generalised Heegner cycles and modular forms over the complex numbers. In Sects. 4 to 7, purely transcendental, or Hodge theoretic, arguments are used for the computation of complex Abel–Jacobi maps. Specific properties of modular forms on modular curves (period lattices, modular symbols) lead to simplifications of the previous Abel–Jacobi computations, culminating in the proof of Theorem 1 in Sect. 9. Section 10, which forms the technical core of the paper, is devoted to the study of the Chow group and Griffiths group of X_r. Section 10.1 singles out a distinguished subcollection of generalised Heegner cycles. The aim is to study the subgroup generated by these in the various cycle groups. Analytic estimates of the explicit line integrals appearing in the Abel–Jacobi formula are used in Sect. 10.2 in order to determine their vanishing (or not), and consequences for the order of the cycles in the relevant groups. Section 10.3 uses class field theory, étale ℓ-adic variants of Abel–Jacobi maps and fundamental properties of étale cohomology to upgrade the previous order estimates and show that infinitely many of the cycles have infinite order. Class field theory and complex multiplication theory as formulated by Shimura are key in Sect. 10.4 where it is proved that the cycles generate a subgroup of infinite rank. Section 10.5 goes through the necessary modifications that allow one to deduce, when $r \geq 2$, the analogous result for the Griffiths group.

2 Generalised Heegner cycles

We begin by briefly recalling the definition of generalised Heegner cycles, following the notations of [1, §2].

Fix an integer $N \geq 5$ and let $\Gamma = \Gamma_1(N)$ be the standard congruence subgroup of level N:

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbf{Z}) \quad \text{such that } a-1,\ d-1,\ c \equiv 0 \pmod{N} \right\}.$$

Let $C^0 := Y_1(N)$ and $C := X_1(N)$ denote the usual (affine and projective, respectively) modular curves of level N, and write W_r for the r-th Kuga–Sato variety over $X_1(N)$ as described for instance in Section 2.1 and the appendix of [1].

Let K be an imaginary quadratic field of discriminant $-d_K$, let $\mathscr{O}_K$ be its ring of integers, and let H denote the Hilbert class field of K. Choose once and for all a complex embedding $\bar{K} \longrightarrow \mathbf{C}$, and let A be a fixed elliptic curve defined over H satisfying $\mathrm{End}_H(A) \cong \mathscr{O}_K$. The generalised Heegner cycles of [1] are an infinite collection of codimension $r+1$ cycles on the smooth projective $(2r+1)$-dimensional variety

$$X_r := W_r \times A^r.$$

To define them precisely, assume that K satisfies the Heegner hypothesis relative to N, namely that N is the norm of an ideal $\mathfrak{N}$ for which $\mathscr{O}_K/\mathfrak{N} \simeq \mathbf{Z}/N\mathbf{Z}$. Let $t_A \in A[\mathfrak{N}]$ be a choice of $\mathfrak{N}$-torsion point on A. Following the moduli description of $X_1(N)$, the pair (A, t_A) corresponds to a complex point on C (defined, in fact, over an abelian extension of K, relative to any of the standard models of $C = X_1(N)$ as a curve over $\mathbf{Q}$). For obvious reasons, the datum of the point t_A on A of order N is sometimes referred to as a $\Gamma_1(N)$-*structure* on A.

Consider the set of pairs (φ, A'), where $\varphi : A \longrightarrow A'$ is an isogeny of A defined over $\bar{K}$. Two pairs (φ_1, A'_1) and (φ_2, A'_2) are said to be isomorphic if there is a $\bar{K}$-isomorphism $\iota : A'_1 \longrightarrow A'_2$ satisfying $\iota\varphi_1 = \varphi_2$. Let

$$\mathrm{Isog}(A) := \{\text{Isomorphism classes of pairs } (\varphi, A')\}.$$

There is a natural bijection between this set and the set of finite subgroups of $A(\bar{H})$. The absolute Galois group $G_H = \mathrm{Gal}(\bar{H}/H)$ acts naturally on $\mathrm{Isog}(A)$ by acting on the corresponding subgroups and a pair (φ, A') admits a representative defined over a field $F \subset \bar{H}$ if it is fixed by the subgroup $G_F \subset G_H$.

The generalised Heegner cycles are naturally indexed by the subset $\mathrm{Isog}^{\mathfrak{N}}(A)$ of $\mathrm{Isog}(A)$ consisting of pairs (φ, A'), where φ is an isogeny whose kernel intersects $A[\mathfrak{N}]$ trivially. An element $(\varphi, A') \in \mathrm{Isog}^{\mathfrak{N}}(A)$ determines a point $P_{A'}$ on C attached to the pair $(A', t_{A'} := \varphi(t_A))$, and an embedding

$$\iota_{A'} : (A')^r \longrightarrow W_r$$

of $(A')^r$ as the fiber of W_r above the point $P_{A'}$ relative to the natural projection $W_r \longrightarrow C$. Given $(\varphi, A') \in \mathrm{Isog}^{\mathfrak{N}}(A)$, let Υ_φ be the codimension $r+1$ cycle on X_r defined by letting $\mathrm{Graph}(\varphi) \subset A \times A'$ be the graph of φ, and setting

$$\Upsilon_\varphi := \mathrm{Graph}(\varphi)^r \subset (A \times A')^r \xrightarrow{\simeq} (A')^r \times A^r \subset W_r \times A^r, \tag{3}$$

where the last inclusion is induced from the pair $(\iota_{A'}, \mathrm{id}_A^r)$.

When $r = 0$, the cycle Υ_φ is just the CM point on the modular curve C attached to the pair $(A', t_{A'})$. The generalised Heegner cycle Δ_φ attached to φ is then obtained by setting

$$\Delta_\varphi := \Upsilon_\varphi - \infty, \tag{4}$$

where ∞ is the standard cusp on $X_1(N)$ (although any fixed choice will do). This modification has the effect of making the cycle Δ_φ homologically trivial.

For general r, we obtain a homologically trivial cycle from Υ_φ by setting

$$\Delta_\varphi := \epsilon_X \Upsilon_\varphi, \tag{5}$$

where ϵ_X is an idempotent in the ring of algebraic correspondences from X_r to itself, which is defined as a product

$$\epsilon_X := \epsilon_W \epsilon_A \tag{6}$$

of two idempotents in the ring of correspondences on W_r and A^r respectively. We now briefly recall the definition of the projectors ϵ_W and ϵ_A.

The projector ϵ_A. Let S_r denote the symmetric group on r letters. Multiplication by -1 on A, combined with the natural permutation action of S_r on A^r, gives rise to an action of the wreath product

$$\Xi_r := (\mu_2)^r \rtimes S_r \tag{7}$$

on A^r. Let $j : \Xi_r \longrightarrow \mu_2$ be the homomorphism which is the identity on μ_2 and the sign character on S_r, and let

$$\epsilon_A := \frac{1}{2^r r!} \sum_{\sigma \in \Xi_r} j(\sigma)\sigma \in \mathbf{Q}[\mathrm{Aut}(A^r)] \tag{8}$$

denote the associated idempotent in the rational group ring of $\mathrm{Aut}(A^r)$.

The projector ϵ_W. Translation by the sections of order N gives rise to an action of $(\mathbf{Z}/N\mathbf{Z})^r$ on W_r (see [1, §2.1] for details). Let σ_a denote the automorphism of W_r associated to $a \in (\mathbf{Z}/N\mathbf{Z})^r$, and let

$$\epsilon_W^{(1)} = \frac{1}{N^r} \sum_{a \in (\mathbf{Z}/N\mathbf{Z})^r} \sigma_a \tag{9}$$

denote the corresponding idempotent in the rational group ring of $(\mathbf{Z}/N\mathbf{Z})^r$. Similarly, the group Ξ_r of (7) can be viewed as a subgroup of $\mathrm{Aut}(W_r/C)$ acting on the fibers of the natural projection from W_r to C. Let $\epsilon_W^{(2)}$ be the idempotent in the group ring $\mathbf{Z}[1/2r!][\mathrm{Aut}(W_r/C)]$ which is defined by the same formula as in (8) with A^r replaced by W_r/C. The idempotents $\epsilon_W^{(1)}$ and $\epsilon_W^{(2)}$ commute, and we define ϵ_W as the product

$$\epsilon_W = \epsilon_W^{(2)} \epsilon_W^{(1)} \tag{10}$$

in the ring of rational correspondences on W_r.

Since the correspondence ϵ_X is compatible with the projection $\pi_r : X_r \longrightarrow C$, the generalised Heegner cycle Δ_φ is supported on the fiber $\pi_r^{-1}(P_{A'})$ of π_r above $P_{A'}$. As in the case where $r = 0$, it is also homologically trivial. This follows from the fact that the image of Δ_φ under the cycle class map belongs to $\epsilon_X H_{\mathrm{dR}}^{2r+2}(X_r/\mathbf{C})$, which is zero by [1, Prop. 2.4]. Section 4 below gives a more explicit description of a chain of real dimension $2r+1$ in $X_r(\mathbf{C})$ having Δ_φ as boundary, which will be used in subsequent calculations.

3 Modular forms and de Rham cohomology

Let $\pi : \mathscr{E} \longrightarrow C^0$ be the universal elliptic curve with level N structure over C^0, and let $\underline{\omega} := \pi_* \Omega^1_{\mathscr{E}/C^0}$ be the coherent sheaf of relative differentials on $\mathscr{E}/C^0$, extended to a coherent sheaf on C in the standard way (cf. [1, §1.1]). Let $\underline{\omega}^r$ be the r-th tensor power of this line bundle. The sheaf $\underline{\omega}^2$ is related to the sheaf $\Omega^1_C(\log \text{cusps})$ of regular differentials on C with logarithmic poles at the cusps by the Kodaira–Spencer isomorphism

$$\sigma : \underline{\omega}^2 \xrightarrow{\sim} \Omega^1_C(\log \text{cusps}), \tag{11}$$

as described for instance in [1, §1.1].

A (holomorphic) modular form of weight $k = r + 2$ is a global section of the sheaf $\underline{\omega}^k$, or—equivalently, by (11)—of $\underline{\omega}^r \otimes \Omega^1_C(\log \text{cusps})$ over C. The global sections of $\underline{\omega}^r \otimes \Omega^1_C$ are called *cusp forms*. Let $M_k(\Gamma)$ and $S_k(\Gamma)$ denote the complex vector spaces of modular forms and cusp forms on Γ, respectively.

The sheaf $\underline{\omega}$ is a subsheaf of the relative logarithmic de Rham cohomology sheaf on C defined by taking the relative hypercohomology of the complex of sheaves

$$\mathscr{L}_1 := \mathbb{R}^1\pi_*(0 \to \mathscr{O}_{\mathscr{E}} \to \Omega^1_{\mathscr{E}/C^0} \to 0),$$

and extending to C following the prescription given in [1, §1.1]. The Hodge filtration gives rise to an exact sequence of coherent sheaves over C:

$$0 \longrightarrow \underline{\omega} \longrightarrow \mathscr{L}_1 \longrightarrow \underline{\omega}^{-1} \longrightarrow 0. \tag{12}$$

The vector bundle $\mathscr{L}_1$ is also equipped with the canonical integrable Gauss–Manin connection

$$\nabla : \mathscr{L}_1 \longrightarrow \mathscr{L}_1 \otimes \Omega^1_C(\log \text{cusps}), \tag{13}$$

and Poincaré duality on the fibres of $\mathscr{L}_1$ gives rise to a canonical pairing

$$\langle\ ,\ \rangle : \mathscr{L}_1 \times \mathscr{L}_1 \longrightarrow \mathscr{O}_C. \tag{14}$$

Let $\mathscr{L}_r := \mathrm{Sym}^r \mathscr{L}_1$ denote the r-th symmetric power of $\mathscr{L}_1$. The natural inclusion $\underline{\omega}^r \longrightarrow \mathscr{L}_r$ gives rise to inclusions

$$S_{r+2}(\Gamma) := H^0(C, \underline{\omega}^r \otimes \Omega^1_C) \hookrightarrow H^0(C, \mathscr{L}_r \otimes \Omega^1_C). \tag{15}$$

The self-duality

$$\langle\ ,\ \rangle : \mathscr{L}_r \times \mathscr{L}_r \longrightarrow \mathscr{O}_C \tag{16}$$

induced by (14) is given by the rule

$$\langle \alpha_1 \cdots \alpha_r, \beta_1 \cdots \beta_r \rangle = \frac{1}{r!} \sum_{\sigma \in S_r} \langle \alpha_1, \beta_{\sigma(1)} \rangle \cdots \langle \alpha_r, \beta_{\sigma(r)} \rangle. \tag{17}$$

We will also have use for further coherent sheaves of $\mathscr{O}_C$-modules arising in the cohomology of the fibers for the natural projection $X_r \longrightarrow C$,

$$\mathscr{L}_{r,r} = \mathscr{L}_r \otimes \mathrm{Sym}^r H^1_{\mathrm{dR}}(A). \tag{18}$$

Note that $\mathscr{L}_{r,r}$ is also equipped with the self-duality

$$\langle\ ,\ \rangle : \mathscr{L}_{r,r} \times \mathscr{L}_{r,r} \longrightarrow \mathscr{O}_C \tag{19}$$

arising from (17), which is discussed in more details in [1, §2.2].

As explained in [1, §1.1], all the notions introduced so far in this section are purely algebraic and make sense over an arbitrary field over which the modular curve C can be defined. In the present note we are interested solely in their complex incarnations. The set $C(\mathbf{C})$ of complex points of C is a compact Riemann surface, and the analytic map

$$\mathrm{pr} : \mathscr{H} \longrightarrow C^0(\mathbf{C}), \qquad \mathrm{pr}(\tau) := \left(\mathbf{C}/\langle 1, \tau\rangle, \frac{1}{N}\right)$$

identifies $C^0(\mathbf{C})$ with the quotient $\Gamma\backslash\mathscr{H}$, where we recall that $\Gamma = \Gamma_1(N)$. The coherent sheaf $\mathscr{L}_r$ gives rise to an analytic sheaf $\mathscr{L}_r^{\mathrm{an}}$ on the Riemann surface $C(\mathbf{C})$; let $\tilde{\mathscr{L}}_r^{\mathrm{an}} := \mathrm{pr}^* \mathscr{L}_r^{\mathrm{an}}$ denote its pullback to $\mathscr{H}$.

Recall the elliptic fibration $\pi : \mathscr{E} \longrightarrow C^0$, and let

$$\mathbb{L}_1^B := R^1\pi_*\mathbf{Z}, \qquad \mathbb{L}_r^B := \mathrm{Sym}^r \mathbb{L}_1^B,$$

be the locally constant sheaves of $\mathbf{Z}$-modules whose fibers at $x \in C^0(\mathbf{C})$ are identified with the Betti cohomology $H_B^1(\mathscr{E}_x, \mathbf{Z})$ and $\mathrm{Sym}^r H_B^1(\mathscr{E}_x, \mathbf{Z})$ respectively. The local system

$$\mathbb{L}_r := \mathbb{L}_r^B \otimes_{\mathbf{Z}} \mathbf{C} \tag{20}$$

is identified with the sheaf of horizontal sections of $(\mathscr{L}_r^{\mathrm{an}}, \nabla)$ over $C^0(\mathbf{C})$. (Cf. [5, thm. 2.17]) Likewise, let

$$\mathbb{L}_{r,r} := \mathbb{L}_r \otimes \mathrm{Sym}^r H_{\mathrm{dR}}^1(A/\mathbf{C}) \tag{21}$$

denote the sheaf of locally constant sections (again, for the complex topology on $C^0(\mathbf{C})$) of the sheaf $\mathscr{L}_{r,r}$.

Let τ denote a point on $\mathscr{H}$ and let w be the standard complex coordinate on the elliptic curve $\mathbf{C}/\langle 1, \tau\rangle$. The Hodge filtration on $H_{\mathrm{dR}}^1(\mathbf{C}/\langle 1, \tau\rangle)$ admits a canonical, functorial (but *not holomorphic*) splitting

$$H_{\mathrm{dR}}^1(\mathbf{C}/\langle 1, \tau\rangle) := \mathbf{C}dw \oplus \mathbf{C}d\bar{w}, \tag{22}$$

called the *Hodge decomposition*. In terms of the coordinates τ, dw, and $d\bar{w}$, one has

$$\nabla dw = \left(\frac{dw - d\bar{w}}{\tau - \bar{\tau}}\right) d\tau, \qquad \sigma((2\pi i dw)^2) = 2\pi i d\tau. \tag{23}$$

(See [1, §1.2] for the details of these calculations.)

A modular form $\omega_f \in M_k(\Gamma)$ gives rise to a holomorphic function on the upper half plane $\mathscr{H}$ by the rule

$$\omega_f(\tau) = f(\tau)(2\pi i dw)^{r+2} = f(\tau)(2\pi i dw)^r \otimes (2\pi i d\tau). \tag{24}$$

This function obeys the familiar transformation rule

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N), \tag{25}$$

and the modular form ω_f is completely determined by the associated function $f(\tau)$.

4 Homological triviality

All Chow groups will henceforth be taken with rational coefficients, so that they consist of $\mathbf{Q}$-linear combinations of cycles modulo rational equivalence.

The goal of this section is to express the generalised Heegner cycles Δ_φ as the boundaries of explicit $(2r+1)$-dimensional topological chains in $X_r^0(\mathbf{C})$. Such a calculation will be useful in calculating the images of these cycles under the complex Abel–Jacobi map.

Let $W_r^0 := W_r \times_C C^0$ and $X_r^0 = X_r \times_C C^0$ denote the complements in W_r and X_r respectively of the fibers above the cusps of C. Let $\tilde{W}_r$ be the r-fold product of the universal elliptic curve over the upper half-plane $\mathscr{H}$ (which we will denote $\mathscr{E}$ by slight abuse of notation). It is isomorphic as an analytic variety to the quotient $\mathbf{Z}^{2r}\backslash(\mathbf{C}^r \times \mathscr{H})$, where $\mathbf{Z}^{2r}$ acts on $\mathbf{C}^r \times \mathscr{H}$ by the rule

$$(m_1, n_1, \ldots, m_r, n_r)(w_1, \ldots, w_r, \tau) := (w_1 + m_1 + n_1\tau, \ldots, w_r + m_r + n_r\tau, \tau). \tag{26}$$

Finally, let

$$\tilde{X}_r = \tilde{W}_r \times A^r(\mathbf{C}).$$

It follows from these definitions that

$$W_r^0(\mathbf{C}) = \Gamma \backslash \tilde{W}_r, \qquad X_r^0(\mathbf{C}) = \Gamma \backslash \tilde{X}_r,$$

where Γ acts on $\tilde{W}_r$ by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (w_1, \ldots, w_r, \tau) = \left(\frac{w_1}{c\tau + d}, \ldots, \frac{w_r}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right), \tag{27}$$

and acts trivially on $A^r(\mathbf{C})$. Write pr for the natural Γ-covering maps $\tilde{X}_r \longrightarrow X_r^0(\mathbf{C})$ and $\mathscr{H} \longrightarrow C^0(\mathbf{C})$, and let $\tilde{\pi}_r$ be the natural fibering $\tilde{\pi}_r : \tilde{X}_r \longrightarrow \mathscr{H}$. These maps fit into the cartesian diagram

$$\begin{array}{ccc} \tilde{X}_r & \xrightarrow{\text{pr}} & X_r^0(\mathbf{C}) \\ \Big\downarrow{\scriptstyle \tilde{\pi}_r} & & \Big\downarrow{\scriptstyle \pi_r} \\ \mathscr{H} & \xrightarrow{\text{pr}} & C^0(\mathbf{C}). \end{array}$$

Given $(\varphi, A') \in \text{Isog}^{\mathfrak{N}}(A)$, set $t' := \varphi(t_A)$, so that $\varphi : (A, t_A) \longrightarrow (A', t')$ is an isogeny of elliptic curves with Γ-level structure, in the obvious sense. Let $P_{A'}$ be the point of $C^0(\mathbf{C})$ associated to the pair (A', t'). The main result of this section, which directly implies the homological triviality of Δ_φ, is the following:

Proposition 1 *Assume $r > 0$. Then there exists a topological cycle $\widetilde{\Delta}_\varphi$ on $\tilde{X}_r$ satisfying*

1. *The pushforward* $\text{pr}_*(\widetilde{\Delta}_\varphi)$ *satisfies*

$$\text{pr}_*(\widetilde{\Delta}_\varphi) = \Delta_\varphi + \partial \xi,$$

   where ξ is a topological $(2r+1)$-chain supported on $\pi_r^{-1}(P_{A'})$.
2. *The cycle $\widetilde{\Delta}_\varphi$ is homologically trivial on $\tilde{X}_r$.*

Proof Choose a point $\tau_{A'} \in \mathscr{H}$ such that $\text{pr}(\tau_{A'}) = P_{A'}$. Since pr induces an isomorphism between $\tilde{\pi}_r^{-1}(\tau_{A'})$ and $\pi_r^{-1}(P_{A'})$, the choice of $\tau_{A'}$ determines cycles $\Upsilon_\varphi^\natural$ and $\Delta_\varphi^\natural$ on $\tilde{X}_r$ supported on $\tilde{\pi}_r^{-1}(\tau_{A'})$ and satisfying

$$\text{pr}_*(\Upsilon_\varphi^\natural) = \Upsilon_\varphi, \qquad \text{pr}_*(\Delta_\varphi^\natural) = \Delta_\varphi. \tag{28}$$

These cycles need not be homologically trivial on $\tilde{X}_r$. In fact, since $\mathscr{H}$ is contractible, the inclusion

$$i_{\tau_{A'}} : \tilde{\pi}_r^{-1}(\tau_{A'}) \longrightarrow \tilde{X}_r$$

induces an isomorphism

$$i_{\tau_{A'}*} : H_{2r}(\tilde{\pi}_r^{-1}(\tau_{A'}), \mathbf{Q}) \xrightarrow{\sim} H_{2r}(\tilde{X}_r, \mathbf{Q}), \tag{29}$$

and the classes $[\Upsilon_\varphi^\natural]$ and $[\Delta_\varphi^\natural]$ of $\Upsilon_\varphi^\natural$ and $\Delta_\varphi^\natural$ in $H_{2r}(\tilde{X}_r, \mathbf{Q})$ are identified with those of Υ_φ and Δ_φ in $H_{2r}((A' \times A)^r(\mathbf{C}), \mathbf{Q})$.

The fundamental group Γ of C^0 acts naturally on $H_{2r}(\tilde{X}_r, \mathbf{Q})$, and the kernel of the pushforward map

$$\mathrm{pr}_* : H_{2r}(\tilde{X}_r, \mathbf{Q}) \longrightarrow H_{2r}(X_r^0(\mathbf{C}), \mathbf{Q})$$

contains the module $I_\Gamma H_{2r}(\tilde{X}_r, \mathbf{Q})$, where I_Γ is the augmentation ideal in the rational group ring $\mathbf{Q}[\Gamma]$.

Note that the projector ϵ_X of (6) acts naturally on $H_{2r}(\tilde{X}_r, \mathbf{Q})$ and $[\Delta_\varphi^\natural] = \epsilon_X[\Upsilon_\varphi^\natural]$ belongs to $\epsilon_X H_{2r}(\tilde{X}_r, \mathbf{Q})$. $\square$

Lemma 1 *For all $r \geq 1$,*

$$\epsilon_X H_{2r}(\tilde{X}_r, \mathbf{Q}) = \mathrm{Sym}^r H_1(\mathscr{E}, \mathbf{Q}) \otimes \mathrm{Sym}^r H_1(A(\mathbf{C}), \mathbf{Q}) \subset I_\Gamma H_{2r}(\tilde{X}_r, \mathbf{Q}).$$

Proof Since multiplication by (-1) acts as -1 on $H^1_{\mathrm{dR}}(A/F)$ and as 1 on $H^0_{\mathrm{dR}}(A/F)$ and $H^2_{\mathrm{dR}}(A/F)$, it follows that ϵ_A annihilates all the terms except $H^1_{\mathrm{dR}}(A/F)^{\otimes r}$ in the Künneth decomposition

$$H^*_{\mathrm{dR}}(A^r/F) = \bigoplus_{(i_1,\ldots,i_r)} H^{i_1}_{\mathrm{dR}}(A/F) \otimes \cdots \otimes H^{i_r}_{\mathrm{dR}}(A/F), \tag{30}$$

(where the direct sum is taken over all r-tuples $(i_1, \ldots, i_r)$ with $0 \leq i_j \leq 2$). The natural action of S_r on $H^1_{\mathrm{dR}}(A/F)^{\otimes r}$ corresponds to the geometric permutation action of S_r on A^r, twisted by the sign character. It follows that the restriction of ϵ_A to $H^1_{\mathrm{dR}}(A/F)^{\otimes r}$ induces the natural projection onto the space $\mathrm{Sym}^r H^1_{\mathrm{dR}}(A/F)$ of symmetric tensors. A similar argument applies to the projector ϵ_W and its action on the homology of the fibers of the natural projection $\tilde{X}_r \longrightarrow \mathscr{H}$. The first equality follows. The second containment is a consequence of the fact that

$$(\mathrm{Sym}^r H_1(\mathscr{E}, \mathbf{Q}) \otimes \mathrm{Sym}^r H_1(A(\mathbf{C}), \mathbf{Q})) \otimes_{\mathbf{Q}} \mathbf{C} = \mathbb{L}_{r,r},$$

where $\mathbb{L}_{r,r}$ is the local system of (21), and that the representation of Γ associated to this local system is isomorphic to a direct sum of $r + 1$ copies of the r-th symmetric power of the standard two-dimensional representation of Γ. Each of these copies is irreducible and, since $r > 0$, is non-trivial and hence has a trivial space of Γ-coinvariants. $\square$

It now follows from Lemma 1 that

$$[\Delta_\varphi] = \mathrm{pr}_*([\Delta_\varphi^\natural]) \in \mathrm{pr}_*(I_\Gamma H_{2r}(\tilde{X}_r, \mathbf{Q})) = 0,$$

and therefore Δ_φ is homologically trivial. $\square$

To produce the cycle $\widetilde{\Delta}_\varphi$ of Proposition 1 explicitly, let

$$[\Delta_\varphi^\natural] = \sum_{j=1}^{t} (\gamma_j^{-1} - 1) Z_j, \quad \begin{array}{l} \gamma_1, \ldots, \gamma_t \in \Gamma, \\ Z_1, \ldots, Z_t \in H_{2r}(\tilde{X}_r, \mathbf{Q}) \end{array} \tag{31}$$

be an expression of $[\Delta_\varphi^\natural]$ as an element of $I_\Gamma H_{2r}(\tilde{X}_r, \mathbf{Q})$. Letting $\mathscr{Z}(\tau, Z)$ denote any topological $2r$-cycle supported on $\tilde{\pi}_r^{-1}(\tau)$ and determined by the class of Z in $H_{2r}(\tilde{X}_r, \mathbf{Q})$ via (29), define:

$$\widetilde{\Delta}_\varphi := \sum_{j=1}^{t} \left(\mathscr{Z}(\gamma_j \tau_{A'}, Z_j) - \mathscr{Z}(\tau_{A'}, Z_j) \right). \tag{32}$$

It is then straightforward to check that $\widetilde{\Delta}_\varphi$ has the required properties. For example, the homological triviality of $\widetilde{\Delta}_\varphi$ follows from the fact that

$$\widetilde{\Delta}_\varphi = \partial \widetilde{\Delta}_\varphi^\sharp, \qquad \text{with} \qquad \widetilde{\Delta}_\varphi^\sharp := \sum_{j=1}^{t} \mathscr{Z}(\tau_{A'} \to \gamma_j \tau_{A'}, Z_j), \tag{33}$$

where

$$\mathscr{Z}(\tau_{A'} \to \gamma_j \tau_{A'}, Z_j) := \mathrm{path}(\tau_{A'} \to \gamma_j \tau_{A'}) \times Z_j \tag{34}$$

and $\mathrm{path}(\tau_{A'} \to \gamma_j \tau_{A'})$ is any continuous path on $\mathscr{H}$ joining $\tau_{A'}$ to $\gamma_j \tau_{A'}$. Note that in (34) we have identified $\tilde{X}_r(\mathbf{C})$ with $\mathscr{H} \times (\mathbf{C}^{2r}/\mathbf{Z}^{4r})$.

Remark 1 Yet another approach to proving the homological triviality of Δ_φ, by deforming these cycles to the fibers supported above the cusps of the modular curve, is described in [14]. The approach we have given adapts more readily to the setting of Shimura curves attached to arithmetic subgroups of $\mathbf{SL}_2(\mathbf{R})$ with compact quotient.

Remark 2 A decomposition as in (31) with $Z_1, \ldots, Z_t \in H_{2r}(\tilde{X}_r, \mathbf{Z})$ is said to be *integral*. Such a decomposition may not always be possible, owing to the possible presence of torsion in $H_{2r}(X_r^0(\mathbf{C}), \mathbf{Z})$. But it may be obtained after replacing $[\Delta_\varphi^\natural]$ by a suitable integer multiple. In the rest of this note, when the image of Δ_φ under the complex Abel–Jacobi map is computed, *it will be tacitly assumed that the Z_i do belong to this integral lattice.*

5 The complex Abel–Jacobi map

The complex Abel–Jacobi map is a function from the Chow group $\mathrm{CH}^{r+1}(X_r)_0(\mathbf{C})$ into a complex torus:

$$\mathrm{AJ}_{\mathbf{C}} : \mathrm{CH}^{r+1}(X_r)_0(\mathbf{C}) \longrightarrow J^{r+1}(X_r/\mathbf{C}) = \frac{\mathrm{Fil}^{r+1} H_{\mathrm{dR}}^{2r+1}(X_r/\mathbf{C})^\vee}{\mathrm{Im}\, H_{2r+1}(X_r(\mathbf{C}), \mathbf{Z})},$$

where the superscript $^\vee$ denotes the dual of complex vector spaces, and the group $\mathrm{Im}\, H_{2r+1}(X_r(\mathbf{C}), \mathbf{Z})$ is viewed as a sublattice of $\mathrm{Fil}^{r+1} H_{\mathrm{dR}}^{2r+1}(X_r\mathbf{C})^\vee$ via integration of closed differential $(2r+1)$-forms against singular integral homology classes of dimension $2r+1$. The linear functional $\mathrm{AJ}_{\mathbf{C}}(\Delta)$ is defined by choosing a continuous integral $(2r+1)$-chain $\Delta^\sharp$ on $X_r(\mathbf{C})$ whose boundary $\partial(\Delta^\sharp)$ is equal to Δ, and setting

$$\mathrm{AJ}_{\mathbf{C}}(\Delta)(\alpha) = \int_{\Delta^\sharp} \alpha, \qquad \text{for all } \alpha \in \mathrm{Fil}^{r+1} H_{\mathrm{dR}}^{2r+1}(X_r/\mathbf{C}). \tag{35}$$

We will be solely interested in the piece of the Abel–Jacobi map that survives after applying the projector ϵ_X defined in (6). By [1, Prop. 2.4],

$$\begin{aligned} \epsilon_X \mathrm{Fil}^{r+1} H_{\mathrm{dR}}^{2r+1}(X_r/\mathbf{C}) &= \mathrm{Fil}^{r+1} H_{\mathrm{par}}^1(C, \mathscr{L}_{r,r}, \nabla) \\ &= H^0(C, \underline{\omega}^r \otimes \Omega_C^1) \otimes \mathrm{Sym}^r H_{\mathrm{dR}}^1(A/\mathbf{C}). \end{aligned} \tag{36}$$

This allows us to view $\mathrm{AJ}_{\mathbf{C}}$ as a map

$$\mathrm{AJ}_{\mathbf{C}} : \epsilon_X \mathrm{CH}^{r+1}(X_r)_0(\mathbf{C}) \longrightarrow \frac{(S_{r+2}(\Gamma) \otimes \mathrm{Sym}^r H_{\mathrm{dR}}^1(A/\mathbf{C}))^\vee}{\Pi_{r,r}},$$

where the lattice $\Pi_{r,r}$ is defined by

$$\Pi_{r,r} := \epsilon_X(\mathrm{Im}\, H_{2r+1}(X_r(\mathbf{C}), \mathbf{Z})). \tag{37}$$

6 Global primitives

We will follow the notations that were introduced in Sect. 3 and in the proof of Proposition 1. Let

$$\tilde{\mathbb{L}}_r := \mathrm{pr}^*(\mathbb{L}_r), \quad \tilde{\mathbb{L}}_{r,r} := \mathrm{pr}^*(\mathbb{L}_{r,r}), \qquad \tilde{\mathscr{L}}_r := \mathrm{pr}^*(\mathscr{L}_r^{\mathrm{an}}), \quad \tilde{\mathscr{L}}_{r,r} := \mathrm{pr}^*(\mathscr{L}_{r,r}^{\mathrm{an}})$$

denote the pullbacks via the analytic projection pr. The local systems $\tilde{\mathbb{L}}_r$ and $\tilde{\mathbb{L}}_{r,r}$ are trivial, i.e., they admit a basis of global sections over $\mathscr{H}$. In other words, if θ is an element of the fiber $\tilde{\mathbb{L}}_{r,r}(\tau)$ of $\tilde{\mathbb{L}}_{r,r}$ at $\tau \in \mathscr{H}$, then there is a unique global horizontal section $\theta^{\nabla} \in H^0(\mathscr{H}, \tilde{\mathscr{L}}_{r,r})^{\nabla=0}$ satisfying $\theta^{\nabla}(\tau) = \theta$.

More generally, if $\mathscr{L}$ is any vector bundle over C^0 equipped with an integrable connection and $\mathbb{L}$ denotes the corresponding local system, we will write $\tilde{\mathbb{L}} := \mathrm{pr}^*(\mathbb{L})$ and $\tilde{\mathscr{L}} := \mathrm{pr}^*(\mathscr{L}^{\mathrm{an}})$, and define *global primitives* in the following way:

Definition 1 Let ω be a global section of $\mathscr{L} \otimes \Omega^1_C$ over C^0. A *primitive* of ω is an element $F \in H^0(\mathscr{H}, \tilde{\mathscr{L}})$ satisfying

$$\nabla F = \mathrm{pr}^*(\omega).$$

Such a primitive always exists, and is well-defined up to elements of the space of global horizontal sections of $\tilde{\mathscr{L}}$ over $\mathscr{H}$.

Definition 2 An $\mathbb{L}$-*valued divisor* on C is a finite formal linear combination of the form $\sum_{j=1}^{t} \theta_j \cdot P_j$ with $P_j \in C(\mathbf{C})$ and $\theta_j \in \mathbb{L}(P_j)$. The module of all such divisors is denoted $\mathrm{Div}(C, \mathbb{L})$.

One defines the notion of a $\tilde{\mathbb{L}}$-valued divisor on $\mathscr{H}$ in a similar way. The analytic projection $\mathrm{pr} : \mathscr{H} \longrightarrow C^0(\mathbf{C})$ induces the natural push-forward map

$$\mathrm{pr}_* : \mathrm{Div}(\mathscr{H}, \tilde{\mathbb{L}}) \longrightarrow \mathrm{Div}(C, \mathbb{L}).$$

Given $G \in H^0(\mathscr{H}, \tilde{\mathscr{L}}_{r,r})$ and $D = \sum_{j=1}^{t} \theta_j \cdot \tau_j \in \mathrm{Div}(\mathscr{H}, \tilde{\mathbb{L}}_{r,r})$, the “value” of G at D is defined by the rule:

$$[G, D] := \sum_{j=1}^{t} \langle G(\tau_j), \theta_j \rangle,$$

where the pairing $\langle\ ,\ \rangle$ on the right is the duality on the fibers at τ_j of the local system $\tilde{\mathbb{L}}_{r,r}$ induced by the pairing of Eq. (19).

For $D = \sum_{j=1}^{t} \theta_j \cdot \tau_j$ as above, the coefficient θ_j belongs to $\tilde{\mathbb{L}}_{r,r}(\tau_j)$ by definition, i.e., to $\mathrm{Sym}^r H^1_{\mathrm{dR}}(\mathscr{E}_{\tau_j}) \otimes \mathrm{Sym}^r H^1_{\mathrm{dR}}(A)$, where $\mathscr{E}_{\tau_j}$ denotes the fibre at τ_j of the pull-back of $\mathscr{E}$ to $\mathscr{H}$ by pr. Calculations similar to those in the proof of Lemma 1 identify $\tilde{\mathbb{L}}_{r,r}(\tau_j)$ with $\varepsilon_X H^{2r}_{\mathrm{dR}}(\tilde{\pi}_r^{-1}(\tau_j))$. Moreover, since $\mathscr{H}$ is contractible, the inclusion of $\tilde{\pi}_r^{-1}(\tau_j)$ in $\tilde{X}_r$ induces a canonical isomorphism of $H^{2r}_{\mathrm{dR}}(\tilde{X}_r)$ onto $H^{2r}_{\mathrm{dR}}(\tilde{\pi}_r^{-1}(\tau_j))$, and hence a canonical identification

$$\varepsilon_X H^{2r}_{\mathrm{dR}}(\tilde{X}_r) = \tilde{\mathbb{L}}_{r,r}(\tau_j). \tag{38}$$

In view of these remarks, the *degree* of an $\tilde{\mathbb{L}}_{r,r}$-valued divisor on $\mathscr{H}$ can be defined by the equation

$$\deg\left(\sum_{j=1}^{t} \theta_j \cdot \tau_j\right) := \sum_{j=1}^{t} \theta_j \in \epsilon_X H^{2r}_{\mathrm{dR}}(\tilde{X}_r).$$

A similar definition could be made for $\mathbb{L}_{r,r}$-valued divisors on C^0, with the degree map taking values in $\epsilon_X H^{2r}_{\mathrm{dR}}(X_r)$. Note that when $r > 0$, this target group is trivial by [1, Prop. 2.4] and hence every $\mathbb{L}_{r,r}$-valued divisor on C^0 (or on C) is of degree 0.

Given $\tau \in \mathscr{H}$ or $P \in C^0$, let

$$\mathrm{cl}_\tau : \mathrm{CH}^r(\mathscr{E}^r_\tau \times A^r) \longrightarrow \tilde{\mathbb{L}}_{r,r}(\tau), \quad \mathrm{cl}_P : \mathrm{CH}^r(\mathscr{E}^r_P \times A^r) \longrightarrow \mathbb{L}_{r,r}(P)$$

denote the (ε_X-components of the) cycle class maps on the associated fibers. The first map is defined by composing the usual cycle class map with isomorphism (38). The second map is defined in terms of the first by identifying $\mathscr{E}_P$ with $\mathscr{E}_\tau$ and $\mathbb{L}_{r,r}(P)$ with $\tilde{\mathbb{L}}_{r,r}(\tau)$ if $P = \mathrm{pr}(\tau)$.

The cycle $\Delta^\natural_\varphi$ that was introduced in Eq. (31) in the proof of Proposition 1 gives rise to the $\tilde{\mathbb{L}}_{r,r}$-valued divisor (which shall be denoted by the same symbol, by abuse of notation):

$$\Delta^\natural_\varphi = \mathrm{cl}_{\tau_{A'}}(\Delta^\natural_\varphi) \cdot \tau_{A'}.$$

Note that $\mathrm{pr}_*(\Delta^\natural_\varphi) = \mathrm{cl}_{P_{A'}}(\Delta_\varphi) \cdot P_{A'}$, but that $\Delta^\natural_\varphi$ is not of degree 0. We will identify the cycle $\tilde{\Delta}_\varphi$ defined in Eq. (32) with the corresponding degree zero divisor on $\mathscr{H}$ with values in $\tilde{\mathbb{L}}_{r,r}$ given by

$$\widetilde{\Delta}_\varphi := \sum_{j=1}^{t} \left(\mathrm{cl}_{\gamma_j \tau_{A'}}(Z_j) \cdot (\gamma_j \tau_{A'}) - \mathrm{cl}_{\tau_{A'}}(Z_j) \cdot \tau_{A'} \right), \tag{39}$$

where, by a slight abuse of notation, $\mathrm{cl}_\tau(Z_j)$ denotes the Poincaré dual of the topological cycle Z_j in the cohomology of the fiber above τ. Let $\omega_f \in S_{r+2}(\Gamma)$ be a cusp form, viewed as an element of $H^0(C, \mathscr{L}_r \otimes \Omega^1_C)$. We remark that given a class $\alpha \in \mathrm{Sym}^r H^1_{\mathrm{dR}}(A/\mathbf{C})$, a primitive of $\omega_f \wedge \alpha \in H^0(C, \mathscr{L}_{r,r} \otimes \Omega^1_C)$ is given by $F_f \wedge \alpha$, where F_f is a primitive of ω_f. This is because α is a horizontal section of the trivial bundle $\mathrm{Sym}^r H^1_{\mathrm{dR}}(A) = \mathrm{Sym}^r \mathscr{H}^1_{\mathrm{dR}}(A \times C/C)$ over C that arises in the identification $\mathscr{L}_{r,r} = \mathscr{L}_r \otimes \mathrm{Sym}^r H^1_{\mathrm{dR}}(A/\mathbf{C})$.

The following proposition gives an explicit formula for $\mathrm{AJ}_\mathbf{C}(\Delta_\varphi)$ in terms of this divisor and a primitive of ω_f.

Proposition 2 *For all $f \in S_{r+2}(\Gamma)$ and all $\alpha \in \mathrm{Sym}^r H^1_{\mathrm{dR}}(A/\mathbf{C})$,*

$$\mathrm{AJ}_\mathbf{C}(\Delta_\varphi)(\omega_f \wedge \alpha) = [F_f \wedge \alpha, \widetilde{\Delta}_\varphi] \pmod{\Pi_{r,r}}, \tag{40}$$

where F_f is any primitive of ω_f.

Remark 3 Both sides in (40) are to be viewed as belonging to the complex vector space $(S_{r+2}(\Gamma) \otimes \mathrm{Sym}^r H^1_{\mathrm{dR}}(A/\mathbf{C}))^\vee$, the equality being up to an element of the lattice $\Pi_{r,r}$ in this vector space. Note also that the right hand side of (40) depends on the choice of a *degree 0* divisor $\tilde{\Delta}_\varphi$ satisfying $\mathrm{pr}_*(\widetilde{\Delta}_\varphi) = \Delta_\varphi$, but only up to an element of $\Pi_{r,r}$.

Proof Recall the $(2r+1)$-cycle $\widetilde{\Delta}^\sharp_\varphi$ arising in Eq. (33). The definition of $\mathrm{AJ}_\mathbf{C}$ and Proposition 1, combined with Fubini's theorem, imply the equalities

$$\begin{aligned}
\mathrm{AJ}_\mathbf{C}(\Delta_\varphi)(\omega_f \wedge \alpha) &= \int_{\mathrm{pr}_*(\widetilde{\Delta}^\sharp_\varphi)} \omega_f \wedge \alpha = \int_{\widetilde{\Delta}^\sharp_\varphi} \mathrm{pr}^*\omega_f \wedge \alpha \pmod{\Pi_{r,r}} \\
&= \sum_{j=1}^{t} \int_{\tau_{A'}}^{\gamma_j \tau_{A'}} \left\langle \mathrm{pr}^*\omega_f \wedge \alpha, \theta^\nabla_{Z_j} \right\rangle \pmod{\Pi_{r,r}},
\end{aligned}$$

where $\theta^\nabla_{Z_j}$ is the horizontal section of $\tilde{\mathscr{L}}_{r,r}$ whose value at $\tau_{A'}$ is equal to $\mathrm{cl}_{\tau_{A'}}(Z_j)$, and the integral is taken over any continuous path in $\mathscr{H}$ joining $\tau_{A'}$ to $\gamma_j \tau_{A'}$. (Note the independence on the choice of paths, which follows from the fact that the expressions $\langle \mathrm{pr}^*\omega_f \wedge \alpha, \theta_{Z_j}^{\nabla}\rangle$ are holomorphic one-forms on $\mathscr{H}$.) Since $\theta_{Z_j}^{\nabla}$ is horizontal, it follows from the definition of the Gauss–Manin connection that

$$\left\langle \mathrm{pr}^*\omega_f \wedge \alpha, \theta_{Z_j}^{\nabla}\right\rangle = \left\langle \nabla F_f \wedge \alpha, \theta_{Z_j}^{\nabla}\right\rangle = d\left\langle F_f \wedge \alpha, \theta_{Z_j}^{\nabla}\right\rangle.$$

Hence Stokes' theorem yields the equalities modulo $\Pi_{r,r}$

$$\begin{aligned}
\mathrm{AJ}_{\mathbf{C}}(\Delta_\varphi)(\omega_f \wedge \alpha) &= \sum_{j=1}^{t}\left(\left\langle F_f(\gamma_j\tau_{A'}) \wedge \alpha, \theta_{Z_j}^{\nabla}\right\rangle - \left\langle F_f(\tau_{A'}) \wedge \alpha, \theta_{Z_j}^{\nabla}\right\rangle\right) \\
&= \sum_{j=1}^{t}\left(\left[F_f \wedge \alpha, \mathrm{cl}_{\gamma_j\tau_{A'}}(Z_j)\cdot(\gamma_j\tau_{A'})\right] - \left[F_f \wedge \alpha, \mathrm{cl}_{\tau_{A'}}(Z_j)\cdot\tau_{A'}\right]\right) \\
&= \left[F_f \wedge \alpha, \widetilde{\Delta}_\varphi\right],
\end{aligned}$$

as was to be shown. □

Remark 4 The expression on the right of Proposition 2 is independent of the choice of primitive F_f for ω_f. This is because the primitive $F_f \wedge \alpha$ is well-defined up to addition of global horizontal sections of the sheaf $\widetilde{\mathscr{L}}_{r,r}$ over $\mathscr{H}$. If θ is such a horizontal section, we have

$$[\theta, \widetilde{\Delta}_\varphi] = \langle \theta, \deg \widetilde{\Delta}_\varphi\rangle = 0.$$

Note that this independence ceases to hold if $\widetilde{\Delta}_\varphi$ is replaced by $\Delta_\varphi^\natural$, because the latter divisor is not of degree 0.

7 Calculation of the primitive

We now turn to the explicit calculation of the primitive F_f that appears in Proposition 2. Let p_1 and p_τ denote the elements of $H_1(\mathscr{E}_\tau, \mathbf{Q})$ corresponding to a closed path from 0 to 1 and from 0 to τ respectively along the fiber $\mathscr{E}_\tau = \mathbf{C}/\langle 1, \tau\rangle$. Write η_1 and η_τ for the associated basis of $H^1_{\mathrm{dR}}(\mathscr{E}_\tau)$, satisfying

$$\langle \omega, \eta_1\rangle = \int_{p_1}\omega, \qquad \langle \omega, \eta_\tau\rangle = \int_{p_\tau}\omega, \qquad \text{for all } \omega \in H^1_{\mathrm{dR}}(\mathscr{E}_\tau). \tag{41}$$

After writing w for the natural holomorphic coordinate on $\mathscr{E}_\tau$, the values of $\langle dw, \xi\rangle$ and $\langle d\bar{w}, \xi\rangle$ for various classes ξ are summarised in the following table:

$$\begin{array}{c|cccc}
 & dw & d\bar{w} & \eta_1 & \eta_\tau \\
\hline
dw & 0 & \frac{-1}{2\pi i}(\tau - \bar{\tau}) & 1 & \tau \\
d\bar{w} & \frac{1}{2\pi i}(\tau - \bar{\tau}) & 0 & 1 & \bar{\tau}
\end{array} \tag{42}$$

It follows directly from this table that

$$2\pi i dw = \tau\eta_1 - \eta_\tau, \quad 2\pi i d\bar{w} = \bar{\tau}\eta_1 - \eta_\tau, \tag{43}$$

and that

$$\left\langle dw^r, \eta_\tau^j\eta_1^{r-j}\right\rangle = \tau^j. \tag{44}$$

It will be convenient to work with the basis for $H^1_{\mathrm{dR}}(\mathscr{E}_\tau)$ given by setting

$$\omega = 2\pi i dw, \qquad \eta = \frac{d\bar{w}}{\bar{\tau} - \tau}. \tag{45}$$

The class η is completely determined (relative to ω) by the conditions

$$\eta \in H^{0,1}_{\mathrm{dR}}(\mathscr{E}_\tau), \qquad \langle \omega, \eta \rangle = 1.$$

A basis for $H^0(\mathscr{H}, \tilde{\mathscr{L}}_r)$ is given by the expressions $\omega^j \eta^{r-j}$, as $0 \le j \le r$.

Proposition 3 *Choose a base point $\tau_0 \in \mathscr{H}$, and let ω, η be given by (45). The section F_f of $\tilde{\mathscr{L}}_r$ over $\mathscr{H}$ satisfying*

$$\langle F_f(\tau), \omega^j \eta^{r-j} \rangle = \frac{(-1)^j (2\pi i)^{j+1}}{(\tau - \bar{\tau})^{r-j}} \int_{\tau_0}^{\tau} (z-\tau)^j (z-\bar{\tau})^{r-j} f(z) dz, \quad (0 \le j \le r)$$

is a primitive of ω_f.

Proof By definition of the Gauss–Manin connection, since the sections $\eta_\tau^j \eta_1^{r-j}$ are horizontal,

$$d\left\langle F_f, \eta_\tau^j \eta_1^{r-j} \right\rangle = \left\langle \nabla F_f, \eta_\tau^j \eta_1^{r-j} \right\rangle = \left\langle \mathrm{pr}^* \omega_f, \eta_\tau^j \eta_1^{r-j} \right\rangle. \tag{46}$$

By formula (24) for $\mathrm{pr}^*\omega_f$, this last expression is equal to

$$\left\langle \mathrm{pr}^* \omega_f, \eta_\tau^j \eta_1^{r-j} \right\rangle = (2\pi i)^{r+1} \left\langle f(\tau) dw^r, \eta_\tau^j \eta_1^{r-j} \right\rangle d\tau = (2\pi i)^{r+1} f(\tau) \tau^j d\tau. \tag{47}$$

Combining (46) and (47) and integrating the resulting identity with respect to τ, we find (after fixing some $\tau_0 \in \mathscr{H}$) that the global section of $\tilde{\mathscr{L}}_r$ over $\mathscr{H}$ defined by the rule

$$\left\langle F_f, \eta_\tau^j \eta_1^{r-j} \right\rangle = (2\pi i)^{r+1} \int_{\tau_0}^{\tau} f(z) z^j dz, \qquad (0 \le j \le r) \tag{48}$$

is a global primitive of ω_f. The defining relation (48) implies that, for all homogenous polynomials $P(x, y)$ of degree r,

$$\langle F_f, P(\eta_\tau, \eta_1) \rangle = (2\pi i)^{r+1} \int_{\tau_0}^{\tau} f(z) P(z, 1) dz.$$

After noting from (42) that

$$\omega^j \eta^{r-j} = Q(\eta_\tau, \eta_1), \quad \text{with } Q(x, y) = \frac{(-1)^j}{(2\pi i(\tau - \bar{\tau}))^{r-j}} (x - \tau y)^j (x - \bar{\tau} y)^{r-j},$$

we obtain

$$\langle F_f, \omega^j \eta^{r-j} \rangle = \frac{(-1)^j (2\pi i)^{r+1}}{(2\pi i(\tau - \bar{\tau}))^{r-j}} \int_{\tau_0}^{\tau} (z-\tau)^j (z-\bar{\tau})^{r-j} f(z) dz,$$

as was to be shown. □

Remark 5 (Relation with the Shimura–Maass operator) Recall the Shimura–Maass differential operator δ_r defined by

$$\delta_r f(\tau) := \frac{1}{2\pi i} \left(\frac{d}{d\tau} + \frac{r}{\tau - \bar{\tau}} \right) f(\tau), \tag{49}$$

which maps real analytic modular forms of weight r to real analytic modular forms of weight $r+2$. The real analytic functions G_j on $\mathscr{H}$ defined by the rule

$$G_j(\tau) := \langle F_f(\tau), \omega^j \eta^{r-j} \rangle = \frac{(-1)^j (2\pi i)^{j+1}}{(\tau - \bar{\tau})^{r-j}} \int_{\tau_0}^{\tau} (z-\tau)^j (z-\bar{\tau})^{r-j} f(z) dz$$

satisfy

$$\delta_r G_0(\tau) = f(\tau), \qquad \delta_{r-2j} G_j(\tau) = j G_{j-1}(\tau), \quad \text{for all } 1 \le j \le r. \tag{50}$$

For example, the integrand in the expression defining G_0 is antiholomorphic in τ, and therefore

$$\begin{aligned}
\delta_r G_0(\tau) &= \frac{1}{2\pi i} \left(\frac{d}{d\tau} + \frac{r}{\tau - \bar{\tau}} \right) \frac{2\pi i}{(\tau - \bar{\tau})^r} \int_{\tau_0}^{\tau} (z - \bar{\tau})^r f(z) dz \\
&= \frac{-r}{(\tau - \bar{\tau})^{r+1}} \int_{\tau_0}^{\tau} (z - \bar{\tau})^r f(z) dz + \frac{1}{(\tau - \bar{\tau})^r} (\tau - \bar{\tau})^r f(\tau) \\
&\quad + \frac{r}{(\tau - \bar{\tau})^{r+1}} \int_{\tau_0}^{\tau} (z - \bar{\tau})^r f(z) dz \\
&= f(\tau).
\end{aligned}$$

A similar direct calculation proves (50) for all $1 \le j \le r$.

An analogous formula in the p-adic context, with δ_r replaced by the operator $\theta = q\frac{d}{dq}$ on p-adic modular forms, is proved in [1, Prop. 3.24]. The reader may find it instructive to compare (50) with its p-adic analogue given in equation (3.8.6) of [1].

8 Integral primitives

Propositions 2 and 3 yield a formula for $\mathrm{AJ}_{\mathbf{C}}(\Delta_\varphi)$, but this formula is not as explicit as one could desire, because it requires evaluating the primitives $F_f \wedge \alpha$ on the divisor $\tilde{\Delta}_\varphi$ instead of the simpler divisors $\Delta_\varphi^\natural$ which are supported on a single point $\tau_{A'}$ (but are not of degree 0). We will now study the relation between $[F_f \wedge \alpha, \widetilde{\Delta}_\varphi]$ and $[F_f \wedge \alpha, \Delta_\varphi^\natural]$. Given $Z \in \tilde{\mathbb{L}}_r(\tau) = H^0(\mathscr{H}, \tilde{\mathscr{L}}_r)^{\nabla=0}$, let $P_Z \in \mathbf{C}[x, y]$ be the homogenous polynomial of degree r satisfying

$$Z = P_Z(\eta_\tau, \eta_1).$$

Lemma 2 *Let F_f be the primitive of f given in Proposition 3. Then for all $\gamma \in \Gamma$,*

$$\langle F_f(\gamma\tau), Z \rangle - \langle \gamma F_f(\tau), Z \rangle = (2\pi i)^{r+1} \int_{\tau_0}^{\gamma\tau_0} P_Z(z, 1) f(z) dz. \tag{51}$$

Proof By (48),

$$\langle F_f(\gamma\tau), Z \rangle = (2\pi i)^{r+1} \int_{\tau_0}^{\gamma\tau} P_Z(z, 1) f(z) dz. \tag{52}$$

The fact that f is a modular form of weight $r+2$ on Γ, coupled with the fact that P_Z is homogenous of degree r, shows that

$$P_Z(\gamma w, 1) f(\gamma w) d(\gamma w) = P_{\gamma^{-1} Z}(w, 1) f(w) dw.$$

Therefore

$$\langle \gamma F_f(\tau), Z\rangle = \langle F_f(\tau), \gamma^{-1} Z\rangle = (2\pi i)^{r+1}\int_{\tau_0}^{\tau} P_{\gamma^{-1}Z}(z,1) f(z)dz$$
$$= (2\pi i)^{r+1}\int_{\gamma\tau_0}^{\gamma\tau} P_Z(z,1) f(z)dz. \tag{53}$$

The lemma follows from (52) and (53). □

Note in particular that the global section $\tau \mapsto F_f(\gamma\tau) - \gamma F_f(\tau)$ does not depend on τ, and can be viewed as a horizontal section of $\tilde{\mathscr{L}_r}$ over $\mathscr{H}$. The function κ_{F_f} defined on Γ by

$$\kappa_{F_f}(\gamma) := F_f(\gamma\tau) - \gamma F_f(\tau)$$

is a one-cocycle on Γ with values in

$$H^0(\mathscr{H}, \tilde{\mathscr{L}_r})^{\nabla=0} = \tilde{\mathbb{L}}_r(\tau) \simeq L_r(\mathbf{C}),$$

where $L_r(\mathbf{C})$ is the space of homogenous polynomials of degree r in two variables with complex coefficients, equipped with its natural action of Γ. The class of κ_{F_f} in $H^1(\Gamma, L_r(\mathbf{C}))$ depends only on the differential ω_f and not on the choice of primitive F_f. This class will therefore be denoted by κ_f.

We briefly recall the definition of the *period lattice* in the space $S_{r+2}(\Gamma)^\vee$. Let $L_r(\mathbf{Q})$ and $L_r(\mathbf{Z})$ be the rational structure and lattice in $L_r(\mathbf{C})$ obtained by considering the polynomials with rational and integer coefficients respectively, and let $L_r(\mathbf{Z})^\vee$ inside $L_r(\mathbf{Q})$ be the dual lattice relative to the inner product on $L_r(\mathbf{C}) = \mathbb{L}_r(\tau)$ arising from Eq. (17). After choosing a basis $f_1, \ldots, f_g$ for $S_{r+2}(\Gamma)$, and a $\mathbf{Z}$-module basis $\kappa_1, \ldots, \kappa_{2g}$ for $H^1_{\mathrm{par}}(\Gamma, L_r(\mathbf{Z})^\vee)$, let (λ_{ij}) be the $g \times 2g$ matrix with complex entries satisfying

$$\begin{aligned}
\kappa_{f_1} &= \lambda_{1,1}\kappa_1 + \cdots + \lambda_{1,2g}\kappa_{2g},\\
\kappa_{f_2} &= \lambda_{2,1}\kappa_1 + \cdots + \lambda_{2,2g}\kappa_{2g},\\
&\vdots \;\vdots\; \vdots\\
\kappa_{f_g} &= \lambda_{g,1}\kappa_1 + \cdots + \lambda_{g,2g}\kappa_{2g}.
\end{aligned} \tag{54}$$

For each $1 \le j \le 2g$, let $\phi_j \in S_{r+2}(\Gamma)^\vee$ be the element defined by the rule

$$\phi_j(f_i) = \lambda_{ij}.$$

Definition 3 The *period lattice* attached to $S_{r+2}(\Gamma)$, denoted Λ_r, is the $\mathbf{Z}$-submodule of $S_{r+2}(\Gamma)^\vee$ generated by the vectors $\phi_1, \ldots, \phi_{2g}$.

Hodge theory asserts that Λ_r is indeed a lattice (of rank $2g$) in the complex vector space $S_{r+2}(\Gamma)^\vee$, justifying this terminology. Note that the module Λ_r does not depend on the choices of complex basis for $S_{r+2}(\Gamma)$ and of integral basis for $H^1_{\mathrm{par}}(\Gamma, L_r(\mathbf{Z})^\vee)$ that were made to define it.

Let $F_1, \ldots, F_g$ be arbitrarily chosen primitives of $\omega_{f_1}, \ldots, \omega_{f_g}$, and let $\tilde{\kappa}_1, \ldots, \tilde{\kappa}_{2g}$ be a choice of one-cocycles on Γ representing $\kappa_1, \ldots, \kappa_{2g}$. The linear equations (54) defining the period lattice imply that there exist vectors $\xi_1, \ldots, \xi_g \in L_r(\mathbf{C})$ such that, for all $\gamma \in \Gamma$

and all $\tau \in \mathscr{H}$:

$$\begin{aligned}
\kappa_{F_1}(\gamma) &= \lambda_{1,1}\tilde{\kappa}_1(\gamma) + \cdots + \lambda_{1,2g}\tilde{\kappa}_{2g}(\gamma) + (\gamma\xi_1 - \xi_1),\\
\kappa_{F_2}(\gamma) &= \lambda_{2,1}\tilde{\kappa}_1(\gamma) + \cdots + \lambda_{2,2g}\tilde{\kappa}_{2g}(\gamma) + (\gamma\xi_2 - \xi_2),\\
&\vdots \quad \vdots \quad \vdots\\
\kappa_{F_g}(\gamma) &= \lambda_{g,1}\tilde{\kappa}_1(\gamma) + \cdots + \lambda_{g,2g}\tilde{\kappa}_{2g}(\gamma) + (\gamma\xi_g - \xi_g).
\end{aligned} \tag{55}$$

After replacing F_j by $F_j + \xi_j$ (viewing the ξ_j as elements of $H^0(\mathscr{H}, \tilde{\mathscr{L}}_r)^{\nabla=0}$), we obtain a new collection of primitives satisfying the following relation, for all $\gamma \in \Gamma$ and $\tau \in \mathscr{H}$:

$$\begin{aligned}
F_1(\gamma\tau) - \gamma F_1(\tau) &= \lambda_{1,1}\tilde{\kappa}_1(\gamma) + \cdots + \lambda_{1,2g}\tilde{\kappa}_{2g}(\gamma),\\
F_2(\gamma\tau) - \gamma F_2(\tau) &= \lambda_{2,1}\tilde{\kappa}_1(\gamma) + \cdots + \lambda_{2,2g}\tilde{\kappa}_{2g}(\gamma),\\
&\vdots \quad \vdots \quad \vdots\\
F_g(\gamma\tau) - \gamma F_g(\tau) &= \lambda_{g,1}\tilde{\kappa}_1(\gamma) + \cdots + \lambda_{g,2g}\tilde{\kappa}_{2g}(\gamma).
\end{aligned} \tag{56}$$

Definition 4 A *collection of integral primitives* is a choice of a primitive F_j of f_j for each $j = 1, \ldots, g$ satisfying (56).

A collection of integral primitives determines, by linearity, a primitive F_f of f for each $f \in S_{r+2}(\Gamma)$. The primitive F_f arising from such a choice will be called an *integral primitive* of ω_f.

Lemma 3 *Let $f \mapsto F_f$ be a choice of integral primitives of f. For each $\gamma \in \Gamma$ and $v \in L_r(\mathbf{Z})$, the assignment*

$$f \mapsto \langle F_f(\gamma\tau) - \gamma F_f(\tau), v\rangle$$

belongs to $\Lambda_r \subset S_{r+2}(\Gamma)^\vee$.

Proof This follows directly from (56) in light of the fact that the scalars

$$\langle\tilde{\kappa}_1(\gamma), v\rangle, \ldots, \langle\tilde{\kappa}_{2g}(\gamma), v\rangle$$

are integers. □

By definition, the $\mathbf{Z}$-module

$$\Lambda_{r,r} := \Lambda_r \otimes \operatorname{Sym}^r H_1(A(\mathbf{C}), \mathbf{Z})$$

is a lattice in $S_{r+2}(\Gamma)^\vee \otimes \operatorname{Sym}^r H^1_{\mathrm{dR}}(A/\mathbf{C})^\vee = \operatorname{Fil}^{r+1}\epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r)^\vee$. It is commensurable with the lattice $\Pi_{r,r}$ appearing in (37). After eventually replacing $\Lambda_{r,r}$ by a larger lattice, we may therefore assume that $\Lambda_{r,r}$ contains $\Pi_{r,r}$. This assumption allows us to replace $\Pi_{r,r}$ by $\Lambda_{r,r}$ in the arguments to follow.

Lemma 3 implies:

$$\langle F_f(\gamma\tau) \wedge \alpha, Z\rangle = \langle F_f(\tau) \wedge \alpha, \gamma^{-1}Z\rangle \pmod{\Lambda_{r,r}}, \tag{57}$$

for all $Z \in L_r(\mathbf{Z}) \otimes \operatorname{Sym}^r H^1(A, \mathbf{Z})$. (Where now both f and α are treated as variables, and the equality is viewed as taking place in $\operatorname{Fil}^{r+1}\epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/\mathbf{C})^\vee$.)

The Abel–Jacobi image of generalised Heegner cycles can be expressed more simply in terms of integral primitives, as follows:

Proposition 4 *Let $f \mapsto F_f$ be a choice of integral primitives, and let Δ_φ be a generalised Heegner cycle attached to $\varphi : A \longrightarrow A'$. Then*

$$\mathrm{AJ}_{\mathbf{C}}(\Delta_\varphi)(\omega_f \wedge \alpha) = \left\langle F_f(\tau_{A'}) \wedge \alpha, \mathrm{cl}_{\tau_{A'}}(\Delta_\varphi^\natural)\right\rangle \pmod{\Lambda_{r,r}},$$

where the pairing is the natural one on $\tilde{\mathbb{L}}_{r,r}(\tau_{A'})$.

Proof By Proposition 2 combined with the formula (32) for $\widetilde{\Delta}_\varphi$,

$$\begin{aligned}
\mathrm{AJ}_{\mathbf{C}}(\Delta_\varphi)(\omega_f \wedge \alpha) &= [F_f \wedge \alpha, \widetilde{\Delta}_\varphi] \pmod{\Lambda_{r,r}}\\
&= \sum_{j=1}^{t} \langle F_f(\gamma_j \tau_{A'}) \wedge \alpha, Z_j\rangle - \langle F_f(\tau_{A'}) \wedge \alpha, Z_j\rangle \pmod{\Lambda_{r,r}}\\
&= \sum_{j=1}^{t} \langle F_f(\tau_{A'}) \wedge \alpha, \gamma_j^{-1} Z_j\rangle - \langle F_f(\tau_{A'}) \wedge \alpha, Z_j\rangle \pmod{\Lambda_{r,r}}\\
&= \left\langle F_f(\tau_{A'}) \wedge \alpha, \sum_{j=1}^{t} (\gamma_j^{-1} - 1) Z_j \right\rangle \pmod{\Lambda_{r,r}},
\end{aligned}$$

where we have used (57) in deriving the penultimate equality. Proposition 4 now follows from Eq. (31) for the class of $\Delta_\varphi^\natural$. □

Proposition 5 *With the same notations as in Proposition 4,*

$$\mathrm{AJ}_{\mathbf{C}}(\Delta_\varphi)(\omega_f \wedge \alpha) = \langle \varphi^* F_f(\tau_{A'}), \alpha\rangle_A \pmod{\Lambda_{r,r}},$$

where the pairing $\langle\ ,\ \rangle_A$ on the right is the Poincaré duality on $\mathrm{Sym}^r H^1_{\mathrm{dR}}(A/\mathbf{C})$.

Proof Let

$$\varrho := (\varphi^r, \mathrm{id}^r) : A^r \longrightarrow \Upsilon_\varphi \subset (A')^r \times A^r.$$

Note that

$$\varrho^*(F_f(\tau_{A'}) \wedge \alpha) = \varphi^*(F_f(\tau_{A'})) \wedge \alpha, \qquad \varrho([A^r]) = \mathrm{cl}_{\tau_{A'}}(\Upsilon_\varphi^\natural),$$

where $[A^r] \in H^0_{\mathrm{dR}}(A^r/\mathbf{C})$ is the fundamental class associated to the variety A^r. Let

$$\langle\ ,\ \rangle_{A,j} : H^{2r-j}_{\mathrm{dR}}(A^r/\mathbf{C}) \times H^j_{\mathrm{dR}}(A^r/\mathbf{C}) \longrightarrow H^{2r}(A^r/\mathbf{C}) = \mathbf{C}$$

denote the Poincaré pairing, so that the restriction of $\langle\ ,\ \rangle_{A,r}$ to the subspace

$$\mathrm{Sym}^r H^1_{\mathrm{dR}}(A/\mathbf{C}) \subset H^r_{\mathrm{dR}}(A/\mathbf{C})$$

agrees with $\langle\ ,\ \rangle_A$. Observe that

$$\left\langle F_f(\tau_{A'}) \wedge \alpha, \mathrm{cl}_{\tau_{A'}}(\Delta_\varphi^\natural)\right\rangle = \left\langle F_f(\tau_{A'}) \wedge \alpha, \mathrm{cl}_{\tau_{A'}}(\Upsilon_\varphi^\natural)\right\rangle = \left\langle F_f(\tau_{A'}) \wedge \alpha, \varrho([A^r])\right\rangle. \tag{58}$$

The functoriality properties of the Poincaré pairing imply that

$$\begin{aligned}
\langle F_f(\tau_{A'}) \wedge \alpha, \varrho([A^r])\rangle &= \langle \varrho^*(F_f(\tau_{A'}) \wedge \alpha), [A^r]\rangle_{A,0}\\
&= \langle \varphi^*(F_f(\tau_{A'})) \wedge \alpha, [A^r]\rangle_{A,0} = \langle \varphi^*(F_f(\tau_{A'})), \alpha\rangle_A.
\end{aligned} \tag{59}$$

Proposition 5 follows by combining Proposition 4 with (58) and (59). □

9 Modular symbols

Propositions 4 and 5 gain in explicitness because they involve the divisor $\Delta_\varphi^\natural$ supported on a single point, rather that the more complicated divisor (31) which is given in terms of a (non-canonical) expression for the class of $\Delta_\varphi^\natural$ as an element of $I_\Gamma H_{2r}(\tilde{X}_r, \mathbf{Q})$. The price one pays is that it becomes necessary to work with *integral primitives* rather than arbitrary primitives.

In the case of a group like $\Gamma_1(N)$ containing parabolic elements, an integral primitive can be defined explicitly by invoking the theory of modular symbols. More precisely, let us define primitives F_f of ω_f by allowing the base point τ_0 appearing in Proposition 3 to tend to a cusp. The integrals appearing in Proposition 3 still converge, by the cuspidality of f. Furthermore, the right-hand term appearing in (51) is of the form

$$J_{s,t,P}(f) := (2\pi i)^{r+1} \int_s^t P(z) f(z) dz, \qquad \text{with } s, t \in \mathbf{P}_1(\mathbf{Q}), \quad P(x) \in \mathbf{Z}[x]^{\deg=r}.$$

Let Λ_r' denote the $\mathbf{Z}$-module generated by Λ_r and the functionals $J_{s,t,P}$ in the complex vector space $S_{r+2}(\Gamma)^\vee$. The following theorem is the basis for the theory of "modular symbols" attached to modular forms of higher weight.

Proposition 6 *The group Λ_r' is a sublattice of $S_{r+2}(\Gamma)^\vee$ which contains Λ_r with finite index.*

Proof The proof of this theorem can be found, for instance, in Proposition 3.5 of [14]. (The statement and proof are given there for $r = 2$, i.e., forms of weight 4, but no serious modification is required to handle the case of general r.) □

After replacing the period lattice Λ_r by the possibly slightly larger lattice Λ_r', and redefining $\Lambda_{r,r}$ accordingly, we obtain Theorem 1 below on the complex Abel–Jacobi images of generalised Heegner cycles, which is one of the two main results of this paper. Because the formula is given modulo a larger lattice, it is slightly less precise, but has the virtue of being more explicit and amenable to numerical calculation.

Theorem 1 *Let*

$$\varphi : A \longrightarrow \mathbf{C}/\langle 1, \tau\rangle$$

be an isogeny of degree $d_\varphi = \deg(\varphi)$, satisfying

$$\varphi(t_A) = \frac{1}{N}, \qquad \varphi^*(2\pi i dw) = \omega_A,$$

and let Δ_φ be the associated generalised Heegner cycle on X_r. Then

$$\mathrm{AJ}_{\mathbf{C}}(\Delta_\varphi)(\omega_f \wedge \omega_A^j \eta_A^{r-j}) = \frac{(-d_\varphi)^j (2\pi i)^{j+1}}{(\tau - \bar{\tau})^{r-j}} \int_{i\infty}^{\tau} (z-\tau)^j (z-\bar{\tau})^{r-j} f(z) dz \pmod{\Lambda_{r,r}}.$$

Proof Let F_f be the integral primitive of ω_f obtained by setting $\tau_0 = i\infty$. By Proposition 5,

$$\mathrm{AJ}_{\mathbf{C}}(\Delta_\varphi)(\omega_f \wedge \omega_A^j \eta_A^{r-j}) = \langle \varphi^* F_f(\tau), \omega_A^j \eta_A^{r-j}\rangle_A \pmod{\Lambda_{r,r}}. \tag{60}$$

But letting $\omega', \eta' \in H^1_{\mathrm{dR}}(\mathbf{C}/\langle 1, \tau\rangle)$ be defined by

$$\omega' = 2\pi i dw, \qquad \eta' \in H_{dR}^{0,1}(\mathbf{C}/\langle 1, \tau\rangle), \qquad \langle \omega', \eta'\rangle = 1,$$

we have

$$\varphi^*(\omega') = \omega_A, \qquad \varphi^*(\eta') = d_\varphi \eta_A. \tag{61}$$

Hence

$$\begin{aligned}\left\langle \varphi^* F_f(\tau), \omega_A^j \eta_A^{r-j} \right\rangle_A &= d_\varphi^{j-r} \langle \varphi^* F_f(\tau), \varphi^*((\omega')^j (\eta')^{r-j})\rangle_A \\ &= d_\varphi^j \langle F_f(\tau), (\omega')^j (\eta')^{r-j}\rangle_{A'}.\end{aligned}$$

The result now follows from Proposition 3 with $\tau_0 = i\infty$. □

10 The Chow group of X_r

Assume in this section that A is isomorphic over $\mathbf{C}$ to the complex torus $\mathbf{C}/\mathscr{O}_K$ and let X_r be the $(2r+1)$-dimensional variety over H defined previously. For simplicity, we assume that $d_K \neq 3, 4$, so that $\mathscr{O}_K^\times = \{\pm 1\}$. For any field F, let

$$\mathrm{Gr}^{r+1}(X_r)(F) := \mathrm{CH}^{r+1}(X_r)_0(F)/\mathrm{CH}^{r+1}(X_r)_{\mathrm{alg}}(F),$$

where $\mathrm{CH}^{r+1}(X_r)_{\mathrm{alg}}(F)$ is the subgroup of null-homologous codimension $r+1$ cycles on X_r that are defined over F and are algebraically equivalent to zero.

The goal of this section is to prove the following:

Theorem 2 *For all $r \geq 0$ the Chow group $\mathrm{CH}^{r+1}(X_r)_0(\bar{H})$ of null-homologous cycles modulo rational equivalence has infinite rank. Furthermore, for all $r \geq 2$, the Griffiths group $\mathrm{Gr}^{r+1}(X_r)(\bar{H})$ also has infinite rank.*

The proof follows closely that of Theorem 4.7 of Chad Schoen's paper [14] which treats the case of "usual" Heegner cycles on a Kuga–Sato threefold, and rests on an ingenious method of Bloch. The most significant difference lies in the setting that is treated: whereas Schoen's cycles are indexed by arbitrary quadratic orders of varying discriminant, ours are forced by necessity to be indexed by (not necessarily maximal) orders of the fixed imaginary quadratic field K. The reader is referred to loc.cit for background material on Heegner cycles and the tools from class field theory, complex multiplication theory, and étale cohomology that are used below, as well as to the introduction of [1] for further background on generalised Heegner cycles beyond the material covered in the earlier sections.

Remark 6 When $r = 0$ the variety X_0 is the modular curve $X_1(N)$ which is defined over $\mathbf{Q}$. Codimension 1 cycles are divisors and rational equivalence corresponds to linear equivalence on divisors, whence $\mathrm{CH}^1(X_1(N)) = \mathrm{Pic}(X_1(N))$. Moreover, a divisor is null-homologous if and only if it has degree zero and any degree zero divisor on a smooth connected curve is algebraically equivalent to zero. It follows that the Griffiths group $\mathrm{Gr}^1(X_1(N))$ is trivial. The content of Theorem 2 is that the Chow group $\mathrm{CH}^1(X_1(N))_0(\bar{\mathbf{Q}})$ has infinite rank, a well-known result. The generalised Heegner cycles in this case are images of Heegner points on the Jacobian variety of $X_1(N)$ and our method consists in showing that the subgroup generated by these Heegner points has infinite rank. In [10, Proposition 2.8], it is shown that $E(\bar{\mathbf{Q}})$ has infinite rank where E is an elliptic curve defined over $\mathbf{Q}$ by proving that the subgroup generated by Heegner points on $X_0(N)$ via a modular parametrisation $X_0(N) \longrightarrow E$ has infinite rank. In particular, this implies Theorem 2 for $r = 0$.

Throughout this section we will adopt the following notational conventions. If X is a variety defined over H and F is any field containing H, then we let X_F denote its base change to F, i.e., $X \times \operatorname{Spec} F$. We fix an algebraic closure $\bar{H}$ of H and we will use the shorthand notation $\bar{X} := X_{\bar{H}}$. Recall that K has discriminant $-d_K$ and $\mathscr{O}_K$ denotes its ring of integers. Let $\tau := (-d_K + \sqrt{-d_K})/2$ be the standard generator of $\mathscr{O}_K = \langle 1, \tau\rangle$. Fix an analytic isomorphism $\xi : \mathbf{C}/\mathscr{O}_K \cong A(\mathbf{C})$ and let $\omega_A \in \Omega^1_{A/H}$ be the regular differential satisfying $\xi^*(\omega_A) = 2\pi i dw$.

10.1 An infinite collection of cycles

We now introduce a distinguished collection of generalised Heegner cycles. The fields of definition of these cycles will play a crucial role in Sect. 10.3 and the understanding of the Galois action on these cycles is key in Sect. 10.4.

Let p and q be distinct odd primes which are congruent to 1 modulo N, and consider the following lattices associated to $\beta \in \mathbf{P}_1(\mathbf{F}_q)$,

$$\Lambda_{p,q,\infty} := \mathbf{Z}\frac{1}{pq} \oplus \mathbf{Z}\tau, \qquad \Lambda_{p,q,\beta} := \mathbf{Z}\frac{1}{p} \oplus \mathbf{Z}\frac{\tau+\beta}{q}, \quad \text{for } 0 \leq \beta \leq q-1,$$

which each contain $\mathscr{O}_K$ with index pq, and let $A_{p,q,\beta}$ be the elliptic curve whose complex points are isomorphic to $\mathbf{C}/\Lambda_{p,q,\beta}$. The natural isogeny

$$\varphi_{p,q,\beta} : A \longrightarrow A_{p,q,\beta}$$

of degree pq gives rise to the generalised Heegner cycle

$$\Delta_{p,q,\beta} := \Delta_{\varphi_{p,q,\beta}}. \tag{62}$$

Let F_{pq} denote the field compositum of $K_{\mathfrak{N}}$ and H_{pq}, where $K_{\mathfrak{N}}$ denotes the ray class field of K of modulus $\mathfrak{N}$ and H_{pq} is the ring class field of K conductor pq.

Proposition 7 *For all $\beta \in \mathbf{P}_1(\mathbf{F}_q)$, the cycle $\Delta_{p,q,\beta}$ is defined over F_{pq}.*

Proof The variety W_r is defined over $\mathbf{Q}$, and the elliptic curve A along with its complex multiplication can be defined over the Hilbert class field H of K. Following the moduli description of $X_1(N)$, the pair (A, t_A) corresponds to a complex point on $X_1(N)$ defined over the abelian extension of K corresponding to the subgroup $K^\times W \subset \mathbb{A}_K^\times$, where

$$W := \left\{x \in \mathbb{A}_K^\times : x\mathscr{O}_K = \mathscr{O}_K,\ x\xi^{-1}(t_A) = \xi^{-1}(t_A)\right\}.$$

This field is the ray class field $K_{\mathfrak{N}}$ of K of conductor $\mathfrak{N}$. The elliptic curves $A_{p,q,\beta}$ have complex multiplication by the order $\mathscr{O}_{pq}$ of conductor pq and can thus be defined over the ring class field H_{pq}. The isogenies $\varphi_{p,q,\beta}$ are also defined over H_{pq}. Note that since $(pq, N) = 1$, we have $(\varphi_{p,q,\beta}, A_{p,q,\beta}) \in \operatorname{Isog}^{\mathfrak{N}}(A)$. The point $(A_{p,q,\beta}, t_{A_{p,q,\beta}})$ on $X_1(N)$ can thus be defined over the field compositum F_{pq}. Since the correspondence ϵ_X that was used to define the generalised Heegner cycle is defined over $\mathbf{Q}$, we can conclude that the cycle $\Delta_{p,q,\beta}$ is defined over F_{pq} as well. □

Remark 7 More generally, let (φ, A') be an element of $\operatorname{Isog}(A)$. Since A has complex multiplication by $\mathscr{O}_K$, the endomorphism ring of A' is an order in $\mathscr{O}_K$. Such an order is completely determined by its conductor, and therefore there is a unique integer $c \geq 1$ such that $\operatorname{End}_{\bar{K}}(A') = \mathscr{O}_c := \mathbf{Z} + c\mathscr{O}_K$. The pair (φ, A') is then said to be *of conductor c* and we set

$$\operatorname{Isog}_c(A) := \{\text{Isomorphism classes of pairs } (\varphi, A') \text{ of conductor } c\}$$

and $\mathrm{Isog}_c^{\mathfrak{N}}(A) := \mathrm{Isog}_c(A) \cap \mathrm{Isog}^{\mathfrak{N}}(A)$. Note that if $(\varphi, A') \in \mathrm{Isog}_c^{\mathfrak{N}}(A)$, then by a similar reasoning as above the associated cycle Δ_φ is defined over the field compositum $F_\varphi := K_{\mathfrak{N}} \cdot H_c$, where $H_c := K(j(\mathscr{O}_c))$ denotes the ring class field of K of conductor c.

10.2 Cycles of large order in the Chow group

Using the explicit formula for the image of generalised Heegner cycles under the complex Abel–Jabobi map given in Theorem 1, we will now prove, following the approach of [14, §3], that many of the cycles $\Delta_{p,q,\beta}$ are of large (possibly infinite) order in the Chow group and even in the Griffiths group (if $r \geq 1$). This part of the argument uses only complex analytic and Hodge theoretic methods, and rests on the following theorem:

Theorem 3 *For all $r \geq 0$ (resp. for all $r \geq 1$) the order of $\Delta_{p,q,\beta}$ in $\mathrm{CH}^{r+1}(X_r)_0(\bar{H})$ (resp. in $\mathrm{Gr}^{r+1}(X_r)(\bar{H})$) tends to ∞ as p/q tends to ∞.*

If $f \in S_{r+2}(\Gamma)$ and $0 \leq j \leq r$, then we will identify, by a slight abuse of notation, $\mathrm{AJ}_{\mathbf{C}}(\Delta_{p,q,\beta})(\omega_f \wedge \omega_A^j \eta_A^{r-j})$ with the complex number appearing in the right hand side of the displayed equation in Theorem 1. This amounts to choosing a fixed representative of $\mathrm{AJ}_{\mathbf{C}}(\Delta_{p,q,\beta})$ in $(S_{r+2}(\Gamma) \otimes \mathrm{Sym}^r H^1_{\mathrm{dR}}(A))^\vee$, and then evaluating it at $\omega_f \wedge \omega_A^j \eta_A^{r-j}$. The proof of Theorem 3 relies on the following lemma:

Lemma 4 *With the above notations and conventions, for any non-zero cusp form f we have*

$$\lim_{p/q \to \infty} \mathrm{AJ}_{\mathbf{C}}(\Delta_{p,q,\beta}) \left(\omega_f \wedge \omega_A^j \eta_A^{r-j} \right) = 0$$

and $\mathrm{AJ}_{\mathbf{C}}(\Delta_{p,q,\beta})(\omega_f \wedge \omega_A^j \eta_A^{r-j}) \neq 0$ for all large enough p/q.

Proof Fix p, q, and $\beta \in \mathbf{P}_1(\mathbf{F}_q)$. The lattice $\Lambda_{p,q,\beta}$ is homothetic to $\langle 1, \tau_{p,q,\beta} \rangle$, where

$$\tau_{p,q,\infty} := pq\tau, \qquad \tau_{p,q,\beta} := \frac{p}{q}(\tau + \beta). \tag{63}$$

Set $\tau_{p,q,\beta} := X_\beta + iY_\beta$, and note that $Y_\beta = pq \cdot \sqrt{d_K}/2$ if $\beta = \infty$, and $Y_\beta = p/q \cdot \sqrt{d_K}/2$ otherwise. By Theorem 1, $\mathrm{AJ}_{\mathbf{C}}(\Delta_{p,q,\beta})(\omega_f \wedge \omega_A^j \eta_A^{r-j})$ is equal to

$$\begin{aligned} &\frac{(-1)^j (2\pi i)^{j+1} \cdot \kappa_\beta}{(\tau - \bar{\tau})^{r-j}} \int_{i\infty}^{\tau_{p,q,\beta}} (z - \tau_{p,q,\beta})^j (z - \bar{\tau}_{p,q,\beta})^{r-j} f(z) dz \\ &= \gamma_\beta \int_{Y_\beta}^{\infty} (y - Y_\beta)^j (y + Y_\beta)^{r-j} f(X_\beta + iy) dy, \end{aligned} \tag{64}$$

where

$$\kappa_\beta := \begin{cases} (pq)^{2j-2r} & \text{if } \beta = \infty, \\ p^{2j-2r} q^r & \text{otherwise,} \end{cases} \qquad \gamma_\beta := (-1)^{j+1} \cdot i^{r+1} \cdot (2\pi i)^{j+1} \cdot \frac{\kappa_\beta}{(\tau - \bar{\tau})^{r-j}},$$

and the equality in (64) is obtained by performing the change of variables $z = X_\beta + iy$.

Assume without loss of generality that f is a normalised cuspidal eigenform. By examining the Fourier expansion of f, one can see that there is an absolute real constant $C_f > 0$ (depending only on f) for which

$$|f(z) - e^{2\pi i z}| \leq C_f \cdot e^{-4\pi \mathrm{Im}(z)}$$

on the domain $\{\mathrm{Im}(z) > 1\}$. Combining this with (64) gives

$$\left|\mathrm{AJ}_{\mathbf{C}}(\Delta_{p,q,\beta})(\omega_f \wedge \omega_A^j \eta_A^{r-j}) - \gamma_\beta \cdot e^{2\pi i X_\beta} \cdot A_\beta\right| \\ \leq \gamma_\beta \cdot C_f \cdot \int_{Y_\beta}^{\infty} (y - Y_\beta)^j (y + Y_\beta)^{r-j} e^{-4\pi y} dy, \tag{65}$$

where

$$A_\beta := \int_{Y_\beta}^{\infty} (y - Y_\beta)^j (y + Y_\beta)^{r-j} e^{-2\pi y} dy \tag{66}$$

is clearly non-zero and positive since the function appearing in the integral is strictly positive on the domain of integration. The error term in (65) is majorised by

$$\left|\gamma_\beta \cdot C_f \cdot \int_{Y_\beta}^{\infty} (y - Y_\beta)^j (y + Y_\beta)^{r-j} e^{-4\pi y} dy\right| \leq C_f \cdot \gamma_\beta \cdot e^{-2\pi Y_\beta} A_\beta. \tag{67}$$

If we let

$$B_\beta := \gamma_\beta \cdot e^{2\pi i X_\beta} \cdot A_\beta, \tag{68}$$

then (67) implies that $\mathrm{AJ}_{\mathbf{C}}(\Delta_{p,q,\beta})(\omega_f \wedge \omega_A^j \eta_A^{r-j})$ is asymptotically equivalent, as a function of p and q, to B_β as p/q tends to infinity, in the sense that the ratio of these two functions tends to 1 as p/q tends to infinity. The result now follows after observing that the quantity B_β is non-zero but tends to 0 as p/q tends to infinity. □

Proof of Theorem 3 As p/q tends to ∞, Lemma 4 shows that $\mathrm{AJ}_{\mathbf{C}}(\Delta_{p,q,\beta})$ tends to the origin in $J^{r+1}(X_r/\mathbf{C})$ without being equal to it. Consequently, the order of $\mathrm{AJ}_{\mathbf{C}}(\Delta_{p,q,\beta})$ tends to ∞ in $J^{r+1}(X_r/\mathbf{C})$. It follows that the order of $\Delta_{p,q,\beta}$ in the Chow group $\mathrm{CH}^{r+1}(X_r)_0(\bar{H})$ tends to ∞ as p/q tends to ∞.

To treat the image of $\Delta_{p,q,\beta}$ in the Griffiths group, let $J^{r+1}(X_r/\mathbf{C})_{\mathrm{alg}}$ denote the complex subtorus of $J^{r+1}(X_r/\mathbf{C})$ which is the intermediate Jacobian of the largest sub-Hodge structure V of $H^{r+1,r}(X_r) \oplus H^{r,r+1}(X_r)$. More precisely,

$$J^{r+1}(X_r/\mathbf{C})_{\mathrm{alg}} = J^{r+1}(V) := V_{\mathbf{C}}/(\mathrm{Fil}^{r+1} V \oplus V_{\mathbf{Z}}). \tag{69}$$

The image of $\mathrm{CH}^{r+1}(X_r)_{\mathrm{alg}}(\mathbf{C})$ under $\mathrm{AJ}_{\mathbf{C}}$ is a complex subtorus of $J^{r+1}(X_r/\mathbf{C})$ which is contained in $J^{r+1}(X_r/\mathbf{C})_{\mathrm{alg}}$ (see [17, §12.2.2]) and has the structure of an abelian variety. One can thus define the transcendental part of the Abel–Jacobi map

$$\mathrm{AJ}_{\mathbf{C},\mathrm{tr}} : \mathrm{Gr}^{r+1}(X_r)(\mathbf{C}) \longrightarrow J^{r+1}(X_r/\mathbf{C})_{\mathrm{tr}} := J^{r+1}(X_r/\mathbf{C})/J^{r+1}(X_r/\mathbf{C})_{\mathrm{alg}} \tag{70}$$

as the factorisation of $\mathrm{AJ}_{\mathbf{C}}$. Note that for $r = 0$, $J^{r+1}(X_r/\mathbf{C}) = J^{r+1}(X_r/\mathbf{C})_{\mathrm{alg}}$ and $\mathrm{Gr}^{r+1}(X_r)(\mathbf{C}) = 0$ by Remark 6, so the transcendental part of the Abel–Jacobi map is trivial in this case. For $r \geq 1$, by (36), we observe that

$$(H^{r+1,r}(X_r) \oplus H^{r,r+1}(X_r)) \cap \epsilon_X H_{\mathrm{dR}}^{2r+1}(X_r/\mathbf{C}) = (S_{r+2}(\Gamma) \otimes \mathbf{C}\eta_A^r) \oplus (\overline{S_{r+2}(\Gamma)} \otimes \mathbf{C}\omega_A^r).$$

The same reasoning as before shows that the order of $\Delta_{p,q,\beta}$ in $\mathrm{Gr}^{r+1}(X_r)(\bar{H})$ tends to ∞ with p/q. □

10.3 Cycles of infinite order in the Chow group

Theorem 3 implies that for sufficiently large p/q, the cycles $\Delta_{p,q,\beta}$ have large (possibly infinite) order in the Chow group. Following [14, §4], we show that for large p/q, the cycles

$\Delta_{p,q,\beta}$ are non-torsion in the Chow group. This section constitutes the algebraic part of the argument, where the fields of definition of our cycles play a crucial role.

Proposition 8 *For all $r \geq 0$, there exists a non-negative integer M_r with the property that if $\Delta \in \langle\{\Delta_{p,q,\beta}\}\rangle \subset \mathrm{CH}^{r+1}(X_r)_0(\bar{H})$ is such that the order of $\mathrm{AJ}_{\mathbf{C}}(\Delta)$ in $J^{r+1}(X_r/\mathbf{C})$ does not divide M_r, then Δ has infinite order in $\mathrm{CH}^{r+1}(X_r)_0(\bar{H})$.*

Before proving this proposition, we deduce the following two corollaries.

Corollary 1 *For p/q sufficiently large, $\Delta_{p,q,\beta}$ has infinite order in the Chow group.*

Proof It suffices to combine Lemma 4 and Proposition 8. □

Corollary 2 *Fix a rational prime q congruent to 1 modulo N. There exist infinitely many rational primes p congruent to 1 modulo N such that the cycle $\Delta_{p,q,\beta} - \Delta_{p,q,\gamma}$ has infinite order in the Chow group when $\beta \neq \gamma$.*

Proof Let f be a normalised cuspidal eigenform and consider $B_\beta = \gamma_\beta \cdot e^{2\pi i X_\beta} \cdot A_\beta$ of (68) defined in the proof of Lemma 4 for all $\beta \in \mathbf{P}_1(\mathbf{F}_q)$. If $\beta = \infty$, then $\gamma \neq \infty$ and a comparison of integrals reveals that

$$\left|\frac{B_\infty}{B_\gamma}\right| \leq e^{-\pi \frac{p}{q}(q^2-1)\sqrt{d_K}} q^{2(j+1)-r}$$

from which we deduce that B_∞/B_γ tends to zero as p/q tends to ∞. In particular, B_∞ and B_γ are not asymptotically equivalent as $p/q \to \infty$ and it follows that for infinitely many p/q,

$$\mathrm{AJ}_{\mathbf{C}}(\Delta_{p,q,\infty})\left(\omega_f \wedge \omega_A^j \eta_A^{r-j}\right) \neq \mathrm{AJ}_{\mathbf{C}}(\Delta_{p,q,\gamma})\left(\omega_f \wedge \omega_A^j \eta_A^{r-j}\right) \tag{71}$$

since asymptotic equivalence is an equivalence relation. Moreover, we have

$$\lim_{p/q\to\infty} \mathrm{AJ}_{\mathbf{C}}(\Delta_{p,q,\infty} - \Delta_{p,q,\gamma}) = 0. \tag{72}$$

Suppose now that $\beta, \gamma \neq \infty$ and observe that $B_\beta = e^{2\pi i \frac{p}{q}(\beta-\gamma)} B_\gamma$, so the complex argument of the ratio B_β/B_γ is greater in absolute value than $2\pi/q$ for all p. In particular, B_β and B_γ are not asymptotically equivalent as p tends to ∞ and thus for infinitely many rational primes p congruent to 1 modulo N,

$$\mathrm{AJ}_{\mathbf{C}}(\Delta_{p,q,\beta})\left(\omega_f \wedge \omega_A^j \eta_A^{r-j}\right) \neq \mathrm{AJ}_{\mathbf{C}}(\Delta_{p,q,\gamma})\left(\omega_f \wedge \omega_A^j \eta_A^{r-j}\right). \tag{73}$$

Moreover, we have $\lim_{p/q\to\infty} \mathrm{AJ}_{\mathbf{C}}(\Delta_{p,q,\beta} - \Delta_{p,q,\gamma}) = 0$.

Consequently, by taking p sufficiently large, one can ensure that the order of $\mathrm{AJ}_{\mathbf{C}}(\Delta_{p,q,\beta} - \Delta_{p,q,\gamma})$ in $J^{r+1}(X_r/\mathbf{C})$ is greater than M_r and thus, by Proposition 8, $\Delta_{p,q,\beta} - \Delta_{p,q,\gamma}$ is non-torsion in $\mathrm{CH}^{r+1}(X_r)_0(\bar{H})$. □

We now turn to the proof of Proposition 8. For any rational prime ℓ, Bloch has defined in [4] a map of Galois modules

$$\lambda_\ell : \mathrm{CH}^{r+1}(X_r)(\bar{H})(\ell) \longrightarrow H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}_\ell/\mathbf{Z}_\ell(r+1)) \tag{74}$$

where $\mathrm{CH}^{r+1}(X_r)(\bar{H})(\ell)$ denotes the ℓ-power torsion subgroup. This map is constructed by studying the coniveau spectral sequence of $\bar{X}_r$ and can be viewed as an arithmetic avatar of the complex Abel–Jacobi map restricted to torsion. In order to justify this claim, recall that

$$J^{r+1}(X_r/\mathbf{C}) = H^{2r+1}(X_r(\mathbf{C}), \mathbf{C}) / \left(\mathrm{Fil}^{r+1} H^{2r+1}_{\mathrm{dR}}(X_r/\mathbf{C}) \oplus \mathrm{Im}\, H^{2r+1}(X_r(\mathbf{C}), \mathbf{Z})\right) \tag{75}$$

and observe that we have an isomorphism of $\mathbf{R}$-vector spaces

$$H^{2r+1}(X_r(\mathbf{C}), \mathbf{R}) \cong H^{2r+1}(X_r(\mathbf{C}), \mathbf{C})/\mathrm{Fil}^{r+1} H^{2r+1}_{\mathrm{dR}}(X_r/\mathbf{C}), \tag{76}$$

so that we may identify

$$J^{r+1}(X_r/\mathbf{C})_{\mathrm{tors}} \cong H^{2r+1}(X_r(\mathbf{C}), \mathbf{Q})/\mathrm{Im}\, H^{2r+1}(X_r(\mathbf{C}), \mathbf{Z}). \tag{77}$$

From the long exact sequence in singular cohomology associated to the short exact sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Q} \longrightarrow \mathbf{Q}/\mathbf{Z} \longrightarrow 0 \tag{78}$$

we deduce a short exact sequence

$$0 \longrightarrow J^{r+1}(X_r/\mathbf{C})_{\mathrm{tors}} \xrightarrow{u} H^{2r+1}(X_r(\mathbf{C}), \mathbf{Q}/\mathbf{Z}) \longrightarrow H^{2r+2}(X_r(\mathbf{C}), \mathbf{Z})_{\mathrm{tors}} \longrightarrow 0. \tag{79}$$

Note that $H^{2r+2}(X_r(\mathbf{C}), \mathbf{Z})$ is a group of finite type and thus its torsion subgroup is finite. We have identified $J^{r+1}(X_r/\mathbf{C})_{\mathrm{tors}}$ up to a finite group with $H^{2r+1}(X_r(\mathbf{C}), \mathbf{Q}/\mathbf{Z})$.

Composing the complex Abel–Jacobi map restricted to torsion with u yields a map

$$u \circ \mathrm{AJ}_{\mathbf{C}} : \mathrm{CH}^{r+1}(X_r)(\mathbf{C})_0(\ell) \longrightarrow H^{2r+1}(X_r(\mathbf{C}), \mathbf{Q}_\ell/\mathbf{Z}_\ell). \tag{80}$$

For each natural number ν, we have a sequence of isomorphisms

$$H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mu_{\ell^\nu}^{\otimes(r+1)}) \cong H^{2r+1}_{\mathrm{et}}(X_{r,\mathbf{C}}, \mu_{\ell^\nu}^{\otimes(r+1)}) \cong H^{2r+1}(X_r(\mathbf{C}), \mu_{\ell^\nu}^{\otimes(r+1)}). \tag{81}$$

For the first isomorphism, apply [12, VI Corollary 4.3] with respect to our fixed complex embedding $\bar{K} \hookrightarrow \mathbf{C}$. The second isomorphism is an application of [12, III Theorem 3.12]. Taking direct limits, we obtain a sequence of isomorphisms

$$\begin{aligned} H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}_\ell/\mathbf{Z}_\ell(r+1)) &\cong H^{2r+1}_{\mathrm{et}}(X_{r,\mathbf{C}}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(r+1)) \\ &\cong H^{2r+1}(X_r(\mathbf{C}), \mathbf{Q}_\ell/\mathbf{Z}_\ell(r+1)). \end{aligned} \tag{82}$$

If we identify $\mathbf{Q}_\ell/\mathbf{Z}_\ell \cong \mathbf{Q}_\ell/\mathbf{Z}_\ell(r+1)$ by taking $e^{\frac{2\pi i}{\ell^\nu}}$ as the generator of the ℓ^ν-th roots of 1, then the diagram

$$\begin{tikzcd} \mathrm{CH}^{r+1}(X_r)_0(\bar{H})(\ell) \arrow[r, "\lambda_\ell"] \arrow[d] & H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}_\ell/\mathbf{Z}_\ell(r+1)) \arrow[d, "\cong"] \\ \mathrm{CH}^{r+1}(X_r)_0(\mathbf{C})(\ell) \arrow[r, "u\circ\mathrm{AJ}_{\mathbf{C}}"] & H^{2r+1}(X_r(\mathbf{C}), \mathbf{Q}_\ell/\mathbf{Z}_\ell). \end{tikzcd} \tag{83}$$

commutes by [4, Proposition 3.7], where the right hand side isomorphism is (82).

Summing over all primes ℓ yields a map of Galois modules

$$\lambda : \mathrm{CH}^{r+1}(X_r)(\bar{H})_{\mathrm{tors}} \longrightarrow H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}/\mathbf{Z}(r+1)) \tag{84}$$

which fits into a commutative diagram

$$\begin{tikzcd} \mathrm{CH}^{r+1}(X_r)_0(\bar{H})_{\mathrm{tors}} \arrow[r, "\lambda"] \arrow[d] & H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}/\mathbf{Z}(r+1)) \arrow[d, "\cong"] \\ \mathrm{CH}^{r+1}(X_r)_0(\mathbf{C})_{\mathrm{tors}} \arrow[r, "u\circ\mathrm{AJ}_{\mathbf{C}}"] & H^{2r+1}(X_r(\mathbf{C}), \mathbf{Q}/\mathbf{Z}). \end{tikzcd} \tag{85}$$

Lemma 5 *For all $r \geq 0$, there exists a non-negative integer M_r that annihilates the group*

$$H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}/\mathbf{Z}(r+1))^{G_{F_n}}$$

for any square-free positive integer n coprime to N, where $F_n = K_{\mathfrak{N}} \cdot H_n$.

Proof Let n be a positive square-free integer. Let q denote a rational prime which remains inert in K. If $q|n$, write $n = qm$ with $(q, m) = 1$. Since q is inert in K and coprime to m, class field theory implies that q splits completely in H_m/K. Again by class field theory, each factor of q in H_m is totally ramified in $H_n = H_q \cdot H_m$. As a consequence, the residual degree of q in H_n/K is 1 and its residual degree in $H_n/\mathbf{Q}$ is equal to 2.

Let us fix once and for all two distinct rational primes q_1 and q_2 which are inert in K and satisfy $(2N, q_1q_2) = 1$ with the property that there exist two primes $\mathfrak{q}_1$ and $\mathfrak{q}_2$ in H which lie above q_1 and q_2 respectively such that X_r has good reduction at $\mathfrak{q}_1$ and $\mathfrak{q}_2$.

Let s_1 and s_2 denote the residual degrees of $\mathfrak{q}_1$ and $\mathfrak{q}_2$ in $K_{\mathfrak{N}}/H$ respectively. We claim that $K_{\mathfrak{N}} \cap H_n = H$. Indeed, if a prime ideal in K ramifies in the abelian extension $K_{\mathfrak{N}} \cap H_n$ over K, then it divides both $\mathfrak{N}$ and $n\mathscr{O}_K$. But these two ideals are coprime since the norm of $\mathfrak{N}$ is N and $(N, n) = 1$. Thus $K_{\mathfrak{N}} \cap H_n$ is everywhere unramified above K and is thus contained in H, hence equal to H. Because the residual degrees of q_1 and q_2 in H_n/K equal 1, the residual degrees of $\mathfrak{q}_1$ and $\mathfrak{q}_2$ in F_n/H are equal to s_1 and s_2 respectively. In particular, these residual degrees are independent of n.

Let H_∞ denote the compositum of all ring class fields of K of square-free conductor coprime to N and define $F_\infty = K_{\mathfrak{N}} \cdot H_\infty$. It follows from the above discussion that the residual degrees of $\mathfrak{q}_1$ and $\mathfrak{q}_2$ in F_∞/H are equal to s_1 and s_2 respectively.

Let ℓ be a rational prime. For $i = 1, 2$ fix $\mathfrak{q}_i^\infty$ a prime of F_∞ above $\mathfrak{q}_i$ and let D_i denote the decomposition group in G_{F_∞} of a prime above $\mathfrak{q}_i^\infty$. Because of our assumption of good reduction, the inertia group $I_i \subset D_i$ acts trivially on the group $H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}_\ell/\mathbf{Z}_\ell(r+1))$ and we have, by [12, VI Corollary 4.2],

$$H^{2r+1}_{\mathrm{et}}\left(\bar{X}_r, \mu_{\ell^\nu}^{\otimes(r+1)}\right)^{D_i} \cong H^{2r+1}_{\mathrm{et}}\left(X_{r,\bar{\mathbf{F}}_{q_i}}, \mu_{\ell^\nu}^{\otimes(r+1)}\right)^{G_{\mathbf{F}_{q_i^{s_i}}}} \tag{86}$$

for all ν, as long as $\ell \neq q_i$. Taking direct limits, we obtain an isomorphism

$$H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}_\ell/\mathbf{Z}_\ell(r+1))^{D_i} \cong H^{2r+1}_{\mathrm{et}}\left(X_{r,\bar{\mathbf{F}}_{q_i}}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(r+1)\right)^{G_{\mathbf{F}_{q_i^{s_i}}}}. \tag{87}$$

From the long exact sequence in ℓ-adic cohomology associated to the short exact sequence

$$0 \longrightarrow \mathbf{Z}_\ell(r+1) \longrightarrow \mathbf{Q}_\ell(r+1) \longrightarrow \mathbf{Q}_\ell/\mathbf{Z}_\ell(r+1) \longrightarrow 0 \tag{88}$$

we obtain a short exact sequence

$$\begin{aligned} 0 \longrightarrow & \frac{H^{2r+1}_{\mathrm{et}}\left(X_{r,\bar{\mathbf{F}}_{q_i}}, \mathbf{Q}_\ell(r+1)\right)}{\mathrm{Im}\left(H^{2r+1}_{\mathrm{et}}\left(X_{r,\bar{\mathbf{F}}_{q_i}}, \mathbf{Z}_\ell(r+1)\right)\right)} \longrightarrow H^{2r+1}_{\mathrm{et}}\left(X_{r,\bar{\mathbf{F}}_{q_i}}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(r+1)\right) \\ & \longrightarrow H^{2r+2}_{\mathrm{et}}\left(X_{r,\bar{\mathbf{F}}_{q_i}}, \mathbf{Z}_\ell(r+1)\right)_{\mathrm{tors}} \longrightarrow 0. \end{aligned} \tag{89}$$

Consequently, the order of $H^{2r+1}_{\mathrm{et}}(X_{r,\bar{\mathbf{F}}_{q_i}}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(r+1))^{G_{\mathbf{F}_{q_i}^{s_i}}}$ is bounded by the product of

$$\left|H^{2r+2}_{\mathrm{et}}\left(X_{r,\bar{\mathbf{F}}_{q_i}}, \mathbf{Z}_\ell(r+1)\right)_{\mathrm{tors}}\right| \qquad \text{and} \qquad \left|\left(\frac{H^{2r+1}_{\mathrm{et}}\left(X_{r,\bar{\mathbf{F}}_{q_i}}, \mathbf{Q}_\ell(r+1)\right)}{\mathrm{Im}\left(H^{2r+1}_{\mathrm{et}}\left(X_{r,\bar{\mathbf{F}}_{q_i}}, \mathbf{Z}_\ell(r+1)\right)\right)}\right)^{G_{\mathbf{F}_{q_i}^{s_i}}}\right|.$$

We claim that both these quantities are finite, and equal to 1 for all but finitely many primes ℓ.

On one hand, we have a sequence of isomorphisms

$$\begin{aligned} H^{2r+2}_{\mathrm{et}}\left(X_{r,\overline{\mathbf{F}_{q_i}}}, \mathbf{Z}_\ell(r+1)\right) &\cong H^{2r+2}_{\mathrm{et}}\left(X_{r,\overline{H_{\mathfrak{q}_i}}}, \mathbf{Z}_\ell(r+1)\right) \\ &\cong H^{2r+2}_{\mathrm{et}}(\bar{X}_r, \mathbf{Z}_\ell(r+1)) \cong H^{2r+2}(X_r(\mathbf{C}), \mathbf{Z})(r+1) \otimes \mathbf{Z}_\ell \end{aligned}$$

where $H_{\mathfrak{q}_i}$ denotes the completion of H at $\mathfrak{q}_i$. The first isomorphism is obtained by using [12, VI Corollary 4.2] and taking inverse limits. For the second one, we fix an embedding $\bar{H} \hookrightarrow \overline{H_{\mathfrak{q}_i}}$, apply [12, VI Corollary 4.3] and take inverse limits. The last one is a consequence of [12, III Theorem 3.12] and taking inverse limits. Since $H^{2r+2}(X_r(\mathbf{C}), \mathbf{Z})$ is finitely generated, its torsion subgroup is finite and thus the torsion subgroup of $H^{2r+2}(X_r(\mathbf{C}), \mathbf{Z})(r+1) \otimes \mathbf{Z}_\ell$ is trivial for all but finitely many ℓ.

On the other hand, we have

$$\begin{aligned} &\left|\left(\frac{H^{2r+1}_{\mathrm{et}}\left(X_{r,\bar{\mathbf{F}}_{q_i}}, \mathbf{Q}_\ell(r+1)\right)}{\mathrm{Im}\left(H^{2r+1}_{\mathrm{et}}\left(X_{r,\bar{\mathbf{F}}_{q_i}}, \mathbf{Z}_\ell(r+1)\right)\right)}\right)^{G_{\mathbf{F}_{q_i}^{s_i}}}\right| \\ &= \left|\ker\left(1 - \mathrm{Frob}_{\mathfrak{q}_i^\infty} \;\middle|\; \frac{H^{2r+1}_{\mathrm{et}}\left(X_{r,\bar{\mathbf{F}}_{q_i}}, \mathbf{Q}_\ell(r+1)\right)}{\mathrm{Im}\left(H^{2r+1}_{\mathrm{et}}\left(X_{r,\bar{\mathbf{F}}_{q_i}}, \mathbf{Z}_\ell(r+1)\right)\right)}\right)\right| \end{aligned} \tag{90}$$

which is equal to the ℓ-part of

$$\left|\det\left(1 - \mathrm{Frob}_{\mathfrak{q}_i^\infty} \;\middle|\; \mathrm{Im}\left(H^{2r+1}_{\mathrm{et}}\left(X_{r,\bar{\mathbf{F}}_{q_i}}, \mathbf{Z}_\ell(r+1)\right)\right)\right)\right|. \tag{91}$$

By the Weil conjectures as proved by Deligne [6], (91) does not depend on ℓ. In particular, (90) is equal to 1 for all but finitely many ℓ.

We conclude that the order of $H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}_\ell/\mathbf{Z}_\ell(r+1))^{G_{F_\infty}}$ is finite and equal to 1 for almost all ℓ. Hence $H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}/\mathbf{Z}(r+1))^{G_{F_\infty}}$ is finite and we may define

$$M_r := \left|H^{2r+1}_{\mathrm{et}}\left(\bar{X}_r, \mathbf{Q}/\mathbf{Z}(r+1)\right)^{G_{F_\infty}}\right|. \tag{92}$$

Then M_r annihilates $H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}_\ell/\mathbf{Z}_\ell(r+1))^{G_{F_n}}$ for all square-free n which are coprime to N. □

Proof of Proposition 8 Let M_r be the non-negative integer of Lemma 5 defined in (92). We will prove the contrapositive of the statement of the proposition.

The cycle Δ is defined over the field $F_n = K_{\mathfrak{N}} \cdot H_n$ for some square-free integer n coprime to N by Proposition 7. Suppose that Δ is a torsion element of the group $\mathrm{CH}^{r+1}(X_r)_0(\bar{H})$. By Lemma 5, the order, say m, of $\lambda(\Delta)$ must divide M_r. By (85), we have $\lambda(\Delta) = u \circ \mathrm{AJ}_{\mathbf{C}}(\Delta)$. Thus $u(m\mathrm{AJ}_{\mathbf{C}}(\Delta)) = 0$ and by injectivity of u, we deduce that $m\mathrm{AJ}_{\mathbf{C}}(\Delta) = 0$. Hence the order of $\mathrm{AJ}_{\mathbf{C}}(\Delta)$ divides m and in particular divides M_r. □

10.4 The Chow group is not finitely generated

We conclude the proof of the statement of Theorem 2 concerning the Chow group by exploiting the action of the Galois group G_H on generalised Heegner cycles.

Proof of Theorem 2 Let ℓ be an arbitrary rational prime which does not divide $d_K N$. Fix a rational odd prime q congruent to 1 modulo N, which remains inert in K and such that ℓ divides the degree of H_q/H, i.e., $q+1 \equiv 0 \mod \ell$.

There are $q+1$ distinct isogenies $\varphi_{q,\beta} : A \longrightarrow A_{q,\beta}$ of degree q with $\beta \in \mathbf{P}_1(\mathbf{F}_q)$ attached to the following lattices $\Lambda_{q,\beta}$ containing $\mathscr{O}_K$ with index q:

$$\Lambda_{q,\infty} := \mathbf{Z}\frac{1}{q} \oplus \mathbf{Z}\tau, \qquad \Lambda_{q,\beta} := \mathbf{Z} \oplus \mathbf{Z}\frac{\tau+\beta}{q}, \quad \text{for } 0 \le \beta \le q-1.$$

The theory of complex multiplication shows that the elliptic curves $A_{q,\beta}$ as well as the isogenies $\varphi_{q,\beta}$ can be taken to be rational over H_q, the ring class field of K of conductor q. As q is assumed to be inert in K, the extension H_q/H is cyclic of degree $q+1$, and we let σ_q denote a fixed generator of its Galois group $G_q = \mathrm{Gal}(H_q/H)$. Recall the analytic isomorphism $\xi : \mathbf{C}/\mathscr{O}_K \cong A(\mathbf{C})$ and define, for all $\beta \in \mathbf{P}_1(\mathbf{F}_q)$, the point

$$t_{q,\beta} := \begin{cases} \xi((\tau+\beta)/q), & \text{if } \beta \neq \infty \\ \xi(1/q), & \text{if } \beta = \infty \end{cases}$$

of $A(\mathbf{C})$ and note that $\ker(\varphi_{q,\beta}) = \langle t_{q,\beta} \rangle$.

For any $\sigma \in \mathrm{Aut}(\mathbf{C}/H)$, observe that $A^\sigma = A$ and $\sigma|_{K^{\mathrm{ab}}} = (s|K)$ is the Artin symbol for an idele s of K which is a unit at all finite places (by the idelic description of the ideal class group and the idelic formulation of class field theory). In particular, for any $\sigma \in G_q$ and any idele s of K with $\sigma = (s|K)|_{H_q}$ and $s_v \in \mathscr{O}_{K,v}^\times$ for all $v \nmid \infty$, there is a unique analytic isomorphism $\xi_\sigma : \mathbf{C}/\mathscr{O}_K \cong A(\mathbf{C})$ such that the diagram

$$\begin{array}{ccc} K/\mathscr{O}_K & \xrightarrow{\xi} & A \\ {\scriptstyle s^{-1}}\downarrow & & \downarrow{\scriptstyle \sigma} \\ K/\mathscr{O}_K & \xrightarrow{\xi_\sigma} & A \end{array} \tag{93}$$

commutes, according to Shimura's formulation of the main theorem of complex multiplication [15, Theorem 5.4]. Observe that $\xi_\sigma = \xi \circ \alpha_\sigma$ for some $\alpha_\sigma \in \mathscr{O}_K^\times = \{\pm 1\}$. Note that $\ker(\varphi_{q,\beta})$ is a subgroup of the q-torsion group of A, and we may thus restrict our focus to the q-torsion subgroup of $K/\mathscr{O}_K$, namely $q^{-1}\mathscr{O}_{K,q}/\mathscr{O}_{K,q}$.

Since $(s|K)|_{H_q}$ is an element of G_q, the fractional ideal (s^{-1}) associated to s^{-1} belongs to the group $(I_K(q) \cap P_K)/P_{K,\mathbf{Z}}(q)$, where P_K denotes the principal ideals of K, $P_{K,\mathbf{Z}}(q)$ the principal ideals admitting a representative which is congruent to a non-zero integer modulo q and $I_K(q)$ the fractional ideals of K coprime to q. This group is isomorphic to the quotient $(\mathscr{O}_K/q\mathscr{O}_K)^\times/(\mathbf{Z}/q\mathbf{Z})^\times$ and acts on $\mathbb{F}_q$-lines in $q^{-1}\mathscr{O}_{K,q}/\mathscr{O}_{K,q} \cong \mathscr{O}_{K,q}/q\mathscr{O}_{K,q}$ by multiplication. In particular, we see that s^{-1} permutes the $\mathbb{F}_q$-lines in $q^{-1}\mathscr{O}_{K,q}/\mathscr{O}_{K,q}$ without preserving any of them. We conclude from (93) that σ permutes the kernels $\langle t_{q,\beta} \rangle$ without preserving any of them. Thus the action of G_q on the set of $q+1$ isogenies $\varphi_{q,\beta}$ is simply transitive.

Let p be a rational prime congruent to 1 modulo N. The isogeny $\varphi_{p,q,\beta}$ corresponds to the subgroup $\langle \xi(1/p), t_{q,\beta} \rangle$ of $A(\bar{H})$ which is defined over H_{pq}. Because p and q are distinct, we have $H_{pq} = H_p \cdot H_q$ and $H_p \cap H_q = H$ so that

$$\mathrm{Gal}(H_{pq}/H_p) \cong \mathrm{Gal}(H_q/H). \tag{94}$$

Recall from Proposition 7 that $\Delta_{p,q,\beta}$ is defined over $F_{pq} = K_{\mathfrak{N}} \cdot H_{pq}$ and since the intersection $K_{\mathfrak{N}} \cap H_{pq}$ is H we have an isomorphism

$$\mathrm{Gal}(F_{pq}/K_{\mathfrak{N}}) \cong \mathrm{Gal}(H_{pq}/H). \tag{95}$$

Consider the cyclic subgroup of $\mathrm{Gal}(H_q/H)$ of order ℓ which exists because of the assumption $q + 1 \equiv 0 \mod \ell$. Let G_ℓ denote the image of this group in $\mathrm{Gal}(F_{pq}/K_{\mathfrak{N}})$ under the above isomorphisms (94) and (95), and let σ_ℓ be a generator of G_ℓ. Consider the homomorphism of $\mathbf{Q}$-vector spaces

$$\psi : \mathbf{Q}[G_\ell] \longrightarrow \mathrm{CH}^{r+1}(X_r)_0(\bar{H}) \otimes \mathbf{Q}, \tag{96}$$

which to $\sigma \in G_\ell$ associates $\sigma(\Delta_{p,q,\beta})$. Note that the kernel of ψ is stable under multiplication by $\mathbf{Q}[G_\ell]$, hence $\ker(\psi)$ is an ideal of $\mathbf{Q}[G_\ell]$. But $\mathbf{Q}[G_\ell]$ has a very simple structure; it is isomorphic to the product of two fields, namely $\mathbf{Q}$ and $\mathbf{Q}(\zeta_\ell)$, where ζ_ℓ is a primitive ℓ-th root of unity. Indeed, the map

$$\mathbf{Q}[G_\ell] \longrightarrow \mathbf{Q} \times \mathbf{Q}(\zeta_\ell), \qquad \sum_{i=0}^{\ell-1} \lambda_i \sigma_\ell^i \mapsto \left(\lambda_0, \sum_{i=0}^{\ell-1} \lambda_i \zeta_\ell^i \right)$$

is an isomorphism of rings. There are exactly two proper ideals of $\mathbf{Q} \times \mathbf{Q}(\zeta_\ell)$, namely $\{0\} \times \mathbf{Q}(\zeta_\ell)$ and $\mathbf{Q} \times \{0\}$, which correspond respectively to the augmentation ideal and the ideal $\mathbf{Q} \cdot N$ of $\mathbf{Q}[G_\ell]$, where $N = \sum_{i=0}^{\ell-1} \sigma_\ell^i$.

By Corollary 1, we may assume, by taking p large enough, that $\Delta_{p,q,\beta}$ is non-torsion in the Chow group. In other words $\psi(1) \neq 0$ and therefore $\ker(\psi)$ is not equal to all of $\mathbf{Q}[G_\ell]$.

Because the action of $\mathrm{Gal}(H_q/H)$ on the set of q-isogenies of A is simply transitive, we see that $(\varphi_{q,\beta}, A_{q,\beta})^{\sigma_\ell} = (\varphi_{q,\gamma}, A_{q,\gamma})$ in $\mathrm{Isog}(A)$ for some $\gamma \neq \beta$ in $\mathbf{P}_1(\mathbf{F}_q)$. Since σ_ℓ fixes H_p it must fix the subgroup $\langle \xi(1/p) \rangle$ of $A(\bar{H})$, and we must have

$$(\varphi_{p,q,\beta}, A_{p,q,\beta})^{\sigma_\ell} = (\varphi_{p,q,\gamma}, A_{p,q,\gamma}) \tag{97}$$

in $\mathrm{Isog}(A)$. It follows that $\sigma_\ell(\Delta_{p,q,\beta}) = \Delta_{p,q,\gamma}$ in $\mathrm{CH}^{r+1}(X_r)_0(\bar{H})$, i.e., $\psi(\sigma_\ell) = \Delta_{p,q,\gamma}$.

By Corollary 2, we may choose p such that $\Delta_{p,q,\beta} - \Delta_{p,q,\gamma}$ is non-torsion in the Chow group. In other words, $\psi(\sigma_\ell - 1) \neq 0$ and thus $\ker(\psi)$ is not equal to the augmentation ideal.

We conclude that the kernel of ψ is either trivial or equal to $\mathbf{Q} \cdot N$. In any case, we have

$$\dim_{\mathbf{Q}} \mathbf{Q}[G_\ell]/\ker(\psi) \geq \ell - 1 \tag{98}$$

and we have thus constructed a subgroup of the Chow group of rank greater or equal to $\ell - 1$. Since ℓ was chosen arbitrarily, this proves that $\mathrm{CH}^{r+1}(X_r)_0(\bar{H})$ has infinite rank. □

10.5 The Griffiths group is not finitely generated

By Theorem 3, we know that many of our generalised Heegner cycles have large (possibly infinite) order in the Griffiths group, at least when $r \geq 1$. In the proof of this theorem, we were able to extract information concerning the Griffiths group by studying the transcendental Abel–Jacobi map (70), a modified version of the complex Abel–Jacobi map which enjoyed the property that it factored through $\mathrm{Gr}^{r+1}(X_r)(\mathbf{C})$. If we wish to apply the algebraic arguments of Sect. 10.3 in order to show that many of our cycles have infinite order in the Griffiths group, we need a modified version of Bloch's map λ of Galois modules (84) which factors through $\mathrm{Gr}^{r+1}(X_r)(\bar{H})$. To this end, we introduce an algebraic projector which we compose with λ.

We use the same conventions and notations for motives as in [7, §0]. Given two nonsingular varieties X and Y, we define the rings of correspondences

$$\mathrm{Corr}^0(X,Y) := \mathrm{CH}^{\dim(X)}(X\times Y) \quad \text{and} \quad \mathrm{Corr}^0(X,Y)_E := \mathrm{Corr}^0(X,Y)\otimes E,$$

if E is a number field.

Proposition 9 *For all $r \geq 2$, there exists an idempotent P_X in $\mathrm{Corr}^0(X_r, X_r)_{\mathbf{Q}}$ with the following properties:*

(i) *The map*

$$\mathrm{CH}^{r+1}(X_r)_0(\mathbf{C}) \xrightarrow{\mathrm{AJ}_{\mathbf{C}}} J^{r+1}(X_r/\mathbf{C}) \xrightarrow{(P_X)_*} J(N)$$

factors through $\mathrm{Gr}^{r+1}(X_r)(\mathbf{C})$, where $J(N)$ denotes the intermediate Jacobian associated to the Betti realisation of the Chow motive $N := (X_r, P_X, r+1)$ over H with coefficients in $\mathbf{Q}$.

(ii) *The map of Galois modules*

$$\mathrm{CH}^{r+1}(X_r)_0(\bar{H})_{\mathrm{tors}} \xrightarrow{\lambda} H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}/\mathbf{Z}(r+1)) \xrightarrow{(P_X)_*} H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}/\mathbf{Z}(r+1))$$

factors through $\mathrm{Gr}^{r+1}(X_r)(\bar{H})_{\mathrm{tors}}$, so we obtain a map of Galois modules

$$(P_X)_*\circ\lambda : \mathrm{Gr}^{r+1}(X_r)(\bar{H})_{\mathrm{tors}} \longrightarrow H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}/\mathbf{Z}(r+1)).$$

We begin with the construction of the projector P_X and assume from now on that $r\geq 2$. Write $[x]$ for $x\in K$ viewed as an element of $\mathrm{End}_H(A)\otimes\mathbf{Q}$. The identification of K with $\mathrm{End}_H(A)\otimes\mathbf{Q}$ is normalised such that $[x]^*\omega_A = x\omega_A$ for all $x\in K$. We shall consider the following idempotents of $\mathrm{End}_H(A)\otimes K$:

$$e = \frac{\sqrt{-d_K}+[\sqrt{-d_K}]}{2\sqrt{-d_K}} \qquad \text{and} \qquad \bar{e} = \frac{\sqrt{-d_K}-[\sqrt{-d_K}]}{2\sqrt{-d_K}}$$

and view them as elements of $\mathrm{Corr}^0(A,A)_K$ by taking their graphs. For all $0\leq j\leq r$, we define the idempotent

$$e^{(j)} := \sum_{\substack{I\subset\{1,\ldots,r\}\\ |I|=j}} e_{1,I}\otimes\cdots\otimes e_{r,I} \in \mathrm{Corr}^0(A^r, A^r)_K,$$

where $e_{i,I} := e$ or $\bar{e}$ depending on whether $i\in I$ or $i\notin I$.

Consider the Chow motive $M := (A^r, e_r, 0)$ over H with coefficients in $\mathbf{Q}$ where

$$e_r := \left(\sum_{0<j<r} e^{(j)}\right)\circ\left(\frac{1-[-1]}{2}\right)^{\otimes r} \in \mathrm{Corr}^0(A^r, A^r)_{\mathbf{Q}}.$$

The Betti realisation M_B of this motive is a Hodge structure of weight r. We have $M_B(\mathbf{C}) = e_r H^r_{\mathrm{dR}}(A^r)$ and its Hodge decomposition is given by

$$H^{j,r-j}(M_B(\mathbf{C})) = \begin{cases} H^{j,r-j}(A^r) & \text{for } 0<j<r \\ 0 & \text{for } j=0 \text{ or } j=r. \end{cases} \tag{99}$$

We will use the same notation for e_r and its pull-back to $\mathrm{Corr}^0(X_r, X_r)_{\mathbf{Q}}$ and define

$$P_X := e_r \circ \epsilon_X \in \mathrm{Corr}^0(X_r, X_r)_{\mathbf{Q}}, \tag{100}$$

which is an idempotent in the ring of correspondences of X_r with coefficients in $\mathbf{Q}$ since e_r and ϵ_X commute.

Remark 8 As in Remark 2, we will assume throughout that the projector P_X has been multiplied by a suitable integer so that it lies in $\mathrm{Corr}^0(X_r, X_r)$.

The correspondence P_X induces morphisms $(P_X)_* = (\mathrm{pr}_2)_* \circ (\cdot P_X) \circ (\mathrm{pr}_1)^*$ between Chow groups, cohomology groups and intermediate Jacobians and acts as a projector on these various objects.

The map of intermediate Jacobians

$$(P_X)_* : J^{r+1}(X_r/\mathbf{C}) \longrightarrow J^{r+1}(X_r/\mathbf{C}) \tag{101}$$

is induced from the map on singular cohomology

$$(P_X)_* : H^{2r+1}(X_r(\mathbf{C}), \mathbf{Z}) \longrightarrow H^{2r+1}(X_r(\mathbf{C}), \mathbf{Z}) \tag{102}$$

which makes sense since the latter is a morphism of Hodge structures of bidegree $(0, 0)$ (see [17, Lemma 11.41]) and thus maps $\mathrm{Fil}^{r+1} H^{2r+1}_{\mathrm{dR}}(X_r/\mathbf{C})$ into itself.

We will henceforth work with the Chow motive $N := (X_r, P_X, r+1)$ over H with coefficients in $\mathbf{Q}$. Its Betti realisation $N_B = (P_X)_*(H^{2r+1}(X_r(\mathbf{C}), \mathbf{Z}))(r+1)$ is a Hodge structure of weight -1 and the 0-th step of its Hodge filtration is given by

$$\begin{aligned} \mathrm{Fil}^0 N_B(\mathbf{C}) &= (P_X)_* \mathrm{Fil}^{r+1} H^{2r+1}_{\mathrm{dR}}(X_r/\mathbf{C}) \\ &= S_{r+2}(\Gamma) \otimes \left(\bigoplus_{j=1}^{r-1} \mathbf{C}\omega_A^j \eta_A^{r-j} \right) \subset \bigoplus_{j=1}^{r-1} H^{r+1+j, r-j}(X_r). \end{aligned} \tag{103}$$

We note that $H^{0,-1}(N_B(\mathbf{C})) = H^{r,-(r+1)}(N_B(\mathbf{C})) = 0$ and in particular we have the crucial property

$$(P_X)_*(H^{r+1,r}(X_r) \oplus H^{r,r+1}(X_r)) = 0. \tag{104}$$

Associated to the Hodge structure N_B is a complex torus

$$J(N) := N_B(\mathbf{C})/(\mathrm{Fil}^0(N_B(\mathbf{C})) \oplus N_B)$$

which is the image of the projection (101). By (103) and Poincaré duality, we have an isomorphism of complex tori

$$J(N) \cong \frac{\left(S_{r+2}(\Gamma) \otimes \left(\bigoplus_{j=1}^{r-1} \mathbf{C}\omega_A^j \eta_A^{r-j}\right)\right)^\vee}{\Pi'_{r,r}}, \tag{105}$$

where the lattice $\Pi'_{r,r}$ is defined by

$$\Pi'_{r,r} := (P_X)_*(\mathrm{Im}\, H_{2r+1}(X_r(\mathbf{C}), \mathbf{Z})). \tag{106}$$

Proof of Proposition 9 Recall from (69) that $J^{r+1}(X_r/\mathbf{C})_{\mathrm{alg}} = J^{r+1}(V)$ where V is the largest sub-Hodge structure of $H^{r+1,r}(X_r) \oplus H^{r,r+1}(X_r)$ and that the image of $\mathrm{CH}^{r+1}(X_r)_{\mathrm{alg}}(\mathbf{C})$ under $\mathrm{AJ}_{\mathbf{C}}$ is a complex subtorus of $J^{r+1}(X_r/\mathbf{C})$ which is contained in $J^{r+1}(X_r/\mathbf{C})_{\mathrm{alg}}$. The morphism of tori $(P_X)_* : J^{r+1}(X_r/\mathbf{C}) \longrightarrow J(N)$ is induced from the morphism of Hodge structures (102). The latter map restricts to a morphism of Hodge structures $(P_X)_* : V_{\mathbf{Z}} \longrightarrow N_B$ which is the zero map when tensored up to $\mathbf{C}$ by (104) since $V_{\mathbf{C}} \subset H^{r+1,r}(X_r) \oplus H^{r,r+1}(X_r)$. Hence the induced map $(P_X)_* : J^{r+1}(V) \longrightarrow J(N)$ is the zero map and statement i) of the proposition follows.

The group $\mathrm{CH}^{r+1}(X_r)_{\mathrm{alg}}(\bar{H})$ is divisible since it is generated by images under correspondences of $\bar{H}$-valued points on Jacobians of curves. Therefore we have an exact sequence

$$0 \longrightarrow \mathrm{CH}^{r+1}(X_r)_{\mathrm{alg}}(\bar{H})_{\mathrm{tors}} \longrightarrow \mathrm{CH}^{r+1}(X_r)_0(\bar{H})_{\mathrm{tors}} \longrightarrow \mathrm{Gr}^{r+1}(X_r)(\bar{H})_{\mathrm{tors}} \longrightarrow 0 \tag{107}$$

and in order to prove ii) it suffices to show that the subgroup $\mathrm{CH}^{r+1}(X_r)_{\mathrm{alg}}(\bar{H})_{\mathrm{tors}}$ lies in the kernel of $(P_X)_* \circ \lambda$. Observe from (85) that

$$(P_X)_* \circ \lambda = (P_X)_* \circ u \circ \mathrm{AJ}_{\mathbf{C}} \tag{108}$$

where we use the compatibility of the comparison isomorphism (82) with correspondences which follows from the compatibility of the cycle class maps with respect to the comparison isomorphism (see [11, §5.3]). Note that the maps (102) and (101) commute with u since the latter is induced from the former and we therefore have

$$(P_X)_* \circ \lambda = u \circ (P_X)_* \circ \mathrm{AJ}_{\mathbf{C}}. \tag{109}$$

It follows from i) that $(P_X)_* \circ \lambda(\mathrm{CH}^{r+1}(X_r)_{\mathrm{alg}}(\bar{H})_{\mathrm{tors}}) = 0$. $\square$

When $r \geq 2$, applying the map $(P_X)_*$ on Chow groups yields the cycles

$$\Xi_{p,q,\beta} := (P_X)_* \Delta_{p,q,\beta}, \tag{110}$$

whose classes in the Griffiths group will be denoted $[\Xi_{p,q,\beta}]$. Since the projector P_X is defined over $\mathbf{Q}$, these cycles and their classes are defined over F_{pq} by Proposition 7.

Proposition 10 *For all $r \geq 2$, the order of $[\Xi_{p,q,\beta}]$ in $\mathrm{Gr}^{r+1}(X_r)(\bar{H})$ tends to ∞ as p/q tends to infinity.*

Proof By functoriality of the complex Abel–Jacobi map (see [8]), we may view $\mathrm{AJ}_{\mathbf{C}}(\Xi_{p,q,\beta})$ as an element of $J(N)$. If $f \in S_{r+2}(\Gamma)$ is non-zero and $0 < j < r$, then

$$\mathrm{AJ}_{\mathbf{C}}(\Xi_{p,q,\beta})(\omega_f \wedge \omega_A^j \eta_A^{r-j}) = \mathrm{AJ}_{\mathbf{C}}(\Delta_{p,q,\beta})(\omega_f \wedge \omega_A^j \eta_A^{r-j}). \tag{111}$$

As p/q tends to ∞, by Lemma 4, $\mathrm{AJ}_{\mathbf{C}}(\Xi_{p,q,\beta})$ becomes arbitrarily close but not equal to the origin in $J(N)$. It follows, by Proposition 9 (i), that the order of $[\Xi_{p,q,\beta}]$ tends to ∞ with p/q. $\square$

Proposition 11 *For all $r \geq 2$, if $\Xi \in \langle\{\Xi_{p,q,\beta}\}\rangle \subset \mathrm{CH}^{r+1}(X_r)_0(\bar{H})$ is such that the order of $\mathrm{AJ}_{\mathbf{C}}(\Xi)$ in $J^{r+1}(X_r/\mathbf{C})$ does not divide M_r, then Ξ has infinite order in $\mathrm{Gr}^{r+1}(X_r)(\bar{H})$.*

Proof Suppose that $[\Xi]$ is a torsion element. The cycle Ξ and its class in the Griffiths group are both defined over the field $F_n = K_{\mathfrak{N}} \cdot H_n$ for some square-free integer n coprime to N by Proposition 7 and we have the identity $(P_X)_*\Xi = \Xi$. By Proposition 9 (ii),

$$(P_X)_* \circ \lambda([\Xi]) \in H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}/\mathbf{Z}(r+1))^{G_{F_n}}$$

and thus by Lemma 5, the order, say m, of $(P_X)_* \circ \lambda([\Xi])$ must divide M_r. By (109), we have

$$(P_X)_* \circ \lambda([\Xi]) = u \circ (P_X)_* \circ \mathrm{AJ}_{\mathbf{C}}([\Xi]) = u \circ (P_X)_* \circ \mathrm{AJ}_{\mathbf{C}}(\Xi).$$

By functoriality of the complex Abel–Jacobi map with respect to correspondences (see [8]), we obtain

$$(P_X)_* \circ \lambda([\Xi]) = u(\mathrm{AJ}_{\mathbf{C}}((P_X)_*\Xi)) = u(\mathrm{AJ}_{\mathbf{C}}(\Xi)).$$

By injectivity of u, the order of $\mathrm{AJ}_{\mathbf{C}}(\Xi)$ must divide m and thus divides M_r. $\square$

Proof of Theorem 2 Proceeding as in Sect. 10.3, one uses Propositions 10 and 11 to deduce the analogue statements of Corollaries 1 and 2 for the Griffiths group and the classes $[\Xi_{p,q,\beta}]$. Using these two statements, the same arguments as in Sect. 10.4 apply, proving that $\mathrm{Gr}^{r+1}(X_r)(\bar{H})$ has infinite rank. $\square$

Remark 9 Applying the construction of the projector P_X in the case $r = 1$ yields nothing interesting. In fact, there is no algebraic splitting of the motive X_1 into its algebraic and transcendental components and for this reason we cannot apply our arguments to show that the Griffiths group is infinitely generated in this case. More precisely, we cannot obtain Proposition 9 ii) and therefore we fail to obtain Proposition 11. As a consequence, even though we can show that many of our cycles have large order in the Griffiths group, we are unable to prove that they generate a group of infinite rank.

Remark 10 Section 2.4 in [1] exhibits a correspondence from X_{2r} to W_{2r} under which generalised Heegner cycles are mapped to (rational multiples of) "traditional" Heegner cycles on Kuga–Sato varieties. While this does not imply directly the analogue of Theorem 2 in the setting of Kuga–Sato varieties, the methods of this paper can be expected to carry over to proving the analogues of Theorems 1 and 2 in this setting.

Remark 11 In [10], Bo-Hae Im exploits Heegner points in an ingenious way to prove that Mordell–Weil groups over *large fields* are of infinite rank, where a field is said to be *large* if it is of the form $\bar{\mathbf{Q}}^{\sigma}$, with σ an element of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. We believe that the techniques used in the proof [10, Prop. 2.9] can be combined with Theorem 2 to show that

$$\dim \mathrm{CH}^{r+1}(X_r)_0(\bar{\mathbf{Q}}^{\sigma}) \otimes \mathbf{Q} = \infty,$$

as well as similar statements for the Griffiths group when $r \geq 2$.

Acknowledgements David Lilienfeldt would like to thank Michele Fornea, James Rickards and Jan Vonk for helpful discussions. The authors thank the anonymous referee for their valuable feedback which helped to improve the quality of the exposition.

References

1. Bertolini, M., Darmon, H., Prasanna, K.: Generalized Heegner cycles and p-adic Rankin L-series. Duke Math. J. **162**(6), 1033–1148 (2013). With an appendix by Brian Conrad
2. Bertolini, M., Darmon, H., Prasanna, K.: Chow-Heegner points on CM elliptic curves and values of p-adic L-functions. Int. Math. Res. Not. IMRN **3**, 745–793 (2014)
3. Bertolini, M., Darmon, H., Prasanna, K.: p-adic L-functions and the coniveau filtration on Chow groups. J. Reine Angew. Math. **731**, 21–86 (2017). With an appendix by Brian Conrad
4. Bloch, S.: Torsion algebraic cycles and a theorem of Roitman. Compos. Math. **39**(1), 107–127 (1979)
5. Deligne, P.: Équations différentielles à points singuliers réguliers. Lecture Notes in Mathematics, vol. 163. Springer, Berlin (1970)
6. Deligne, P.: La conjecture de Weil. I. Inst. Hautes Études Sci. Publ. Math. **43**, 273–307 (1974)
7. Deligne, P.: Valeurs de fonctions L et périodes d'intégrales. In: Automorphic Forms, Representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, pp. 313–346. Amer. Math. Soc., Providence (1979). With an appendix by N. Koblitz and A, Ogus

8. El Zein, F., Zucker, S.: Extendability of normal functions associated to algebraic cycles. In: Topics in transcendental algebraic geometry (Princeton, N.J., 1981/1982), Ann. of Math. Stud., vol. 106, pp. 269–288. Princeton Univ. Press, Princeton (1984)
9. Hopkins, K.: Higher-weight Heegner points. Exp. Math. **19**(3), 257–266 (2010)
10. Im, B.H.: Heegner points and Mordell–Weil groups of elliptic curves over large fields. Trans. Am. Math. Soc. **359**(12), 6143–6154 (2007)
11. Jannsen, U.: Mixed Motives and Algebraic K-theory. Lecture Notes in Mathematics, vol. 1400. Springer, Berlin (1990)
12. Milne, J.S.: Étale cohomology, Princeton Mathematical Series, vol. 33. Princeton University Press, Princeton (1980)
13. Nekovář, J.: On the p-adic height of Heegner cycles. Math. Ann. **302**(4), 609–686 (1995)
14. Schoen, C.: Complex multiplication cycles on elliptic modular threefolds. Duke Math. J. **53**(3), 771–794 (1986)
15. Shimura, G.: Introduction to the arithmetic theory of automorphic functions. Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton (1971). Kanô Memorial Lectures, No. 1
16. Shnidman, A.: p-adic heights of generalized Heegner cycles. Ann. Inst. Fourier **66**(3), 1117–1174 (2016)
17. Voisin, C.: Hodge theory and complex algebraic geometry. I, *Cambridge Studies in Advanced Mathematics*, vol. 76. Cambridge University Press, Cambridge (2002). Translated from the French original by Leila Schneps
18. Zhang, S.: Heights of Heegner cycles and derivatives of L-series. Invent. Math. **130**(1), 99–152 (1997)